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Numer. Math. 25, 1—14 (1975)

On the Zeros and Poles of Padé Approximants to e^z

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Received February 4, 1975

Summary. In this paper, we study the location of the zeros and poles of general Padé approximants to e^z. The location of these zeros and poles is useful in the analysis of stability for related numerical methods for solving systems of ordinary differential equations.

1. Introduction

The study of the location of zeros and poles of the Padé approximants to e^z has proved to be of much significance because of its application to the analysis of stability of numerical methods used in solving certain systems of ordinary differential equations (cf. [1, 3, 12]). Indeed, the essence of Ehle's paper [3], concerning A-stability, was in showing that the entries in the first and second superdiagonals of the Padé table for e^z have all their zeros in the open left half-plane.

In the present paper, we substantially improve upon Ehle's results by studying the behavior of the zeros of approximants along *all* (super-as well as sub-) diagonals of the Padé table for e^z. In particular, we show that the approximants from the first four superdiagonals have all zeros in the open left half-plane, and this result is best possible. Furthermore, we obtain "close to sharp" zero-free infinite sectors for *every* Padé approximant. These new results and others are stated explicitly in § 2, with their proofs being given in § 3. For the remainder of this section, we introduce necessary notation and cite existing results.

Let π_m denote the collection of all polynomials in the variable z having degree at most m, and let $\pi_{n,\nu}$ be the set of all complex rational functions $r_{n,\nu}(z)$ of the form

$$r_{n,\nu}(z)=\frac{p_{n,\nu}(z)}{q_{n,\nu}(z)}, \qquad \text{where } p_{n,\nu}\in\pi_n,\ q_{n,\nu}\in\pi_\nu,\ q_{n,\nu}(0)=1.$$

Then, the (n,ν)-th Padé approximant to e^z is defined as that element $R_{n,\nu}(z)\in\pi_{n,\nu}$ for which

$$e^z-R_{n,\nu}(z)=\mathcal{O}(|z|^{n+\nu+1}) \quad \text{as} \quad |z|\to 0.$$

In explicit form, it is known [8, p. 245] that

$$R_{n,\nu}(z)=P_{n,\nu}(z)/Q_{n,\nu}(z),$$

* Research supported in part by the Air Force Office of Scientific Research under Grant AFOSR-74-2688, and by the University of South Fla. Research Council.

** Research supported in part by the Air Force Office of Scientific Research under Grant AFOSR-74-2729, and by the Atomic Energy Commission under Grant AT(11-1)-2075.

where

(1.1) $$P_{n,\nu}(z)=\sum_{j=0}^{n}\frac{(n+\nu-j)!\,n!\,z^j}{(n+\nu)!\,j!\,(n-j)!},$$

(1.2) $$Q_{n,\nu}(z)=\sum_{j=0}^{\nu}\frac{(n+\nu-j)!\,\nu!\,(-z)^j}{(n+\nu)!\,j!\,(\nu-j)!}.$$

We shall refer respectively to the polynomials $P_{n,\nu}(z)$ and $Q_{n,\nu}(z)$ as the *Padé numerator* and *denominator* of type (n,ν) for e^z.

Generally, one is interested in both the zeros and the poles of the Padé approximants $R_{n,\nu}(z)$. However, since the polynomials of (1.1) and (1.2) are related by the obvious identity

(1.3) $$Q_{n,\nu}(z)=P_{\nu,n}(-z),$$

it suffices then to investigate only the *zeros* of the Padé approximants $R_{n,\nu}(z)$; we leave it to the reader to supply all corresponding theorems for the poles.

The approximants $R_{n,\nu}(z)$ are typically displayed in the following doubly infinite array, known as the *Padé table*:

(1.4) $$\begin{bmatrix} R_{0,0}(z) & R_{1,0}(z) & R_{2,0}(z) & \cdots \\ R_{0,1}(z) & R_{1,1}(z) & R_{2,1}(z) & \cdots \\ R_{0,2}(z) & R_{1,2}(z) & R_{2,2}(z) & \cdots \\ \cdot & \cdot & \cdot & \cdots \\ \cdot & \cdot & \cdot & \cdots \\ \cdot & \cdot & \cdot & \cdots \end{bmatrix}.$$

Note that the entries $R_{n,0}(z)$, $n=0,1,2,\ldots$, in the first row are simply the partial sums of the Maclaurin expansion for e^z, i.e.,

$$R_{n,0}(z)=\frac{P_{n,0}(z)}{Q_{n,0}(z)}=\sum_{j=0}^{n}z^j/j!,\qquad n\geqq 0.$$

The convergence properties of sequences from table (1.4) were studied by Padé (cf. [8, § 75]). In particular, he showed that as $n+\nu\to\infty$, all the zeros (poles) of $R_{n,\nu}(z)$ approach infinity. A more recent result concerning the zeros of $R_{n,\nu}(z)$ is the following:

Theorem 1.1. (Ehle [3], Van Rossum [13]). If $n\leqq\nu+2$, the Padé approximant $R_{n,\nu}(z)$ for e^z has all its zeros in the open left half-plane.

Theorem 1.1 includes as a special case the result of Birkhoff and Varga [1] concerning the main diagonal entries $\{R_{n,n}(z)\}_{n=0}^{\infty}$ of the Padé table for e^z, and the result of Wimp [15] concerning the first superdiagonal entries $\{R_{n,n-1}(z)\}_{n=1}^{\infty}$.

Essential to the proof of the main theorems given in § 2 is the following result of the authors concerning zero-free parabolic regions.

Theorem 1.2. (Saff and Varga [9]). For each fixed $\nu\geqq 0$, and every $n\geqq 0$, the Padé approximant $R_{n,\nu}(z)$ for e^z has no zeros in the parabolic region

(1.5) $$\mathscr{P}_{\nu+1}:=\{z=x+iy:\ y^2\leqq 4(\nu+1)\,(x+\nu+1),\quad x>-(\nu+1)\}.$$

Consequently, every Padé approximant to e^z is zero-free in

$$\mathscr{P}_1=\{z=x+iy:\ y^2\leqq 4(x+1),\quad x>-1\}.$$

In the context of table (1.4), the above theorem asserts that along every row of the Padé table the zeros omit a parabolic region, and the size of the region increases with the row number. As discussed in [9], Theorem 1.2 improves upon the work of Newman and Rivlin (cf. [6, 7]) and Dočev [2], the last author having established that $R_{n,\nu}(z)$ is zero-free in $|z| < \nu + 1$.

2. Statements of New Results

We now list and discuss our main results, deferring their proofs to the next section.

Theorem 2.1. For every $n \geqq 2$, $\nu \geqq 0$, the Padé approximant $R_{n,\nu}(z)$ for e^z has no zeros in the infinite sector

$$\mathscr{S}_{n,\nu} := \left\{ z\colon\ |\arg z| \leqq \cos^{-1}\left(\frac{n-\nu-2}{n+\nu}\right)\right\}. \tag{2.1}$$

Consequently, for any (fixed) $\sigma > 0$, the sequence of Padé approximants $\{R_{n\,[\sigma^n]}(z)\}_{n=1}^{\infty}$, where $[\cdot]$ denotes the greatest integer function, is zero-free in the infinite sector

$$\mathscr{S}_\sigma := \left\{ z\colon |\arg z| \leqq \cos^{-1}\left(\frac{1-\sigma}{1+\sigma}\right)\right\}. \tag{2.2}$$

Note that if $n \leqq \nu + 2$, then the sector $\mathscr{S}_{n,\nu}$ in (2.1) contains the closed right half-plane, and thus Theorem 2.1 includes as a special case the known Theorem 1.1.

The second part of Theorem 2.1 has the following (informal) geometric interpretation: If one proceeds down the table (1.4) along a line from $R_{0,0}$ which makes an angle ϕ, $0 < \phi < \pi/2$, with the first row, then all entries encountered will be zero-free in the infinite sector

$$|\arg z| \leqq \cos^{-1}\left(\frac{1-\tan\phi}{1+\tan\phi}\right). \tag{2.3}$$

Note that as ϕ increases from 0 to $\pi/2$, the right hand member of (2.3) increases from 0 to π. We remark that with this interpretation, the case $\phi = 0$ corresponds to the sequence of Taylor sections $\{R_{n,0}(z)\}_{n=0}^{\infty}$ for e^z, and it is well known [11] that this sequence is *not* zero-free in *any* proper sector with vertex at the origin. Similarly, the case $\phi = \pi/2$ corresponds to the first column $\{R_{0,\nu}(z)\}_{\nu=0}^{\infty}$ of the Padé table for e^z, and, as these rational functions all have numerators unity (cf. (1.1)), it is evident that this sequence is zero-free in the whole plane, i.e., in $|\arg z| \leqq \pi$. Hence, Theorem 2.1 is sharp at the endpoints $\sigma = 0+$ and $\sigma = +\infty$.

To graphically illustrate the contents of the second part of Theorem 2.1, we have plotted in Figure 1 the zeros of $\{R_{n,[n/3]}(z)\}_{n=1}^{36}$ in the upper half-plane, corresponding to the choice $\sigma = 1/3$. In this case, the sector $\mathscr{S}_{1/3} = \left\{z\colon |\arg z| \leqq \cos^{-1}\left(\frac{1-1/3}{1+1/3}\right) = 60°\right\}$ contains no zeros from this sequence, and $\mathscr{S}_{1/3}$ is also indicated in Fig. 1. Similarly, we have plotted in Fig. 2 the zeros of $\{R_{n,3n}(z)\}_{n=1}^{21}$ in the upper half-plane, corresponding to the choice $\sigma = 3$, along with the sector $\mathscr{S}_3 = \left\{z\colon |\arg z| \leqq \cos^{-1}\left(\frac{1-3}{1+3}\right) = 120°\right\}$.

To give some indication as to how sharp Theorem 2.1 is in determining a zero-free sector for Padé approximants for e^z, the following computations were

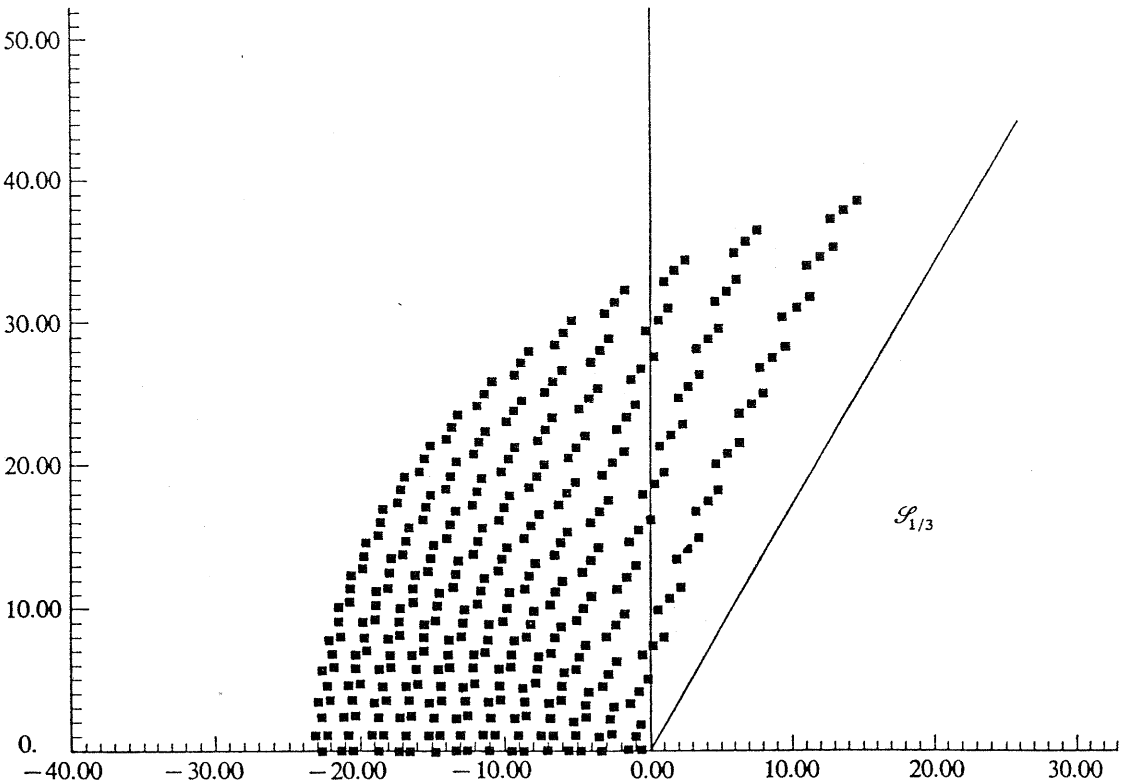

Fig. 1. Zeros of $R_{n,[n/3]}(z)$, $n = 1, 2, \ldots, 36$, and zero-free sector

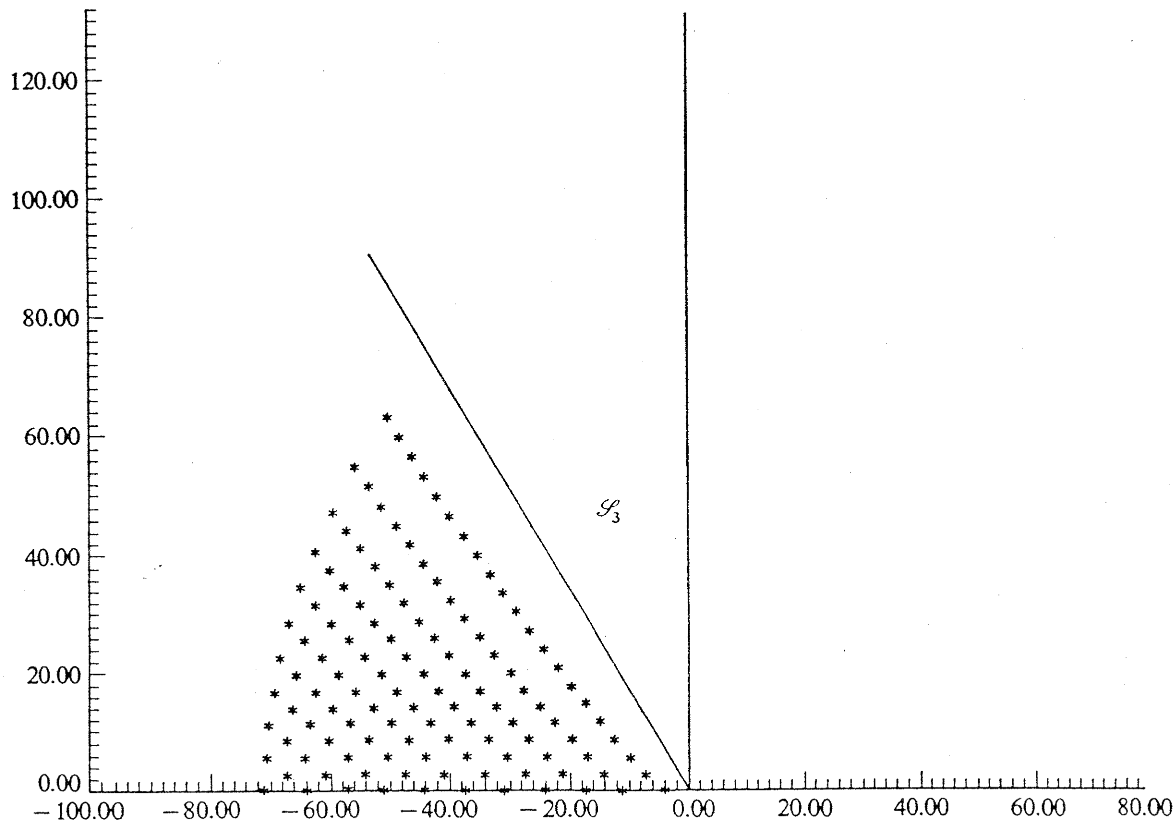

Fig. 2. Zeros of $R_{n,3n}(z)$, $n = 1, 2, \ldots, 21$, and zero-free sector

performed. First, let $\theta_{n,\nu}$ be defined as the minimum (positive) argument of all the zeros of the Padé numerator $P_{n,\nu}(z)$, i.e.,

$$\theta_{n,\nu} := \min\{|\arg z| : P_{n,\nu}(z) = 0\}.$$

In the second column of Table 1, the values of $\{\theta_{n,[n/3]};\ n = 3m\}_{m=11}^{21}$ are given. Because these values were apparently converging, though very slowly, repeated applications (5 in all) of Shanks' extrapolation (cf. [10]) were made to speed convergence, i.e., if the original sequence is $\{\alpha_n^{(0)}\}$, these extrapolations are defined successively by

$$\alpha_n^{(j+1)} := \frac{\alpha_{n+1}^{(j)} \cdot \alpha_{n-1}^{(j)} - (\alpha_n^{(j)})^2}{\alpha_{n+1}^{(j)} + \alpha_{n-1}^{(j)} - 2\alpha_n^{(j)}}.$$

Table 1. $\{\theta_{n,[n/3]};\ n = 3m\}_{m=11}^{21}$ and Shanks' Extrapolations

n	$\theta_{n,[n/3]}$	$\theta_{n,[n/3]}^{(1)}$	$\theta_{n,[n/3]}^{(2)}$	$\theta_{n,[n/3]}^{(3)}$	$\theta_{n,[n/3]}^{(4)}$	$\theta_{n,[n/3]}^{(5)}$
33	72.089492°					
36	71.370532°	66.707901°				
39	70.747621°	66.344545°	63.754270°			
42	70.201913°	66.025889°	63.567382°	62.116535°		
45	69.719275°	65.743796°	63.401820°	62.018951°	61.200693°	
48	69.288887°	65.492029°	63.253961°	61.931765°	61.149062°	60.685213°
51	68.902318°	65.265720°	63.120973°	61.853318°	61.102603°	
54	68.552893°	65.061011°	63.000613°	61.782292°		
57	68.235254°	64.874803°	62.891074°			
60	67.945045°	64.704574°				
63	67.678691°					

Note that the number appearing in the last column of Table 1 agrees rather closely with the value, namely $60° = \cos^{-1}\left(\frac{1-1/3}{1+1/3}\right)$, given by (2.2) of Theorem 2.1 for the case $\sigma = 1/3$. Table 2 gives the analogous results for the case $\sigma = 3$.

Table 2. $\{\theta_{n,3n}\}_{n=11}^{21}$ and Shanks' Extrapolations

n	$\theta_{n,3n}$	$\theta_{n,3n}^{(1)}$	$\theta_{n,3n}^{(2)}$	$\theta_{n,3n}^{(3)}$	$\theta_{n,3n}^{(4)}$	$\theta_{n,3n}^{(5)}$
11	132.965855°					
12	132.241924°	127.358009°				
13	131.611448°	126.979700°	124.194874°			
14	131.056510°	126.646637°	123.994906°	122.401052°		
15	130.563599°	126.350738°	123.817230°	122.294286°	121.379223°	
16	130.122319°	126.085785°	123.658117°	122.198675°	121.321687°	120.797293°
17	129.724527°	125.846903°	123.514645°	122.112464°	121.269840°	
18	129.363745°	125.630216°	123.384491°	122.034254°		
19	129.034755°	125.432596°	123.265780°			
20	128.733298°	125.251494°				
21	128.455861°					

Concerning entries along diagonals of the table (1.4), we establish the following two results.

Theorem 2.2. If $n \leqq \nu+4$, then the Padé approximant $R_{n,\nu}(z)$ for e^z has all its zeros in the open left half-plane.

As stated in the introduction, this theorem extends Ehle's results [3] from the first two to the first four superdiagonals of table (1.4). Moreover, we remark that the first entry of the fifth superdiagonal, i.e., $R_{5,0}(z)=\sum_{j=0}^{5} z^j/j!$, for which $n=\nu+5$, does in fact have a zero in the right half-plane (cf. [Table 3]). Hence, Theorem 2.2 is sharp in this sense.

Table 3

$P_{n,\nu}(z)$	Zeros of $P_{n,\nu}(z)$	$P_{n,\nu}(z)$	Zeros of $P_{n,\nu}(z)$
$P_{4,0}(z)$	$-1.729444 \pm i \cdot 0.888974$ $-0.270556 \pm i \cdot 2.504776$	$P_{5,0}(z)$	-2.180607 $-1.649503 \pm i \cdot 1.693933$ $+0.239806 \pm i \cdot 3.128335$
$P_{5,1}(z)$	-3.237113 $-2.678150 \pm i \cdot 2.181221$ $-0.703293 \pm i \cdot 4.260145$	$P_{6,1}(z)$	$-3.424888 \pm i \cdot 1.047551$ $-2.458301 \pm i \cdot 3.102173$ $-0.116811 \pm i \cdot 5.006586$
$P_{6,2}(z)$	$-4.454039 \pm i \cdot 1.217795$ $-3.464309 \pm i \cdot 3.639366$ $-1.081651 \pm i \cdot 6.023443$		

The next result states that on proceeding sufficiently far enough along *any* diagonal of the table (1.4), the entries have all zeros in the left half-plane. More precisely, we shall establish

Theorem 2.3. Given any integer τ, there exists an integer $m=m(\tau)$ such that the Padé approximants $\{R_{n,n-\tau}(z)\}_{n=m}^{\infty}$ to e^z have all their zeros in the open left half-plane.

The final two results concern half-planes and disks containing all the zeros of $R_{n,\nu}(z)$.

Theorem 2.4. If

$$1<n<3\nu+4, \tag{2.4}$$

then all the zeros of the Padé approximant $R_{n,\nu}(z)$ for e^z lie in the half-plane

$$\operatorname{Re} z<n-\nu-2. \tag{2.5}$$

Theorem 2.5. For any $n \geqq 3$, $\nu \geqq 0$, all the zeros of $R_{n,\nu}(z)$ lie in the disk

$$|z| \leqq \frac{2(n+\nu)(n+\nu-1)}{(n+2\nu+1)}. \tag{2.6}$$

Furthermore, all those zeros of $R_{n,\nu}(z)$ in $\operatorname{Re} z \geqq 0$ satisfy the inequality

$$|z| \leqq 2(n-3). \tag{2.7}$$

3. Proofs of New Results

In proving the theorems of § 2, we shall make use of the following lemma which is in the spirit of Wall [14].

Lemma 3.1. Let $\lambda_0 := 1$, and let the real numbers λ_k, $k = 1, 2, \ldots, n$, satisfy the inequalities

$$0 < \lambda_k < 1,\ 1 \leqq k \leqq n-1, \qquad 0 \leqq \lambda_n < 1, \tag{3.1}$$

for any fixed $n \geqq 1$. Then, the Padé numerator $P_{n,\nu}(z)$ of (1.1) is different from zero at any point z which satisfies the inequalities

$$|z| + \frac{\operatorname{Re} z}{|z|}\left\{\frac{[2(k+\nu)\,\lambda_{k-1} - k + 1]}{2\lambda_{k-1}(1-\lambda_k)}\right\} > \frac{(k-1)}{2\lambda_{k-1}(1-\lambda_k)}, \tag{3.2}$$

for every $k = 1, 2, \ldots, n$.

Proof. Let $z \neq 0$ be any fixed complex number satisfying (3.2) for all $1 \leqq k \leqq n$, and define (in terms of the Padé numerators $P_{n,\nu}$ of (1.1)) the quantity

$$\mu_{k,\nu} = \mu_{k,\nu}(z) := P_{k,\nu}(z)/z\,P_{k-1,\nu}(z), \qquad k = 1, 2, \ldots, n. \tag{3.3}$$

We shall show inductively that

$$\operatorname{Re} \mu_{k,\nu} > \lambda_k/(k+\nu), \qquad \text{for } k = 1, 2, \ldots, n. \tag{3.4}$$

For $k = 1$, we have from (3.3) and (1.1) that

$$\mu_{1,\nu} = \frac{P_{1,\nu}(z)}{z\,P_{0,\nu}(z)} = \frac{1 + \dfrac{z}{(\nu+1)}}{z} = \frac{1}{z} + \frac{1}{\nu+1},$$

from which it follows that $\operatorname{Re} \mu_{1,\nu} > \lambda_1/(\nu+1)$ if and only if

$$\frac{\operatorname{Re} z}{|z|^2} + \frac{1}{(\nu+1)} > \frac{\lambda_1}{\nu+1}. \tag{3.5}$$

Since by hypothesis $\lambda_1 < 1$, inequality (3.5) can be written in the equivalent form

$$|z| + \frac{\operatorname{Re} z}{|z|}\left\{\frac{(\nu+1)}{(1-\lambda_1)}\right\} > 0,$$

which is the same as the case $k = 1$ of inequality (3.2). Hence, $\operatorname{Re} \mu_{1,\nu} > \lambda_1/(1+\nu)$, the case $k = 1$ of (3.4).

Now, assume inductively that $\operatorname{Re} \mu_{k-1,\nu} > \lambda_{k-1}/(k-1+\nu)$ for some k with $2 \leqq k \leqq n$. From the following known three-term recurrence relation of Frobenius [4] (which can be directly verified from (1.1)):

$$P_{k,\nu}(z) = \left[\frac{z}{k+\nu} + 1\right] P_{k-1,\nu}(z) - \frac{(k-1)}{(k+\nu)(k+\nu-1)}\, z\, P_{k-2,\nu}(z), \tag{3.6}$$

we can express $\mu_{k,\nu}$, using (3.3), as

$$\begin{aligned}
\mu_{k,\nu} = \frac{P_{k,\nu}(z)}{z\,P_{k-1,\nu}(z)} &= \frac{\left[\dfrac{z}{k+\nu} + 1\right] P_{k-1,\nu}(z) - \dfrac{(k-1)}{(k+\nu)(k+\nu-1)}\, z\, P_{k-2,\nu}(z)}{z\,P_{k-1,\nu}(z)} \\
&= \frac{1}{k+\nu} + \frac{1}{z} - \frac{(k-1)}{(k+\nu)(k+\nu-1)} \cdot \frac{P_{k-2,\nu}(z)}{P_{k-1,\nu}(z)} \\
&= \frac{1}{k+\nu} + \frac{1}{z} - \frac{(k-1)}{(k+\nu)(k+\nu-1)} \cdot \frac{1}{z\,\mu_{k-1,\nu}}.
\end{aligned}$$

In other words, we can write

$$\mu_{k,\nu} = T_{k,\nu}(\mu_{k-1,\nu}), \tag{3.7}$$

where $T_{k,\nu}(w)$ is the bilinear transformation defined by

$$\xi = T_{k,\nu}(w) := \frac{1}{k+\nu} + \frac{1}{z} - \frac{(k-1)}{(k+\nu)(k+\nu-1)zw}. \tag{3.8}$$

Since $\operatorname{Re}\mu_{k-1,\nu} > \lambda_{k-1}/(k-1+\nu)$ by hypothesis, $\mu_{k,\nu}$ lies in the image of the half-plane $\operatorname{Re} w > \lambda_{k-1}/(k-1+\nu)$ under the transformation $T_{k,\nu}$.
Now, as the pole of $T_{k,\nu}$ is at $w=0$, and as $\lambda_{k-1}>0$ from (3.1), $T_{k,\nu}$ maps $\operatorname{Re} w > \lambda_{k-1}/(k-1+\nu)$ onto an open disk D_k in the ξ-plane. The center ξ_k of this disk is, by the symmetry principle, the image, under $T_{k,\nu}$, of the point in the w-plane, viz. $2\lambda_{k-1}/(k-1+\nu)$, which is the reflection of the pole $w=0$ in the line $\operatorname{Re} w = \lambda_{k-1}/(k-1+\nu)$, i.e.,

$$\xi_k = T_{k,\nu}\left(\frac{2\lambda_{k-1}}{k-1+\nu}\right) = \frac{1}{k+\nu} + \frac{1}{z} - \frac{(k-1)}{2(k+\nu)z\lambda_{k-1}}.$$

Furthermore, since the point

$$T_{k,\nu}(\infty) = \frac{1}{k+\nu} + \frac{1}{z}$$

evidently lies on the boundary of D_k, the radius r_k of this disk is given by

$$r_k = |\xi_k - T_{k,\nu}(\infty)| = \frac{(k-1)}{2(k+\nu)|z|\lambda_{k-1}}.$$

Consequently, the real part of any point in D_k must exceed the quantity

$$\begin{aligned} \operatorname{Re}\xi_k - r_k = {} & \frac{1}{k+\nu} + \frac{\operatorname{Re} z}{|z|^2}\left(1 - \frac{(k-1)}{2(k+\nu)\lambda_{k-1}}\right) \\ & - \frac{(k-1)}{2(k+\nu)|z|\lambda_{k-1}}, \end{aligned} \tag{3.9}$$

which one can directly verify is greater than $\lambda_k/(k+\nu)$ because z satisfies (3.2) and λ_k satisfies (3.1). This then establishes (3.4) for every $k=1, 2, \ldots, n$.

In particular, when $k=n$, we have

$$\operatorname{Re}\mu_{n,\nu} = \operatorname{Re}\left[\frac{P_{n,\nu}(z)}{z P_{n-1,\nu}(z)}\right] > \frac{\lambda_n}{(n+\nu)} \geqq 0. \tag{3.10}$$

Finally, since $P_{n,\nu}(z)$ and $P_{n-1,\nu}(z)$ have no zeros in common (cf. [9]), it follows from (3.10) that $P_{n,\nu}(z) \neq 0$ for any z satisfying the inequalities (3.2). ▪

Now, if $z = re^{i\theta}$ is any nonreal point in the plane, the constants

$$\lambda_k := \tfrac{1}{2}(1+\cos\theta), \; k=1, 2, \ldots, n \tag{3.11}$$

obviously satisfy the inequalities (3.1). Furthermore, on substituting $\cos\theta$ for $\operatorname{Re} z/|z|$, and on substituting (3.11) for λ_k, the inequalities (3.2) become, after some algebraic manipulations,

$$|z| > 2(k+\nu) - \frac{2(\nu+1)}{(1-\cos\theta)}, \; k=1, 2, \ldots, n. \tag{3.12}$$

As the strongest of the inequalities (3.12) occurs when $k=n$, Lemma 3.1 yields

Corollary 3.2. The Padé numerator $P_{n,\nu}(z)$ of (1.1) is different from zero at any nonreal point z which satisfies the inequality

$$|z| > 2(n+\nu) - \frac{2(\nu+1)}{(1-\cos\theta)}, \qquad \theta = \arg z. \tag{3.13}$$

Using Corollary 3.2 and Theorem 1.2, we now give the

Proof of Theorem 2.1. We first show that $P_{n,\nu}(z)$ is zero-free in that portion of the sector $\mathscr{S}_{n,\nu}$ (cf. (2.1)) defined by

$$\mathscr{G}_{n,\nu} := \left\{ z = re^{i\theta} : r > n+\nu,\ 0 < |\theta| \leqq \cos^{-1}\left(\frac{n-\nu-2}{n+\nu}\right)\right\}.$$

Indeed, if $z = re^{i\theta} \in \mathscr{G}_{n,\nu}$, then $1 > \cos\theta \geqq (n-\nu-2)/(n+\nu)$, and thus

$$2(n+\nu) - \frac{2(\nu+1)}{(1-\cos\theta)} \leqq 2(n+\nu) - (n+\nu) = n+\nu < |z|,$$

i.e., inequality (3.13) holds. Hence, by Corollary 3.2, $P_{n,\nu}(z) \neq 0$ for any $z \in \mathscr{G}_{n,\nu}$.

Now, according to Theorem 1.2, $P_{n,\nu}(z)$ is also zero-free in the parabolic region $\mathscr{P}_{\nu+1}$ defined in (1.5). This region can be written in the equivalent form

$$\mathscr{P}_{\nu+1} = \{z = re^{i\theta} : r \leqq 2(\nu+1)/(1-\cos\theta),\ \theta \neq \pi\}.$$

But, for $n \geqq 2$, it is easy to verify that $(\mathscr{S}_{n,\nu} \setminus \mathscr{G}_{n,\nu}) \subset \mathscr{P}_{\nu+1}$. Consequently, $P_{n,\nu}(z) \neq 0$ for any $z \in \mathscr{S}_{n,\nu} \setminus \mathscr{G}_{n,\nu}$, and hence, $P_{n,\nu}(z) \neq 0$ for any $z \in \mathscr{S}_{n,\nu}$. Since the zeros of the Padé approximant $R_{n,\nu}(z)$ are the same as the zeros of $P_{n,\nu}(z)$ then $R_{n,\nu}(z)$ has no zeros in $\mathscr{S}_{n,\nu}$, which establishes the first part of Theorem 2.1.

Finally, applying the above part of Theorem 2.1 to $R_{n,[\sigma n]}(z)$ where $\sigma > 0$, then $R_{n,[\sigma n]}(z)$ has no zeros in $\mathscr{S}_{n,[\sigma n]}$. But, it is readily verified that

$$\frac{n-[\sigma n]-2}{n+[\sigma n]} < \frac{1-\sigma}{1+\sigma} \qquad \text{for all } n \geqq 1,$$

whence (cf. (2.2)) $\mathscr{S}_\sigma \subset \mathscr{S}_{n,[\sigma n]}$ for all $n \geqq 1$. Thus, $\{R_{n,[\sigma n]}(z)\}_{n=1}^{\infty}$ is zero-free in $\mathscr{S}_\sigma$, completing the proof of Theorem 2.1. ▪

It is interesting to note that Theorem 2.1, as stated, fails for the case $n=1$, since the sector $\mathscr{S}_{1,\nu}$ of (2.1) in this case is the entire complex plane. On the other hand, since $R_{1,\nu}(z)$ has its sole zero in the point $z = -(1+\nu)$, Theorem 2.1 is *valid* for the case $n=1$ if the inequality in the definition (2.1) of $\mathscr{S}_{1,\nu}$ is replaced by a strict inequality, i.e., if

$$\mathscr{S}_{1,\nu} := \left\{ z : |\arg z| < \cos^{-1}\left(\frac{-1-\nu}{1+\nu}\right) = \pi \right\}.$$

Proof of Theorem 2.2. Because of Theorem 1.1 (or 2.1), we need only show that the two sequences $\{P_{n,n-3}(z)\}_{n=3}^{\infty}$ and $\{P_{n,n-4}(z)\}_{n=4}^{\infty}$ are zero-free in the closed right half-plane.

To deal with the first sequence, let $n \geqq 3$ and $\nu \geqq 0$ be fixed, and define

$$\lambda_k := \tfrac{1}{2}, \qquad \text{for } k = 1, 2, \ldots, n-1,\ \lambda_n := 0.$$

Then, the inequalities (3.2) become

$$|z| + \frac{\operatorname{Re} z}{|z|}\{2(\nu+1)\} > 2(k-1), \qquad \text{for } k = 1, 2, \ldots, n-1,$$

$$|z| + \frac{\operatorname{Re} z}{|z|}(\nu+1) > (n-1), \qquad \text{for } k = n.$$

Thus, when $\operatorname{Re} z \geqq 0$, all of these inequalities are satisfied for $|z| > 2(n-2)$. Hence, by Lemma 3.1, $P_{n,\nu}(z) \neq 0$ for all z in the set

$$\mathscr{B}_n := \{z : \operatorname{Re} z \geqq 0, |z| > 2(n-2)\}, n \geqq 3, \nu \geqq 0.$$

In particular, $P_{n,n-3}(z)$ is zero-free in $\mathscr{B}_n$. On the other hand, by Theorem 1.2, $P_{n,n-3}(z) \neq 0$ for any z in the parabolic region $\mathscr{P}_{n-2}$ of (1.5), and this region contains the half-disk $|z| \leqq 2(n-2)$, $\operatorname{Re} z \geqq 0$. Thus, $P_{n,n-3}(z)$ is zero-free in the entire closed right half-plane for every $n \geqq 3$.

To deal with the fourth superdiagonal, we proceed in an analogous fashion. Let $n \geqq 7$ and $\nu \geqq 1$ be fixed, and define

$$\lambda_k := \frac{1}{2}, \; 1 \leqq k \leqq n-2, \; \lambda_{n-1} := \frac{n-4}{2(n-3)}, \; \lambda_n := \frac{n-7}{2(n-4)}.$$

Then, the constants λ_k satisfy (3.1), and, as can be directly verified, the associated inequalities (3.2) will all be satisfied for z in the set

$$\mathscr{C}_n := \{z : \operatorname{Re} z \geqq 0, |z| > 2(n-3)\}.$$

Hence, for $n \geqq 7$, $\nu \geqq 1$, $P_{n,\nu}(z)$ is zero-free in $\mathscr{C}_n$. In particular, this is true for the polynomial $P_{n,n-4}(z)$, provided that $n \geqq 7$. Again, as a consequence of Theorem 1.2, $P_{n,n-4}(z)$ is also zero-free in the half-disk $|z| \leqq 2(n-3)$, $\operatorname{Re} z \geqq 0$. Thus, the polynomials $\{P_{n,n-4}(z)\}_{n=7}^{\infty}$ are zero-free in the entire closed right half-plane. Finally, computer computations verify that the same is true of the polynomials $P_{4,0}(z)$, $P_{5,1}(z)$, and $P_{6,2}(z)$. (For the reader's convenience, we have listed the zeros of these polynomials in Table 3.) Hence, the entire sequence $\{P_{n,n-4}\}_{n=4}^{\infty}$ is zero-free in the closed right half-plane. ▪

The method used in the above proof is generalized in

Lemma 3.3. Given any integer $\tau \geqq 4$, there exists integers $N = N(\tau)$ and $M = M(\tau)$ such that for all $n \geqq N(\tau)$, $\nu \geqq M(\tau)$, the Padé numerator $P_{n,\nu}(z)$ of (1.1) is zero-free in the region

$$\mathscr{D}_n^{(\tau)} := \{z : \operatorname{Re} z \geqq 0, |z| > 2(n-\tau+1)\}. \tag{3.14}$$

Proof. From the inequalities (3.2) of Lemma 3.1, it is sufficient to show that there exist constants λ_k, $k = 1, 2, \ldots, n$, which satisfy the inequalities (3.1) and have the following additional properties:

$$\frac{(k-1)}{2\lambda_{k-1}(1-\lambda_k)} \leqq 2(n-\tau+1), \quad \text{for } k = 1, 2, \ldots, n, \tag{3.15}$$

$$0 \leqq 2(k+\nu)\lambda_{k-1} - k + 1, \quad \text{for } k = 1, 2, \ldots, n. \tag{3.16}$$

For this purpose, we let $\alpha := n - \tau$ and define recursively for $\alpha > -1$

$$\lambda_k := \frac{1}{2}, \; 1 \leqq k \leqq \alpha+2, \; \lambda_{\alpha+s} := 1 - \frac{\alpha+s-1}{4\lambda_{\alpha+s-1}(\alpha+1)}, \; 3 \leqq s \leqq \tau. \tag{3.17}$$

We shall show that, for n and ν sufficiently large, the constants of (3.17) satisfy (3.1), (3.15), and (3.16).

This is obviously the case for $1 \leqq k \leqq \alpha+2$, so we deal only with the constants $\lambda_{\alpha+s}$ for $3 \leqq s \leqq \tau$. From the recursive definition of the $\lambda_{\alpha+s}$'s in (3.17), we see that

$$\lambda_{\alpha+3} = \frac{\alpha}{2(\alpha+1)}, \quad \lambda_{\alpha+4} = \frac{\alpha-3}{2\alpha},$$

and in general, by induction,

$$(3.18)\qquad \begin{aligned} \lambda_{\alpha+2j-1} &= \frac{\alpha^{j-1}+p_{j-2}(\alpha)}{2(\alpha+1)\{\alpha^{j-2}+q_{j-3}(\alpha)\}}, \quad j=2,3,\ldots,[(\tau+1)/2],\\ \lambda_{\alpha+2j} &= \frac{\alpha^{j-1}+q_{j-2}(\alpha)}{2\{\alpha^{j-1}+p_{j-2}(\alpha)\}}, \quad j=2,3,\ldots,[\tau/2], \end{aligned}$$

where p_i and q_i are polynomials in α of degree at most i, whose coefficients are absolute constants (independent of τ), with $q_{-1}(\alpha):=0$ and $p_0(\alpha):=0$. Next, as the polynomials

$$\alpha^{j-1}+p_{j-2}(\alpha) \quad \text{and} \quad \alpha^{j-1}+q_{j-2}(\alpha),$$

for each $2\leqq j\leqq[(\tau+1)/2]$, are evidently both positive for all sufficiently large values α, it follows from (3.18) that there exists an $\alpha_0=\alpha_0(\tau)>0$ such that $\lambda_{\alpha+s}>0$ for all $3\leqq s\leqq\tau$ and all $\alpha\geqq\alpha_0$. Moreover, from the second definition of (3.17), we further see that the inequalities (3.1) hold for all $3\leqq s\leqq\tau$, $\alpha\geqq\alpha_0$. Also, direct substitution of (3.17) shows that (3.15) holds with equality for all $3\leqq s\leqq\tau$. Thus, it only remains to verify (3.16) for $k=\alpha+s$, $3\leqq s\leqq\tau$, i.e.,

$$(3.19)\qquad \frac{\alpha+s-1}{2(\alpha+s+\nu)}\leqq\lambda_{\alpha+s-1}.$$

For this purpose, we first use (3.17) to write (3.19) in the equivalent form

$$(3.20)\qquad \frac{(\alpha+s+\nu)(\alpha+s-2)}{2(\alpha+1)(\alpha+s+2\nu+1)}\leqq\lambda_{\alpha+s-2},\ 3\leqq s\leqq\tau,$$

and then we consider separately the cases when s is even, and when s is odd. When s is even, i.e., $s-2=2j$, inequality (3.20) becomes

$$\frac{(\alpha+2j+2+\nu)(\alpha+2j)}{2(\alpha+1)(\alpha+2j+2\nu+3)}\leqq\lambda_{\alpha+2j},$$

which, using the representation (3.18), can be written after some minor manipulations in the equivalent form

$$(3.21)\qquad 0\leqq(\nu-2j+c_j)\alpha^j+\sum_{i=0}^{j-1}a_i^{(j)}(\nu)\cdot\alpha^i,$$

where c_j is an absolute constant, and where the coefficients $a_i^{(j)}(\nu)$ are polynomials of degree at most unity in ν. Now, as $j\leqq[(\tau+1)/2]$, there exists a positive constant $\nu_0=\nu_0(\tau)$ such that for $\nu\geqq\nu_0$, the coefficient, $(\nu-2j+c_j)$, of α^j in (3.21) is positive for every $2\leqq j\leqq[(\tau+1)/2]$. Thus, on dividing by the positive quantity $(\nu-2j+c_j)\alpha^j$, (3.21) becomes

$$(3.22)\qquad 0\leqq1+\sum_{i=0}^{j-1}\frac{a_i^{(j)}(\nu)}{(\nu-2j+c_j)}\alpha^{i-j},\ \nu\geqq\nu_0,$$

and since $a_i^{(j)}(\nu)$ is a polynomial of degree at most unity in ν, the ratios $a_i^{(j)}(\nu)/(\nu-2j+c_j)$ are uniformly bounded below for all $\nu\geqq\nu_0$. Consequently, there exists a positive constant $\alpha_1=\alpha_1(\tau)$ (with $\alpha_1\geqq\alpha_0$) such that for all $\alpha\geqq\alpha_1$, all $\nu\geqq\nu_0$, and for all $2\leqq j\leqq[(\tau+1)/2]$, inequality (3.22) is satisfied. In other words, (3.20) holds for all $n\geqq\alpha_1+\tau$ and all $\nu\geqq\nu_0$ when s is even. Similar reasoning applies in the case when s is odd, so that (3.20) is valid for all $n\geqq N(\tau)$, $\nu\geqq M(\tau)$, $3\leqq s\leqq\tau$. ▪

Proof of Theorem 2.3. From Theorem 2.2, we know that Theorem 2.3 is true for each $\tau \leqq 4$ with $m(\tau) = \min(\tau, 0)$. For $\tau > 4$, it follows from Lemma 3.3 that the Padé numerator $P_{n,n-\tau}(z)$ of (1.1) has no zeros in the region $\mathcal{D}_n^{(\tau)}$ of (3.14) for every $n \geqq m(\tau) := \max(N(\tau), M(\tau)+\tau)$. Furthermore, the result of Theorem 1.2 implies that $P_{n,n-\tau}(z)$ is zero-free in the half-disk $|z| \leqq 2(n-\tau+1)$, $\operatorname{Re} z \geqq 0$. Therefore, $P_{n,n-\tau}(z)$ is zero-free in the entire closed right half-plane for every $n \geqq m(\tau)$. ▪

If we call $\tilde{m}(\tau)$ the least nonnegative integer for which the sequence $\{P_{n,n-\tau}(z)\}_{n=\tilde{m}(\tau)}^{\infty}$ is zero-free in the closed right half-plane, then numerical computations indicate that

$$\tilde{m}(5) = 6;\ \tilde{m}(6) = 9,\ \tilde{m}(7) = 14;\ \tilde{m}(8) = 19,\ \tilde{m}(9) = 26,$$

so that $\tilde{m}(\tau)$ appears to be a monotone increasing function of τ.

Proof of Theorem 2.4. With the assumption of (2.4), we consider first the case when $n > \nu + 2$, and show that $P_{n,\nu}(z)$ is zero-free for $\operatorname{Re} z \geqq n-\nu-2$. From Theorem 2.1, it suffices to deal with only those z which satisfy

$$\operatorname{Re} z \geqq n-\nu-2,\ 0 < \cos\theta < \left(\frac{n-\nu-2}{n+\nu}\right),\ \theta = \arg z. \tag{3.23}$$

Since

$$|z| = \frac{\operatorname{Re} z}{\cos\theta} \geqq \left(\frac{n-\nu-2}{\cos\theta}\right),$$

the inequality (3.13) of Corollary 3.2 will hold if

$$\left(\frac{n-\nu-2}{\cos\theta}\right) > 2(n+\nu) - \frac{2(\nu+1)}{(1-\cos\theta)},$$

i.e., if

$$\left(\cos\theta - \frac{1}{2}\right)\left\{\cos\theta - \left(\frac{n-\nu-2}{n+\nu}\right)\right\} > 0. \tag{3.24}$$

Now, the second inequality of (3.23) gives us that the second factor above is negative, while the assumption of (2.4) coupled with the second inequality of (3.23) gives us that

$$\cos\theta < \left(\frac{n-\nu-2}{n+\nu}\right) < \frac{1}{2},$$

i.e., the first factor of (3.24) is also negative, whence (3.24) is valid. Thus, by Corollary 3.2, $P_{n,\nu}(z) \neq 0$ for all z satisfying (3.23), and therefore $P_{n,\nu}(z)$ is zero-free for $\operatorname{Re} z \geqq n-\nu-2$.

Now if $n = \nu + 2$, then the assertion of Theorem 2.4 follows immediately from Theorem 2.1. Finally, if $1 < n < \nu + 2$, then again from Theorem 2.1 we need only show that $P_{n,\nu}(z) \neq 0$ for all z satisfying

$$n-\nu-2 \leqq \operatorname{Re} z < 0,\ -1 \leqq \cos\theta < \left(\frac{n-\nu-2}{n+\nu}\right),\ \theta = \arg z. \tag{3.25}$$

But, any such z lies in the parabolic region $\mathcal{P}_{\nu+1}$ of (1.5), which is free of zeros of $P_{n,\nu}(z)$ from Theorem 1.2. Hence, $P_{n,\nu}(z)$ is zero-free for $\operatorname{Re} z \geqq n-\nu-2$. ▪

Proof of Theorem 2.5. The first part of this result follows by defining

$$\lambda_k := \frac{k}{2(k+\nu+1)},\ 1 \leqq k \leqq n-1,\ \lambda_0 := 1,\ \lambda_n := 0,$$

and applying Lemma 3.1. Indeed, for these values of λ_k, the inequalities of (3.2) of Lemma 3.1 reduce to

i) $\left\{x+\dfrac{(\nu+1)(\nu+2)}{(2\nu+3)}\right\}^2+y^2>\left\{\dfrac{(\nu+1)(\nu+2)}{(2\nu+3)}\right\}^2$, $\quad k=1,\ (z=x+iy)$,

ii) $|z|>\dfrac{2(k+\nu)(k+\nu+1)}{(k+2\nu+2)}$ $\quad, 2\leqq k\leqq n-1$,

iii) $|z|>n+\nu$ $\quad, k=n$.

Now, it is easily verified that, when $n\geqq 3$, the inequality of ii) for $k=n-1$ implies the remaining inequalities above. Thus, from Lemma 3.1, $|z|>2(n+\nu-1)\cdot(n+\nu)/(n+2\nu+1)$ contains no zeros of $P_{n,\nu}(z)$, which gives (2.6) of Theorem 2.5.

To establish (2.7) of Theorem 2.5, we know from the proof of Theorem 2.2 that $P_{n,\nu}(z)$ is zero-free in $\mathscr{C}_n:=\{z\colon \operatorname{Re} z\geqq 0, |z|>2(n-3)\}$, for all $n\geqq 7$ and all $\nu\geqq 1$. Similarly, from Theorem 2.2, $P_{n,\nu}(z)$ has all its zeros in the open left half-plane for all $n\leqq \nu+4$. Thus, it remains to consider the following three cases: $P_{n,0}(z)$ for $n\geqq 6$, $P_{5,0}(z)$, and $P_{6,1}(z)$. For any $n\geqq 1$, it is well known (cf. [5]) that all the zeros $\{z_j^{(n)}\}_{j=1}^n$ of $P_{n,0}(z)$ satisfy $|z_j^{(n)}|\leqq n$. Hence, $P_{n,0}(z)$ has no zeros in $\mathscr{C}_n$ for any $n\geqq 6$. That $P_{5,0}(z)$ and $P_{6,1}(z)$ have no zeros in $\mathscr{C}_n$ follows directly from Table 3. ▪

It is interesting to note that (2.6) of Theorem 2.5 fails for the excluded cases $n=1$ and $n=2$, because, using (1.1), $P_{1,\nu}(z)$ has its sole zero in $-(\nu+1)$, while the zeros of $P_{2,\nu}(z)$ are $-(\nu+1)\pm i(\nu+1)^{1/2}$, which have moduli $(\nu^2+3\nu+2)^{1/2}$. On the other hand, since the zeros for $P_{1,\nu}(z)$ and $P_{2,\nu}(z)$ are all in the open left half-plane, (2.7) of Theorem 2.5 vacuously holds for the cases $n=1$ and $n=2$.

There are two consequences of Theorem 2.5 worthy of mention. For $n\geqq \nu+4$, the half-disk $\{z\colon \operatorname{Re} z\geqq 0$ and $|z|\leqq 2(n-3)\}$ of (2.7) of Theorem 2.5 intersects the parabolic region $\mathscr{P}_{\nu+1}$ of (1.5) in the point $(\hat{x},\hat{y})$, where

(3.26) $$\hat{x}=2(n-\nu-4);\ \hat{y}=2[(\nu+1)(2n-\nu-7)]^{1/2}.$$

As a consequence of Theorem 2.5 and Theorem 1.2, we easily deduce

Corollary 3.3. For every n and ν with $n\geqq \nu+4$, the Padé approximant $R_{n,\nu}(z)$ for e^z has no zeros in the infinite sector

(3.27) $$\tilde{\mathscr{S}}_{n,\nu}:=\left\{z\colon |\arg z|\leqq \cos^{-1}\left(\frac{n-\nu-4}{n-3}\right)\right\},$$

of the right half-plane. Moreover (cf. (2.1)),

(3.28) $$\tilde{\mathscr{S}}_{n,\nu}\supseteqq \mathscr{S}_{n,\nu}$$

only when $n\leqq \nu+6$.

Finally, from the point of intersection of (3.26), it evidently follows that all the zeros of $P_{n,\nu}(z)$ lie in the half-plane

$$\operatorname{Re} z<2(n-\nu-4)\quad \text{when } n\geqq \nu+4.$$

On the other hand, with the result of Theorem 2.2, we deduce the following result which complements Theorem 2.4.

Corollary 3.4. For any $n\geqq 0$ and $\nu\geqq 0$, the Padé approximant $R_{n,\nu}(z)$ for e^z has all its zeros in the left half-plane defined by

(3.29) $$\operatorname{Re} z<\max\{2(n-\nu-4);\ 0\}.$$

Acknowledgement. We are indebted to A. Price and P. Comadoll of the University of South Florida, and A. Ruttan of Kent State University for their computations, respectively on an IBM 360/65 and a Burroughs B-5700, which were used in preparing Figures 1 and 2, and Tables 1–3.

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