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On Diagonal Dominance Arguments for Bounding $\|A^{-1}\|_\infty$

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ABSTRACT

In a recent paper by J. M. Varah, an upper bound for $\|A^{-1}\|_\infty$ was determined, under the assumption that A is strictly diagonally dominant, and this bound was then used to obtain a lower bound for the smallest singular value for A. In this note, this upper bound for $\|A^{-1}\|_\infty$ is sharpened, and extended to a wider class of matrices. This bound is then used to obtain an improved lower bound for the smallest singular value of a matrix.

1. INTRODUCTION

In a recent paper, Varah [5] established

THEOREM A. *Assume that* $A=[a_{i,j}]\in\mathbf{C}^{n,n}$ *is strictly diagonally dominant* (*cf.* [6, *p.* 23]), *and set*

$$\alpha=\min_{1\leqslant i\leqslant n}\left\{|a_{i,i}|-\sum_{\substack{j=1\\ j\neq i}}^{n}|a_{i,j}|\right\}.$$

Then

$$\|A^{-1}\|_\infty\leqslant\frac{1}{\alpha}. \tag{1}$$

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THEOREM B. *Assume that* $A=[a_{i,j}]\in\mathbf{C}^{n,n}$ *and* A^T *are both strictly diagonally dominant, and set*

$$\beta=\min_{1\leqslant i\leqslant n}\left\{|a_{i,i}|-\sum_{\substack{j=1\\ j\neq i}}^{n}|a_{j,i}|\right\}.$$

Then, the smallest singular value, $\sigma_n(A)$, *of* A *satisfies*

$$\|A^{-1}\|_2^{-1}=\sigma_n(A)\geqslant\sqrt{\alpha\beta}\ . \tag{2}$$

Our interest here is in both generalizing Theorems A and B, and considering when equality is possible in (1) and (2). It should be remarked here that Theorem A is known in the literature, and can be traced explicitly back at least to Ahlberg and Nilson [1, p. 96].

In the case that $a_{i,i}=1$ for all $1\leqslant i\leqslant n$, so that A can be expressed as $A:=I-B$, Eq. (1) of Theorem A becomes the classical result:

$$\left\|(I-B)^{-1}\right\|_{\infty}\leqslant(1-\|B\|_{\infty})^{-1}.$$

We first introduce some notation. Let n be a positive integer with $n\geqslant 2$, and set $N:=\{1,2,\ldots,n\}$, and $N_i:=N\backslash\{i\}$ for any $i\in N$. Let $\mathbf{C}^{n,n}$ ($\mathbf{R}^{n,n}$) denote the collection of all $n\times n$ complex (real) matrices $A=[a_{i,j}]$, and let $\mathbf{R}^n_+$ denote the collection of all real column vectors $\mathbf{v}=[v_1,v_2,\ldots,v_n]^T$ with $v_i\geqslant 0$ for all $i\in N$. Denoting by $\mathring{\mathbf{R}}^n_+$ the interior of $\mathbf{R}^n_+$, we write $\mathbf{u}>\mathbf{0}$ for $\mathbf{u}\in\mathring{\mathbf{R}}^n_+$. Next, given any $A=[a_{i,j}]\in\mathbf{C}^{n,n}$, define $\mathfrak{M}(A)=[\alpha_{i,j}]\in\mathbf{R}^{n,n}$ by

$$\alpha_{i,i}=|a_{i,i}|;\qquad \alpha_{i,j}=-|a_{i,j}|,\quad i\neq j;\quad i,j\in N, \tag{3}$$

and define the possibly empty set $U_A\subseteq\mathring{\mathbf{R}}^n_+$ by

$$U_A:=\{\boldsymbol{u}>\mathbf{0}:\mathfrak{M}(A)\boldsymbol{u}>\mathbf{0}\text{ and }\|\boldsymbol{u}\|_{\infty}=1\}, \tag{4}$$

where as usual $\|\mathrm{v}\|_{\infty}:=\max\{|v|_i:i\in N\}$.

2. MAIN RESULTS

Given any $A=[a_{i,j}]\in\mathbf{C}^{n,n}$, then A is defined [4] to be a nonsingular H-matrix if $\mathfrak{M}(A)$ is a nonsingular M-matrix, i.e., if $\mathfrak{M}(A)$ is nonsingular and all entries of $[\mathfrak{M}(A)]^{-1}$ are nonnegative. Further, of the many known characterizations (cf. [2, 3, 7]) of a nonsingular M-matrix, one states that $\mathfrak{M}(A)$ is a nonsingular M-matrix if and only if the set U_A, as defined in (4), is nonempty, so that the following statements are all equivalent:

$$\begin{aligned} &A \text{ is a nonsingular H-matrix;} \\ &\mathfrak{M}(A) \text{ is a nonsingular M-matrix;} \\ &U_A \text{ is nonempty.} \end{aligned} \tag{5}$$

Thus, assuming that A is a nonsingular H-matrix implies from (4) and (5) that

$$f_A(\mathbf{u}) := \min_{i \in N} \{ (\mathfrak{M}(A) \cdot \mathbf{u})_i \} > 0 \qquad \text{for any} \quad \mathbf{u} \in U_A. \tag{6}$$

It is readily seen that f_A is continuous on the set U_A, and that f_A can be extended continuously on $\overline{U}_A$, the closure of U_A. However, f_A necessarily vanishes on ∂U_A, the boundary of U_A, so that

$$0 < \max\{ f_A(\mathbf{u}) : \mathbf{u} \in \overline{U}_A \} = f_A(\hat{\mathbf{u}}) \qquad \text{for some} \quad \hat{\mathbf{u}} \in U_A.$$

As we shall see, $\hat{\mathbf{u}}$ will be explicitly given in (11).

This brings us to

Lemma 1. *If $A = [a_{i,j}] \in \mathbf{C}^{n,n}$ is a nonsingular H-matrix, then*

$$\|A^{-1}\|_\infty \leqslant \frac{1}{\max\{ f_A(\mathbf{u}) : \mathbf{u} \in \overline{U}_A \}}. \tag{7}$$

Proof. For any $\mathbf{u} \in U_A$, it follows from (3) and (4) that

$$|a_{i,i}| u_i - \sum_{j \in N_i} |a_{i,j}| u_j > 0, \qquad i \in N.$$

With $D := \operatorname{diag}[u_1, u_2, \ldots, u_n]$, the above inequalities imply simply that $A \cdot D = [a_{i,j} u_j]$ is strictly diagonally dominant. It therefore follows from Theorem A that

$$\|(AD)^{-1}\|_\infty \leqslant \frac{1}{f_A(\mathbf{u})}.$$

Next, write $A^{-1} := [c_{i,j}]$. Then, as is known, $\|(AD)^{-1}\|_\infty$ is given by

$$\|D^{-1}A^{-1}\|_\infty = \max_{i\in N}\left\{\sum_{j\in N}\frac{|c_{i,j}|}{u_i}\right\}.$$

But

$$\max_{i\in N}\left\{\sum_{j\in N}\frac{|c_{i,j}|}{u_i}\right\} \geqslant \frac{\max_{i\in N}\left\{\sum_{j\in N}|c_{i,j}|\right\}}{\max_{j\in N}\{u_j\}} = \frac{\|A^{-1}\|_\infty}{\max_{j\in N}\{u_j\}} = \|A^{-1}\|_\infty,$$

the last relation following from the normalization in (4). Combining the above inequalities then gives

$$\|A^{-1}\|_\infty \leqslant \frac{1}{f_A(\mathbf{u})} \qquad \text{for any} \quad \mathbf{u}\in U_A.$$

Then with the above-mentioned properties of f_A, it follows that minimizing the right side of the above inequality over $\overline{U}_A$ yields the desired result of (7). ■

Note that if $A = [a_{i,j}] \in \mathbf{C}^{n,n}$ is a strictly diagonally dominant matrix, then by definition $\zeta := [1, 1, \ldots, 1]^T \in U_A$, and also A is a nonsingular H-matrix from (5). Thus, we see that Theorem A is a special case of Lemma 1.

Next, note that the result of Lemma 1 applies equally well to every matrix in the set Ω_A of matrices *equimodular* to $A = [a_{i,j}]$:

$$\Omega_A := \left\{B = \left[b_{i,j}\right] \in \mathbf{C}^{n,n} : |b_{i,j}| = |a_{i,j}|,\ i,j \in N\right\}, \tag{8}$$

i.e.,

$$\|B^{-1}\|_\infty \leqslant \frac{1}{\max\left\{f_A(\mathbf{u}) : \mathbf{u}\in \overline{U}_A\right\}} \qquad \text{for any} \quad B\in\Omega_A,$$

whence

$$\sup\left\{\|B^{-1}\|_\infty : B\in\Omega_A\right\} \leqslant \frac{1}{\max\left\{f_A(\mathbf{u}) : \mathbf{u}\in \overline{U}_A\right\}}. \tag{9}$$

Note that $\mathfrak{M}(A)$ is by definition an element of Ω_A.

It is now natural to ask if equality holds throughout (9). That this is so is proved in

THEOREM 1. *If $A=[a_{i,j}]\in\mathbf{C}^{n,n}$ is a nonsingular H-matrix, then*

$$\sup\{\|B^{-1}\|_\infty : B\in\Omega_A\}=\|[\mathfrak{M}(A)]^{-1}\|_\infty=\frac{1}{\max\{f_A(\mathbf{u}):\mathbf{u}\in\overline{U}_A\}}. \tag{10}$$

Proof. The hypothesis implies [cf. (5)] that $\mathfrak{M}(A)$ is a nonsingular M-matrix. Hence, with $\boldsymbol{\zeta}:=[1,1,\ldots,1]^T$, define $\hat{\mathbf{u}}$ by

$$\hat{\mathbf{u}}:=\frac{[\mathfrak{M}(A)]^{-1}\boldsymbol{\zeta}}{\|[\mathfrak{M}(A)]^{-1}\boldsymbol{\zeta}\|_\infty}. \tag{11}$$

Since $\mathfrak{M}(A)$ is a nonsingular M-matrix, it is known (cf. [4]) that $[\mathfrak{M}(A)]^{-1}$ has only nonnegative entries, whence $\hat{\mathbf{u}}>\mathbf{0}$. Moreover, as $\mathfrak{M}(A)\cdot\hat{\mathbf{u}}=\boldsymbol{\zeta}/\|[\mathfrak{M}(A)]^{-1}\boldsymbol{\zeta}\|_\infty>\mathbf{0}$, we know that $\hat{\mathbf{u}}$ is an element of U_A. Hence, from the definition in (6), we deduce that

$$f_A(\hat{\mathbf{u}})=\frac{1}{\|[\mathfrak{M}(A)]^{-1}\boldsymbol{\zeta}\|_\infty}=\frac{1}{\|[\mathfrak{M}(A)]^{-1}\|_\infty}.$$

On the other hand, we know from (9) that

$$\|[\mathfrak{M}(A)]^{-1}\|_\infty\leqslant\sup\{\|B^{-1}\|_\infty : B\in\Omega_A\}\leqslant\frac{1}{\max\{f_A(\mathbf{u}):u\in\overline{U}_A\}}\leqslant\frac{1}{f_A(\hat{\mathbf{u}})},$$

whence, with the previous equality, the desired result of (10) follows. ■

Of course, the same analysis applies directly to A^T, since A is a nonsingular H-matrix if and only if A^T is. Thus, since $\|A\|_1=\|A^T\|_\infty$, we have as an immediate consequence of Theorem 1 the following

COROLLARY 1. *If $A=[a_{i,j}]\in\mathbf{C}^{n,n}$ is a nonsingular H-matrix, then*

$$\sup\{\|B^{-1}\|_1 : B\in\Omega_A\}=\|[\mathfrak{M}(A)]^{-1}\|_1=\frac{1}{\max\{f_{A^T}(\mathbf{u}):\mathbf{u}\in\overline{U}_{A^T}\}}. \tag{12}$$

We now consider an application of Theorem 1 and Corollary 1 to a generalization of Theorem B. Given any $A=[a_{i,j}]\in \mathbf{C}^{n,n}$, its smallest singular value, $\sigma_n(A)$, can be defined by $\sigma_n(A):=(\|A^{-1}\|_2)^{-1}$. Since, for any $B\in \mathbf{C}^{n,n}$, $\|B\|_2^2\leqslant \|B\|_1\cdot\|B\|_\infty$, we directly have from Theorem 1 and Corollary 1 the following generalization of Theorem B:

THEOREM 2. *If $A=[a_{i,j}]\in\mathbf{C}^{n,n}$ is a nonsingular H-matrix, then*

$$\sigma_n(A)\geqslant \inf\{\sigma_n(B):B\in\Omega_A\}\geqslant\left\{\|[\mathfrak{M}(A)]^{-1}\|_1\cdot\|[\mathfrak{M}(A)]^{-1}\|_\infty\right\}^{-1/2}$$

$$\geqslant\{f_A(\mathbf{u})\cdot f_{A^T}(\mathbf{v})\}^{1/2}\qquad \text{for any}\quad \mathbf{u}\in U_A,\quad \text{any}\quad \mathbf{v}\in U_{A^T}. \tag{13}$$

3. REMARKS

We remark that the second inequality of (13) cannot in general be replaced by equality, as the next simple example shows. Consider

$$A=\mathfrak{M}(A)=\begin{bmatrix}2 & -1\\ 0 & 3\end{bmatrix},$$

which is a nonsingular M-matrix. In this case,

$$\|[\mathfrak{M}(A)]^{-1}\|_1=\tfrac{1}{2};\quad \|[\mathfrak{M}(A)]^{-1}\|_\infty=\tfrac{2}{3};$$

$$\inf\{\sigma_2(B):B\in\Omega_A\}=\sigma_2(A)\doteq 1.8424>\left\{\|[\mathfrak{M}(A)]^{-1}\|_1\cdot\|[\mathfrak{M}(A)]^{-1}\|_\infty\right\}^{-1/2}$$

$$=\sqrt{3}\doteq 1.7321.$$

We finally remark that Varah [5] gives block diagonally dominant extensions of Theorems A and B. Similar extensions of Lemma 1, Theorem 1, and Theorem 2 are also possible, but the analogous case of equality, as considered in (10) of Theorem 1, remains an open question for the block partitioned case.

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