ZERO-FREE PARABOLIC REGIONS FOR SEQUENCES OF POLYNOMIALS*

Dedicated to Nicholas C. Metropolis on the Occasion of his 60th Birthday, June 11, 1975

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Abstract. In this paper, we show that certain sequences of polynomials $\{p_k(z)\}_{k=0}^n$, generated from three-term recurrence relations, have no zeros in parabolic regions in the complex plane of the form $y^2 \leq 4\alpha(x + \alpha)$, $x > -\alpha$. As a special case of this, no partial sum $s_n(z) = \sum_{k=0}^n z^k/k!$ of e^z has a zero in $y^2 \leq 4(x+1)$, $x > -1$, for any $n \geq 1$. Such zero-free parabolic regions are obtained for Padé approximants of certain meromorphic functions, as well as for the partial sums of certain hypergeometric functions.

1. Introduction. In his thesis [11] and in [12], the second author obtained results concerning the existence of unbounded zero-free regions in the complex plane $\mathbb{C}$ for the partial sums of special entire functions $f(z) = \sum_{k=0}^\infty a_k z^k$, with $a_k \geq 0$ for all k. In particular, it was shown in [12] that the partial sums $s_n(z)$ of the exponential function $f(z) = e^z$, i.e.,

$$(1.1) \qquad s_n(z) := \sum_{k=0}^n z^k/k!$$

have no zeros in the infinite half-strip $|\mathrm{Im}\, z| \leq \sqrt{6}$, $\mathrm{Re}\, z \geq 0$, for any $n = 1, 2, \cdots$. More recently, Newman and Rivlin [8] stated that the parabolic-like domain

$$(1.2) \qquad |y| \leq -\frac{\pi}{2} + \tau\left(x + \frac{\pi^2}{4\tau^2}\right)^{1/2}, \qquad x \geq 0, \qquad \tau \doteq 1.637\,017,$$

is free of zeros of the $s_n(z)$ in (1.1) for all n sufficiently large. However, in [9] this result of (1.2) was retracted, and, using different methods, Newman and Rivlin proved that the smaller region

$$(1.3) \qquad y^2 \leq dx, \qquad x \geq 0, \qquad d \doteq 0.745\,407,$$

is zero-free for every $s_n(z)$.

The purpose of the present paper is to establish the existence of zero-free parabolic regions for certain *general* sequences of polynomials. As a special case of our main result, we deduce that the parabolic region

$$(1.4) \qquad y^2 \leq 4(x+1), \qquad x > -1,$$

is zero-free for all the partial sums of the exponential function. As the unbounded set of (1.4) contains the region of (1.3) (and in fact the region of (1.2) as well), we thereby improve upon Newman and Rivlin's results. Furthermore, we obtain zero-free parabolic regions for Padé approximants of certain meromorphic

functions, as well as for the partial sums of certain $_1F_1$ hypergeometric functions. Also, we improve upon the result of Dočev [3], concerning the location of the zeros of generalized Bessel polynomials.

The essential technique of proof utilizes continued fraction expansions, in the spirit of Wall [13].

2. A parabola theorem. Our main result is the following theorem.

THEOREM 2.1. *Let* $\{p_k(z)\}_{k=0}^n$ *be a sequence of polynomials of respective degrees* k *which satisfy the three-term recurrence relation*

$$(2.1) \qquad p_k(z) = \left(\frac{z}{b_k}+1\right)p_{k-1}(z) - \frac{z}{c_k}p_{k-2}(z), \qquad k = 1, 2, \cdots, n,$$

where the b_k*'s and* c_k*'s are positive real numbers for all* $1 \leq k \leq n$, *and where* $p_{-1}(z):=0, p_0(z):=p_0 \neq 0$. *Set*

$$(2.2) \qquad \alpha := \min\{b_k(1 - b_{k-1}c_k^{-1}): k = 1, 2, \cdots, n\}, \qquad b_0:=0.$$

Then, if $\alpha > 0$, *the parabolic region*

$$(2.3) \qquad \mathscr{P}_\alpha := \{z = x + iy \in \mathbb{C}: y^2 \leq 4\alpha(x+\alpha), x > -\alpha\}$$

contains no zeros of $p_1(z), p_2(z), \cdots, p_n(z)$.

Proof. Let $z \in \mathscr{P}_\alpha$ be any fixed point which is not a zero of any $p_k(z)$, $1 \leq k \leq n$, and define

$$(2.4) \qquad \mu_k = \mu_k(z):=\frac{zp_{k-1}(z)}{b_kp_k(z)} \quad \text{for } k = 1, 2, \cdots, n.$$

We shall show inductively that

$$(2.5) \qquad \operatorname{Re}\mu_k \leq 1 \quad \text{for } k = 1, 2, \cdots, n.$$

This is certainly true for $k = 1$; indeed, from (2.4), (2.1) and the fact that $p_0(z):=p_0 \neq 0$, we have that

$$\mu_1 = \frac{zp_0(z)}{b_1p_1(z)} = \frac{zp_0(z)}{b_1(z/b_1+1)p_0(z)} = \frac{z}{z+b_1},$$

from which it follows that $\operatorname{Re}\mu_1 \leq 1$ if and only if $\operatorname{Re}z \geq -b_1$. But as $z \in \mathscr{P}_\alpha$ and $b_1 \geq \alpha$ from (2.2), this last condition holds; i.e., $\operatorname{Re}z > -\alpha \geq -b_1$.

Now, assume inductively that $\operatorname{Re}\mu_{k-1} \leq 1$ for some k satisfying $2 \leq k \leq n$. We can express μ_k from (2.4) and (2.1) as

$$\mu_k = \frac{zp_{k-1}(z)}{b_kp_k(z)} = \frac{zp_{k-1}(z)}{(z+b_k)p_{k-1}(z) - b_kc_k^{-1}zp_{k-2}(z)}$$
$$= \frac{z}{z+b_k - b_kc_k^{-1}b_{k-1}\mu_{k-1}}.$$

In other words,

$$(2.6) \qquad \mu_k = T_k(\mu_{k-1}).$$

where $T_k(w)$ is the bilinear transformation defined by

$$(2.7) \qquad \xi = T_k(w) := \frac{z}{z + b_k - b_k c_k^{-1} b_{k-1} w}.$$

Hence, since Re $\mu_{k-1} \leqq 1$ by hypothesis, then μ_k lies in the image of the half-plane Re $w \leqq 1$ under the transformation T_k. Now, T_k has its pole at

$$w_k := \frac{z + b_k}{b_k c_k^{-1} b_{k-1}},$$

and since Re $z > -\alpha \geqq -(b_k - b_k c_k^{-1} b_{k-1})$ from (2.2), it follows that Re $w_k > 1$. Therefore, T_k maps Re $w \leqq 1$ onto a closed disk D_k in the ξ-plane. The center ξ_k of this disk is the image, under T_k, of the point in the w-plane symmetric to the pole w_k with respect to the line Re $w = 1$, i.e.,

$$\xi_k = T_k(2 - \bar{w}_k) = T_k\left(2 - \frac{\bar{z} + b_k}{b_k c_k^{-1} b_{k-1}}\right) = \frac{z}{2 \operatorname{Re} z + 2b_k(1 - b_{k-1} c_k^{-1})}.$$

Furthermore, since $T_k(\infty) = 0$ lies on the boundary of D_k, the radius r_k of this disk is given by

$$r_k = |\xi_k| = \frac{|z|}{2 \operatorname{Re} z + 2b_k(1 - b_{k-1} c_k^{-1})}.$$

Consequently, the real part of any point in D_k does not exceed the sum

$$\operatorname{Re} \xi_k + r_k = \frac{\operatorname{Re} z + |z|}{2 \operatorname{Re} z + 2b_k(1 - b_{k-1} c_k^{-1})}.$$

Again from (2.2), an upper bound for this last quantity is

$$\frac{\operatorname{Re} z + |z|}{2 \operatorname{Re} z + 2\alpha},$$

which one can directly verify is at most unity because $z \in \mathscr{P}_\alpha$. In particular, since $\mu_k \in D_k$, we have Re $\mu_k \leqq 1$. This completes the induction for establishing (2.5).

Next, we observe that $p_k(0) \neq 0$ for all $k = 0, 1, \cdots, n$; indeed, from (2.1) we have

$$0 \neq p_0(0) = p_1(0) = \cdots = p_k(0).$$

Furthermore, if $p_k(z_0) = p_{k-1}(z_0) = 0$ for some $k \geqq 1$, then evidently $z_0 \neq 0$, so that from (2.1), we deduce that $p_{k-j}(z_0) = 0$ for all $0 \leqq j \leqq k$. In particular, this would imply that $p_0(z_0) = 0$, which is a contradiction. Hence, $p_k(z)$ and $p_{k-1}(z)$ have no zeros in common for each k, $1 \leqq k \leqq n$.

Finally, suppose on the contrary that $p_k(z_0) = 0$ for some $z_0 \in \mathscr{P}_\alpha$, and some k with $1 \leqq k \leqq n$. Clearly, since $p_1(z) = (p_0/b_1)(z + b_1)$ from (2.1), then p_1 has its sole zero at $-b_1$. But as $-b_1 \leqq -\alpha$ from (2.2), this zero by definition (cf. (2.3)) is not in $\mathscr{P}_\alpha$. Thus, $2 \leqq k \leqq n$. Next, $p_k(z_0) = 0$ implies from (2.1) that $(z_0/b_k + 1)p_{k-1}(z_0) = (z_0/c_k)p_{k-2}(z_0)$, and as $p_k(z)$ and $p_{k-1}(z)$ have no common zeros, then on dividing,

$$(2.8) \qquad \frac{c_k}{b_{k-1}b_k}(z_0 + b_k) = \frac{z_0 p_{k-2}(z_0)}{b_{k-1}p_{k-1}(z_0)} = \mu_{k-1}(z_0).$$

Now, $z_0 \in \mathscr{P}_\alpha$ implies from (2.5) and continuity considerations that Re $\mu_{k-1}(z_0) \leqq$ 1. Thus, taking real parts in (2.8), we obtain

$$\text{Re } z_0 \leqq -b_k(1 - b_{k-1}c_k^{-1}) \leqq -\alpha,$$

the last inequality following from (2.2). On the other hand, $z_0 \in \mathscr{P}_\alpha$ implies from (2.3) that Re $z_0 > -\alpha$, which contradicts the above inequality. Thus, $p_k(z)$ has no zeros in $\mathscr{P}_\alpha$ for each k, $1 \leqq k \leqq n$. $\quad\square$

Note, using (2.6), that

$$\mu_k = T_k(\mu_{k-1}) = T_k(T_{k-1}(\mu_{k-2})) = \cdots = T_k T_{k-1} \cdots T_2(\mu_1), \quad 2 \leqq k \leqq n.$$

Hence, the above technique of proof of Theorem 2.1 essentially depends on the finiteness of a continued fraction expansion of μ_k, namely, from (2.7),

$$\mu_k(z) = \cfrac{z}{z + b_k - \cfrac{b_k c_k^{-1} b_{k-1} z}{z + b_{k-1} - \cfrac{b_{k-1} c_{k-1}^{-1} b_{k-2} z}{z + b_{k-2} - \cdots}}}.$$

There is in fact a well-known "parabola theorem" due to Wall [13, p. 57] for continued fractions, but it does not appear to the authors that the finiteness of the above continued fraction expansion for $\mu_k(z)$ with $z \in \mathscr{P}_\alpha$ follows from Wall's parabola theorem.

We remark that, in a certain sense, the result of Theorem 2.1 is sharp. For, consider any three-term recurrence relation (2.1) for which

$$\alpha = b_1.$$

Then, as $p_1(z) = (p_0/b_1)(z + b_1)$, it has its sole zero at $-b_1 = -\alpha$. Therefore, since the parabola $y^2 = 4\alpha(x + \alpha)$ has its vertex at $x = -\alpha$, the parabolic region of (2.3) cannot be enlarged to include the boundary point $z = -\alpha$ of $\mathscr{P}_\alpha$ and still exclude the zeros of $p_1(z), \cdots, p_n(z)$.

We remark further that Theorem 2.1 has an obvious extension to an *infinite* sequence of polynomials $\{p_k(z)\}_{k=0}^\infty$ which satisfy (2.1). In such a case, we define

$$\alpha := \inf\{b_k(1 - b_{k-1}c_k^{-1}): k = 1, 2, \cdots\}.$$

Then, the conclusion that the region $\mathscr{P}_\alpha$ of (2.3) is zero-free for *every* $p_k(z)$, $k = 1, 2, \cdots$, remains valid provided that $\alpha > 0$. If, in addition, such an infinite sequence $p_k(z)$ converges uniformly on all compact subsets of $\mathscr{P}_\alpha$ to an analytic function $f(z)$ which is not identically zero, then by the theorem of Hurwitz, $f(z)$ must also be zero-free in the interior of $\mathscr{P}_\alpha$.

Some concrete applications of the parabola theorem will be given in the next sections. For the remainder of the present section, we consider sufficient conditions under which the hypotheses of Theorem 2.1 are fulfilled. We deal first with the partial sums of a power series expansion.

COROLLARY 2.2. *Let* $s_k(z) := \sum_{j=0}^k a_j z^j$, $k = 0, 1, \cdots, n$, *and assume that* $a_j > 0$ *for all* $j = 0, 1, \cdots, n$, *and that*

$$(2.9) \qquad \alpha := \min\left\{\left(\frac{a_{k-1}}{a_k} - \frac{a_{k-2}}{a_{k-1}}\right): k = 1, 2, \cdots, n\right\} > 0,$$

where $a_{-1}/a_0 := 0$. Then, the polynomials $s_k(z)$, $k = 1, 2, \cdots, n$, have no zeros in the parabolic region $\mathscr{P}_\alpha$ defined in (2.3).

Proof. One easily verifies that the partial sums $s_k(z)$ satisfy the three-term recurrence relation

$$(2.10) \qquad s_k(z) = \left(\frac{z}{b_k} + 1\right) s_{k-1}(z) - \frac{z}{b_k} s_{k-2}(z), \qquad k = 1, 2, \cdots, n,$$

where $s_{-1} := 0$, and where

$$(2.11) \qquad b_k := a_{k-1}/a_k, \qquad k = 0, 1, \cdots, n.$$

Consequently, (2.1) holds with $c_k = b_k$, and (2.2) becomes

$$\alpha = \min \{(b_k - b_{k-1}): k = 1, \cdots, n\},$$

which from (2.11) is the same as (2.9). Applying Theorem 2.1 then establishes the corollary. □

The partial sums of a formal power series

$$(2.12) \qquad f(z) = \sum_{j=0}^{\infty} a_j z^j, \qquad a_0 \neq 0,$$

can be regarded as special cases of the so-called Padé approximants to $f(z)$ (see Perron [10], or Baker [2]). More precisely, given any pair of nonnegative integers (n, ν), the Padé approximant of type (n, ν) is that rational function $R_{n,\nu}(z)$ of the form

$$(2.13) \qquad R_{n,\nu}(z) = P_{n,\nu}(z)/Q_{n,\nu}(z)$$

for which the following conditions are satisfied:

(i) $P_{n,\nu}(z)$ is a polynomial of degree $\leq n$;

(ii) $Q_{n,\nu}(z)$ is a polynomial of degree $\leq \nu$ with $Q_{n,\nu}(z) \not\equiv 0$;

(iii) The power series development of $f(z)Q_{n,\nu}(z) - P_{n,\nu}(z)$ about $z = 0$ begins with the $(n + \nu + 1)$st power of z.

In particular, for $\nu = 0$, these conditions are satisfied by

$$P_{n,0}(z) = \sum_{j=0}^{n} a_j z^j, \qquad Q_{n,0}(z) = 1, \qquad n = 0, 1, \cdots.$$

Corresponding to the power series (2.12), we define the Hankel determinants

$$(2.14) \qquad A_n^{(0)} := 1, \; n \geq 0, \; A_n^{(\nu)} := \det \begin{bmatrix} a_n & a_{n-1} & \cdots & a_{n-\nu+1} \\ a_{n+1} & a_n & \cdots & a_{n-\nu+2} \\ \vdots & \vdots & & \vdots \\ a_{n+\nu-1} & a_{n+\nu-2} & \cdots & a_n \end{bmatrix}, \qquad n \geq 0, \quad \nu \geq 1,$$

with the convention that

$$a_{-j} := 0 \quad \text{for } j = 1, 2, \cdots.$$

These determinants play an important role in the study of Padé approximants. Indeed, if

$$(2.15) \qquad A_n^{(\nu)} \neq 0,$$

then the conditions (i), (ii) and (iii) above are satisfied by the polynomials

$$(2.16) \qquad P_{n,\nu}(z) = \frac{1}{A_n^{(\nu)}} \sum_{j=0}^{n} \det \begin{bmatrix} a_j & a_{j-1} & \cdots & a_{j-\nu} \\ a_{n+1} & a_n & \cdots & a_{n-\nu+1} \\ a_{n+2} & a_{n+1} & \cdots & a_{n-\nu+2} \\ \vdots & \vdots & & \vdots \\ a_{n+\nu} & a_{n+\nu-1} & \cdots & a_n \end{bmatrix} \cdot z^j,$$

$$(2.17) \qquad Q_{n,\nu}(z) = \frac{1}{A_n^{(\nu)}} \det \begin{bmatrix} 1 & z & z^2 & \cdots & z^\nu \\ a_{n+1} & a_n & a_{n-1} & \cdots & a_{n-\nu+1} \\ a_{n+2} & a_{n+1} & a_n & \cdots & a_{n-\nu+2} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n+\nu} & a_{n+\nu-1} & a_{n+\nu-2} & \cdots & a_n \end{bmatrix}.$$

In such a case, we refer to the polynomial $P_{n,\nu}(z)$ in (2.16) as the *Padé numerator* of type (n, ν), and to $Q_{n,\nu}(z)$ in (2.17) as the *Padé denominator* of type (n, ν).

We now prove a generalization of Corollary 2.2 for the Padé numerators.

COROLLARY 2.3. *Let $f(z) = \sum_{j=0}^{\infty} a_j z^j$ be a formal power series, and assume that, for a fixed $\nu \geq 0$, the corresponding Hankel determinants defined in (2.14) satisfy*

$$(2.18) \qquad \begin{aligned} & A_k^{(\nu)} > 0,\ A_k^{(\nu+1)} > 0, \quad \text{for } k = 0, 1, \cdots, n, \\ & A_k^{(\nu+2)} > 0 \quad \text{for } k = 0, 1, \cdots, n-1. \end{aligned}$$

Then, defining the positive constant α by

$$(2.19) \qquad \alpha := \min \left\{ \frac{A_k^{(\nu)} A_{k-1}^{(\nu+2)}}{A_{k-1}^{(\nu+1)} A_k^{(\nu+1)}} : k = 1, 2, \cdots, n \right\},$$

we find that the Padé numerators $P_{1,\nu}(z), P_{2,\nu}(z), \cdots, P_{n,\nu}(z)$ for $f(z)$ have no zeros in the parabolic region $\mathcal{P}_\alpha$ defined in (2.3).

Proof. A classical identity of Frobenius [5] asserts that

$$(2.20) \qquad P_{k,\nu}(z) = \left(\frac{z}{b_{k,\nu}} + 1 \right) P_{k-1,\nu}(z) - \frac{z}{c_{k,\nu}} P_{k-2,\nu}(z),$$

where

$$(2.21) \qquad \begin{aligned} b_{k,\nu} &:= \frac{A_k^{(\nu)} A_{k-1}^{(\nu+1)}}{A_{k-1}^{(\nu)} A_k^{(\nu+1)}}, \qquad k \geq 1, \\ c_{k,\nu} &:= \frac{A_{k-1}^{(\nu)} A_{k-1}^{(\nu+1)}}{A_{k-2}^{(\nu)} A_k^{(\nu+1)}}, \qquad k \geq 2. \end{aligned}$$

(For notational convenience we set $c_{1,\nu} := 1$.) By assumption (2.18), the $b_{k,\nu}$'s and $c_{k,\nu}$'s are positive real numbers for $k = 1, 2, \cdots, n$. Consequently, the recurrence

relation (2.1) holds with $b_k = b_{k,\nu}$, $c_k = c_{k,\nu}$, and (2.2) of Theorem 2.1 becomes

$$(2.22) \qquad \alpha = \min\left\{\frac{A_k^{(\nu)}A_{k-1}^{(\nu+1)}}{A_{k-1}^{(\nu)}A_k^{(\nu+1)}}\left(1 - \frac{A_{k-2}^{(\nu+1)}A_k^{(\nu+1)}}{[A_{k-1}^{(\nu+1)}]^2}\right): k = 1, \cdots, n\right\}.$$

However, using the known identity

$$(2.23) \qquad [A_{k-1}^{(\nu+1)}]^2 - A_{k-2}^{(\nu+1)}A_k^{(\nu+1)} = A_{k-1}^{(\nu+2)}A_{k-1}^{(\nu)}$$

in (2.22), we obtain

$$\alpha = \min\left\{\frac{A_k^{(\nu)}A_{k-1}^{(\nu+1)}A_{k-1}^{(\nu+2)}A_{k-1}^{(\nu)}}{A_{k-1}^{(\nu)}A_k^{(\nu+1)}[A_{k-1}^{(\nu+1)}]^2}: k = 1, 2, \cdots, n\right\},$$

which is the same as the defining formula (2.19).   □

In a similar manner, we deduce the following result for the Padé denominators.

COROLLARY 2.4. *Suppose that, for fixed $n \geq 0$, the Hankel determinants corresponding to the formal power series $f(z) = \sum_{j=0}^{\infty} a_j z^j$ satisfy*

$$(2.24) \qquad \begin{aligned} A_n^{(k)} &> 0, \quad A_{n+1}^{(k)} > 0, \quad for\ k = 1, 2, \cdots, \nu, \\ A_{n+2}^{(k)} &> 0 \quad for\ k = 1, 2, \cdots, \nu - 1. \end{aligned}$$

Then, defining the positive constant α by

$$(2.25) \qquad \alpha := \min\left\{\frac{A_n^{(k)}A_{n+2}^{(k-1)}}{A_{n+1}^{(k-1)}A_{n+1}^{(k)}}: k = 1, 2, \cdots, \nu\right\},$$

the Padé denominators $Q_{n,1}(z), Q_{n,2}(z), \cdots, Q_{n,\nu}(z)$ for $f(z)$ have no zeros in the parabolic region

$$(2.26) \qquad \hat{\mathscr{P}}_\alpha := \{z = x + iy \in \mathbb{C}: y^2 \leq 4\alpha(\alpha - x), \alpha > x\}.$$

The proof of Corollary 2.4 follows in an analogous fashion from the Frobenius identity

$$(2.27) \qquad Q_{n,k}(-z) = \left(1 + \frac{z}{\hat{b}_{n,k}}\right)Q_{n,k-1}(-z) - \frac{z}{\hat{c}_{n,k}}Q_{n,k-2}(-z),$$

where

$$(2.28) \qquad \hat{b}_{n,k} := \frac{A_n^{(k)}A_{n+1}^{(k-1)}}{A_{n+1}^{(k)}A_n^{(k-1)}}, \qquad \hat{c}_{n,k} := \frac{A_n^{(k-1)}A_{n+1}^{(k-1)}}{A_{n+1}^{(k)}A_n^{(k-2)}}.$$

In concluding this section we remark that the hypotheses (2.18) and (2.24) of the preceding corollaries will be satisfied for all n and ν if $f(z)$ is a meromorphic function of the form

$$(2.29) \qquad f(z) = a_0 e^{\gamma z} \cdot \frac{\prod_{j=1}^{\infty}(1 + \lambda_j z)}{\prod_{j=1}^{\infty}(1 - \beta_j z)},$$

where $a_0 > 0$, $\gamma \geq 0$, $\lambda_j \geq 0$, $\beta_j \geq 0$ and $\sum_j (\lambda_j + \beta_j) < \infty$. The convergence properties of the Padé approximants of such functions were studied by Arms and Edrei in [1]

3. Partial sums and Padé approximants of e^z. As a concrete application of the results in § 2, we now obtain zero-free regions for the Padé numerators $P_{n,\nu}(z)$ and denominators $Q_{n,\nu}(z)$ for e^z. Explicitly, these polynomials are given by (cf. Perron [10, p. 433])

$$(3.1) \qquad P_{n,\nu}(z) = \sum_{j=0}^{n} \frac{(n+\nu-j)!\,n!\,z^j}{(n+\nu)!\,j!\,(n-j)!},$$

$$(3.2) \qquad Q_{n,\nu}(z) = \sum_{j=0}^{\nu} \frac{(n+\nu-j)!\,\nu!}{(n+\nu)!\,j!\,(\nu-j)!}(-z)^j.$$

COROLLARY 3.1. *For any $\nu \geq 0$, each element of the sequence of Padé numerators $\{P_{n,\nu}(z)\}_{n=1}^{\infty}$ for e^z has no zeros in the region*

$$(3.3) \qquad \mathscr{P}_{\nu+1} = \{z = x+iy \in \mathbb{C}: y^2 \leq 4(\nu+1)(x+\nu+1), x > -(\nu+1)\}.$$

Furthermore, for any $n \geq 0$, each element of the sequence of Padé denominators $\{Q_{n,\nu}(z)\}_{\nu=1}^{\infty}$ has no zeros in the region

$$(3.4) \qquad \hat{\mathscr{P}}_{n+1} = \{z = x+iy \in \mathbb{C}: y^2 \leq 4(n+1)(n+1-x), x < (n+1)\}.$$

Proof. The Hankel determinants $A_m^{(s)}$ for $s \geq 1$ for e^z are given (cf. [1]) by

$$(3.5) \qquad A_m^{(s)} = \prod_{j=1}^{s} \frac{1}{j(j+1)\cdots(j+m-1)}.$$

Thus, for any $n \geq 0$, the constant α defined in (2.19) is easily verified to be

$$(3.6) \qquad \alpha = \min\{(\nu+1): k = 1, 2, \cdots, n\} = \nu+1,$$

and so, by Corollary 2.3, the region $\mathscr{P}_{\nu+1}$ is zero-free for every $P_{n,\nu}(z)$, $n = 1, 2, \cdots$.

Similarly, for any $\nu \geq 0$, the constant α defined in (2.25) equals $n+1$, so that by Corollary 2.4, the region $\hat{\mathscr{P}}_{n+1}$ is zero-free for every $Q_{n,\nu}(z)$, $\nu = 1, 2, \cdots$. □

In particular, for $\nu = 0$, we obtain Corollary 3.2.

COROLLARY 3.2. *No partial sum $P_{n,0}(z) = \sum_{j=0}^{n} z^j/j!$ of e^z, for any $n \geq 1$, has a zero in the parabolic region*

$$\mathscr{P}_1 = \{z = x+iy \in \mathbb{C}: y^2 \leq 4(x+1), x > -1\}.$$

This result is sharp at $x = -1$, and, as discussed in the introduction, it improves upon an analogous result due to Newman and Rivlin [9].

In Figs. 1 and 2 we plot, respectively, the zeros (shown as asterisks) in the upper half-plane of the Padé polynomials $\{P_{n,0}(z)\}_{n=1}^{40}$ and of $\{P_{n,6}(z)\}_{n=1}^{40}$ for e^z, together with the corresponding bounding parabolas for $\mathscr{P}_1$ and $\mathscr{P}_7$. The computations and the ones mentioned below were carried out by A. Price and P. Comadoll on an IBM 360/65 using a modified version of SUBROUTINE POLRT from the IBM Scientific Subroutine Package. The plots were done on a Calcomp Model 563 plotter.

We remark that the largest parabolic region of the form $y^2 < \lambda(x+1)$ which omits the zeros of the Padé polynomials $\{P_{n,0}(z)\}_{n=1}^{40}$ for e^z is approximately given by
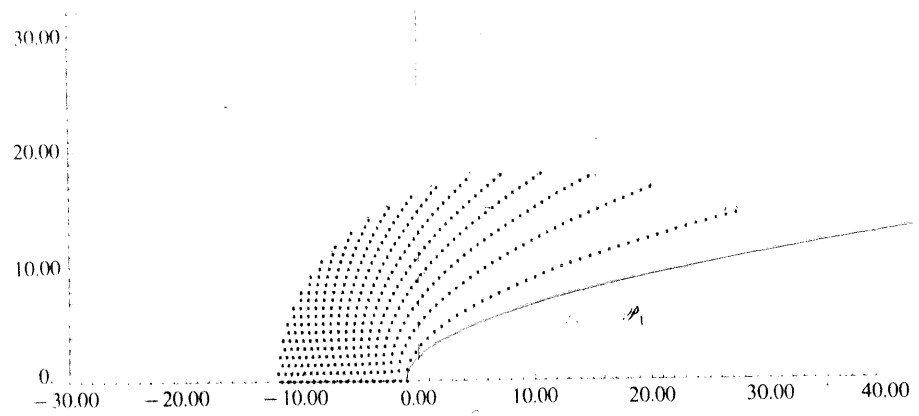
$$y^2 < 7.1940(x+1), \qquad x > -1.$$

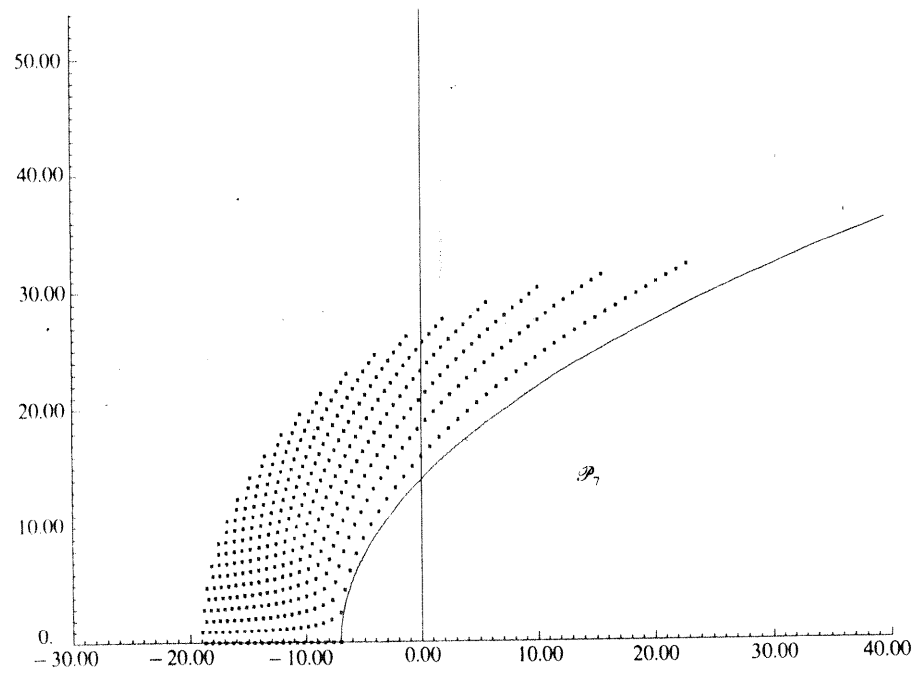FIG. 1. *Zeros of the polynomials* $P_{n,0}(z)$, $n = 1, 2, \cdots, 40$

FIG. 2. *Zeros of the polynomials* $P_{n,6}(z)$, $n = 1, 2, \cdots, 40$

On the other hand, Newman and Rivlin [8] have (correctly) established that

$$\left\{ \frac{P_{n,0}(n + \sqrt{2n} \cdot w)}{\exp(n + \sqrt{2n} \cdot w)} \right\}_{n=1}^{\infty}$$

converges uniformly to

$$\frac{1}{\sqrt{\pi}} \int_{w}^{\infty} e^{-t^2} \, dt := \tfrac{1}{2} \operatorname{erfc}(w),$$

on any compact set in Im $w \geqq 0$. If t_1 denotes the zero of erfc (w), having real part negative and smallest (positive) imaginary part, then t_1 is given approximately (cf. Fettis et al. [4]) by

$$t_1 \doteq -1.354\,810 + i(1.991\,467).$$

Becase of the uniform convergence above, it then follows from Hurwitz's theorem that, for all n sufficiently large, $P_{n,0}$ has a zero of the form

$$n + \sqrt{2n}\, w_n := x_n + iy_n \quad \text{with } \lim_{n \to \infty} w_n = t_1.$$

From this, we easily deduce that for each fixed β,

$$\lim_{n \to \infty} \frac{y_n^2}{(x_n + \beta)} = 2(\operatorname{Im} t_1)^2 \doteq 7.931\,880.$$

In other words, any parabola of the form

$$y^2 < K(x + \beta), \qquad x > -\beta,$$

which is devoid of zeros of $P_{n,0}(z)$ of e^z, for all n sufficiently large, must evidently satisfy

$$K \leqq 2(\operatorname{Im} t_1)^2 \doteq 7.931\,880.$$

4. $_1F_1$ hypergeometric functions. Using the notation

(4.1) $$(a)_j := a(a+1) \cdots (a+j-1), \qquad j \geqq 1, \quad (a)_0 := 1,$$

for any complex number a, the hypergeometric function $_1F_1(c; d; z)$ is defined by

(4.2) $${}_1F_1(c; d; z) := \sum_{j=0}^{\infty} \frac{(c)_j}{(d)_j} \cdot \frac{z^j}{j!},$$

and is an entire function of z, for any choice of c and d with $d \neq 0, -1, -2, \cdots$. For example,

(4.3) $$e^z = {}_1F_1(c; c; z), \qquad c \neq 0, -1, -2, \cdots,$$

and

(4.4) $$e^z - \sum_{k=0}^{n-1} z^k/k! = \frac{z^n}{n!} {}_1F_1(1; n+1; z), \qquad n = 1, 2, \cdots.$$

Concerning zero-free regions for certain $_1F_1$'s and their partial sums, we prove the next corollary.

COROLLARY 4.1. *With the notation* (2.3) *for the parabolic region* $\mathscr{P}_\alpha$, *all the partial sums*

(4.5) $$s_n(z) = \sum_{j=0}^{n} \frac{(c)_j}{(d)_j} \frac{z^j}{j!}, \qquad n \geqq 1,$$

of $_1F_1(c; d; z)$ have no zeros in the region

(i) $\mathscr{P}_{d/c}$, *if* $0 < d \leq c$,

(ii) $\mathscr{P}_1$, *if* $1 \leq c \leq d$,

(iii) $\mathscr{P}_\alpha$, $\alpha = (2c - d + cd)/(c^2 + c)$, *if* $0 < c < 1$ *and* $c \leq d < 2c/(1-c)$.

Consequently, the entire function $_1F_1(c; d; z)$ has no zeros in the corresponding interior region.

Proof. Putting

$$(4.6) \qquad a_j := \frac{(c)_j}{(d)_j j!}, \qquad a_{-1} := 0,$$

we apply Corollary 2.2 with the constant α being defined by

$$(4.7) \qquad \alpha = \inf\left\{\left(\frac{a_{k-1}}{a_k} - \frac{a_{k-2}}{a_{k-1}}\right): k = 1, 2, \cdots\right\}.$$

On substituting (4.6) in (4.7), we obtain

$$(4.8) \qquad \alpha = \inf\left\{\frac{d}{c}, g(k): k = 2, 3, \cdots\right\},$$

where

$$(4.9) \qquad g(t) := \frac{t^2 + (2c-3)t + (c-1)(d-2)}{t^2 + (2c-3)t + (c-1)(c-2)} \quad \text{for all } t \geq 2.$$

Next, we observe that

$$g(2) = \frac{2c - d + cd}{c^2 + c}, \qquad \lim_{t \to +\infty} g(t) = 1,$$

and

$$g'(t) = \frac{(2t + 2c - 3)(c-1)(c-d)}{[t^2 + (2c-3)t + (c-1)(c-2)]^2}.$$

From these facts, it follows that the constant α of (4.8) is positive, and is given by d/c, 1, and $(2c - d + cd)/(c^2 + c)$, in the respective cases (i), (ii) and (iii). Applying Corollary 2.2 then proves that all the partial sums have no zeros in the corresponding region $\mathscr{P}_\alpha$, and consequently, the limit function $_1F_1$ has no zeros in the interior of $\mathscr{P}_\alpha$ (see the remarks following the proof of Theorem 2.1). □

We remark that when c is not a nonpositive integer and $c - d$ is not a nonnegative integer, then it is known [6] that $_1F_1(c; d; z)$ has infinitely many zeros in the complex plane.

In Fig. 3, we plot the zeros in the upper half-plane of the partial sums $\{s_n(z)\}_{n=1}^{40}$ in (4.5) of the hypergeometric function $_1F_1(1; 4; z)$, i.e., when $c = 1$, $d = 4$. The corresponding zero-free parabolic region $\mathscr{P}_1$ from (ii) of Corollary 4.1 is also sketched. Two accumulation points of zeros are evident in the figure, and these are necessarily zeros of $_1F_1(1; 4; z)$.

COROLLARY 4.2. *For all* $n \geq 1$, *the remainder*
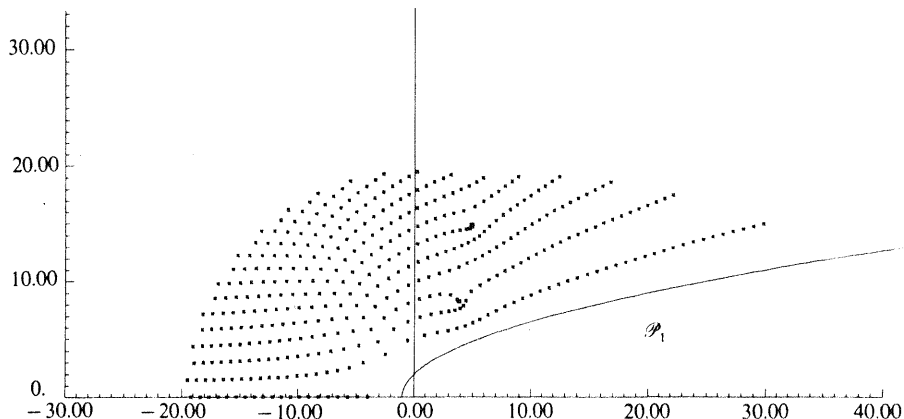
$$(4.10) \qquad e^z - \sum_{k=0}^{n-1} z^k/k!$$

FIG. 3. *Zeros of the partial sums* $s_n(z)$, $n = 1, 2, \cdots, 40$, *for* $_1F_1(1; 4; z)$

has no zeros in the region

$$(4.11) \quad \mathscr{P}_1^0 \cup \hat{\mathscr{P}}_1^0 := \{z = x + iy \in \mathbb{C}: y^2 < 4(x+1)\} \cup \{z = x + iy \in \mathbb{C}: y^2 < 4(1-x)\},$$

except at $z = 0$.

Proof. Applying Corollary 4.1 in the case when $c = 1$, $d = n + 1$, we deduce from conclusion (ii) that the function $_1F_1(1; n+1; z)$ is zero-free in $\mathscr{P}_1^0$, the interior of $\mathscr{P}_1$. Furthermore, the identity (cf. [6])

$$_1F_1(1; n+1; -z) = e^{-z} {}_1F_1(n; n+1, z),$$

together with Corollary 4.1, imply that $_1F_1(1; n+1; z)$ is zero-free in $\hat{\mathscr{P}}_1^0$, the interior of $\hat{\mathscr{P}}_1$, defined in (2.26). Hence, by virtue of the representation (4.4), the remainder (4.10) is zero-free in $(\mathscr{P}_1^0 \cup \hat{\mathscr{P}}_1^0) \setminus \{0\}$. $\square$

5. Generalized Bessel polynomials. In this section, we consider the generalized Bessel polynomials

$$(5.1) \quad Y_n^{(\delta)}(z) := \sum_{j=0}^n \binom{n}{j}(n+\delta+1)_j \left(\frac{-z}{2}\right)^j,$$

where $(n+\delta+1)_j$ is defined as in (4.1). These polynomials were first introduced by Krall and Frink [7], and in their notation,

$$Y_n^{(\delta)}(z) = y_n(-z, \delta+2, 2).$$

Several authors have investigated the location of the zeros of the polynomials (5.1); among them, Dočev [3] appears to have obtained the strongest result. We state his theorem for real δ as follows.

THEOREM 5.1. *If* $n+\delta+1 > 0$, $\delta \neq -2, -3, -4, \cdots$, *then all the zeros of* $Y_n^{(\delta)}(z)$ *lie in the closed disk*

$$(5.2) \quad D_{n+\delta+1} := \left\{z \in \mathbb{C}: |z| \leq \frac{2}{n+\delta+1}\right\}.$$

Using Theorem 2.1, we now improve upon Dočev's result.

THEOREM 5.2. *If* $n + \delta + 1 > 0$, *then all the zeros of* $Y_n^{(\delta)}(z)$ *lie in the cardioidal region*

$$(5.3) \quad C_{n+\delta+1} := \left\{ z = r e^{i\theta} \in \mathbb{C} : 0 < r < \frac{1 + \cos \theta}{n + \delta + 1}, \ -\pi < \theta < \pi \right\} \cup \left\{ \frac{2}{n + \delta + 1} \right\}.$$

Notice that $C_{n+\delta+1} \subset D_{n+\delta+1}$, and this containment is proper, except for $z = 2/(n + \delta + 1)$.

Proof of Theorem 5.2. It is convenient to introduce the polynomials

$$(5.4) \quad P_m^{(\tau)}(z) := \frac{\Gamma(m + \tau + 1)}{\Gamma(2m + \tau + 1)} z^m Y_m^{(\tau)}\left(\frac{-2}{z} \right)$$

$$= \frac{\Gamma(m + \tau + 1)}{\Gamma(2m + \tau + 1)} \sum_{j=0}^{m} \binom{m}{j} (m + \tau + 1)_j z^{m-j}, \quad m + \tau + 1 > 0.$$

As can be directly verified, for fixed n and δ, the polynomials $\{P_k^{(n+\delta-k)}(z)\}_{k=0}^{n}$ satisfy the recurrence relation

$$(5.5) \quad P_k^{(n+\delta-k)}(z) = \left(\frac{z}{b_k} + 1 \right) P_{k-1}^{(n+\delta-k+1)}(z) - \frac{z}{c_k} P_{k-2}^{(n+\delta-k+2)}(z), \quad k \geq 1,$$

where $P_{-1}^{(n+\delta+1)}(z) := 0$, and

$$(5.6) \quad b_k = n + \delta + k, \quad k \geq 1; \quad c_k = \frac{(n + \delta + k)(n + \delta + k - 1)}{(k - 1)}, \quad k \geq 2; \quad c_1 := 1.$$

Since, by hypothesis, $n + \delta + 1 > 0$, the constants b_k and c_k in (5.6) are positive for all $k \geq 1$. Furthermore, a simple computation shows that

$$b_k(1 - b_{k-1}c_k^{-1}) = n + \delta + 1 \quad \text{for all } k = 1, 2, \cdots, n, \quad b_0 := 0.$$

Hence, the constant α defined in (2.2) is given by

$$\alpha = n + \delta + 1,$$

and so from Theorem 2.1, we deduce that all the polynomials $\{P_k^{(n+\delta-k)}(z)\}_{k=1}^{n}$ are zero-free in the region

$$\mathcal{P}_{n+\delta+1} = \{ z = x + iy \in \mathbb{C} : y^2 \leq 4(n + \delta + 1)(x + n + \delta + 1), \ x > -(n + \delta + 1) \}$$

$$= \{ z \in \mathbb{C} : |z| \leq \text{Re}(z) + 2(n + \delta + 1), \ \text{Re} \ z > -(n + \delta + 1) \}.$$

In particular, taking $k = n$, we have that $P_n^{(\delta)}(z)$ is zero-free in $\mathcal{P}_{n+\delta+1}$.

Finally, from (5.4) (with $m = n$, $\tau = \delta$) it follows that no zero of $Y_n^{(\delta)}(w)$ is of the form $w = -2/z$ with $z \in \mathcal{P}_{n+\delta+1}$. In other words, all the zeros of $Y_n^{(\delta)}(w)$ must lie in the region

$$\left\{ w \in \mathbb{C} : \left| \frac{2}{w} \right| > \text{Re}\left(\frac{-2}{w} \right) + 2(n + \delta + 1) \right\} \cup \left\{ \frac{2}{n + \delta + 1} \right\},$$

which is the same as the region $C_{n+\delta+1}$ in (5.3). ☐

We remark that the Padé polynomials in (3.1) are related to the polynomials in (5.4) by the formula

$$P_{n,\nu}(z) = P_n^{(\nu-n)}(z).$$

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