Nonsolvable Groups All of Whose Character Degrees Are Odd-Square-Free

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Abstract

A finite group G is odd-square-free if no irreducible complex character of G has degree divisible by the square of an odd prime. We determine all odd-square-free groups G satisfying $S \leq G \leq \text{Aut}(S)$ for a finite simple group S . More generally, we show that if G is any nonsolvable odd-square-free group, then G has at most two nonabelian chief factors and these must be simple odd-square-free groups. If the alternating group A_7 is involved in G , the structure of G can be further restricted.

1 Introduction

Throughout this paper, G is a finite group and $\text{Irr}(G)$ is the set of irreducible complex characters of G . If $\chi \in \text{Irr}(G)$, then $\chi(1)$ is the *degree* of χ . The set of character degrees of the group G is denoted $\text{cd}(G) = \{\chi(1) \mid \chi \in \text{Irr}(G)\}$ and we use the notation $\rho(G)$ to denote the set of primes that divide degrees in $\text{cd}(G)$. If n is an integer, we use $\pi(n)$ to represent the set of prime divisors of n . If N is a subgroup of G , we write $\pi(G : N)$ for $\pi(|G : N|)$.

In [9], Huppert and Manz studied both solvable and nonsolvable groups all of whose irreducible character degrees are square-free. They proved that for a nonsolvable group G , all the degrees in $\text{cd}(G)$ are square-free if and only if $G \cong A_7 \times S$, where S is a solvable group with all degrees in $\text{cd}(S)$ square-free and $\rho(S) \cap \{2, 3, 5, 7\} = \emptyset$.

In [11], the first author studied groups where 4 divides no irreducible character degree. He proved a similar result. In particular, he proved that G is a nonsolvable group with 4 dividing no degree in $\text{cd}(G)$ if and only if $G \cong A_7 \times S$, where S is solvable with $2 \notin \rho(S)$. Notice that A_7 is the only simple group where 4 divides no irreducible character degree.

In this note, we study a related problem. At the conclusion of [8], Huppert remarks that it would be interesting to determine the simple groups all of whose character degrees are of the form $2^a p_1 p_2 \cdots p_n$, with the p_i distinct odd primes. We say that a positive integer of this form is *odd-square-free* (OSF). A finite group G is odd-square-free if $\chi(1)$ is OSF for every $\chi \in \text{Irr}(G)$. We determine here all odd-square-free simple groups, thereby answering Huppert's question. Moreover, we also determine all odd-square-free groups H satisfying $S \leq H \leq \text{Aut}(S)$ for S a nonabelian finite simple group. In the situations considered in [9] and [11], the only simple group that arose

was A_7 . The following theorem shows that a number of other simple groups are odd-square-free. Except for A_7 , however, the simple groups S that arise all have the property that $|S|$ is OSF.

Theorem A. *Let S be a finite simple group and let $S \leq H \leq \text{Aut}(S)$. The group H is odd-square-free if and only if one of the following holds:*

- i. $S \cong J_1$.
- ii. $S \cong A_5$ or $S \cong A_7$.
- iii. $S \cong \text{PSL}_2(p)$, where $p \geq 5$ is an odd prime and $p^2 - 1$ is odd-square-free.
- iv. $S \cong \text{PSL}_2(q)$, where $q = 2^a$, $a \geq 2$ and $(q^2 - 1)|H : S|$ is odd-square-free.
- v. $S \cong {}^2B_2(q^2)$, where $q^2 = 2^{2m+1}$, $m \geq 1$, and $(q^2 - 1)(q^4 + 1)|H : S|$ is square-free.

The group J_1 is the first sporadic Janko group and $\text{Aut}(J_1) = J_1$. If S is one of A_5 , A_7 , or $\text{PSL}_2(p)$ with p odd, then $|\text{Aut}(S) : S| = 2$. Moreover, $(p-1, p+1) = 2$, hence $p^2 - 1 = (p-1)(p+1)$ is OSF if and only if both of $p-1$ and $p+1$ are OSF.

If S is $\text{PSL}_2(q)$, $q = 2^a$, then the outer automorphism group of S is cyclic of order a . Thus $|\text{Aut}(S) : S| = a$ and $|H : S|$ is a divisor of a . In this case, $(q-1, q+1) = 1$, and $(q^2 - 1)|H : S|$ is OSF if and only if both of $(q-1)|H : S|$ and $(q+1)|H : S|$ are OSF, or equivalently, $q-1$, $q+1$, and $|H : S|$ are OSF, $(q-1, |H : S|) = 1$, and $(q+1, |H : S|) = 1$.

Similarly, if S is ${}^2B_2(q^2)$, $q^2 = 2^{2m+1}$, then the outer automorphism group of S is cyclic of order $2m+1$. Thus $|\text{Aut}(S) : S| = 2m+1$ and $|H : S|$ is a divisor of $2m+1$. Again, $(q^2 - 1, q^4 + 1) = 1$, and $(q^2 - 1)(q^4 + 1)|H : S|$ is OSF if and only if both of $(q^2 - 1)|H : S|$ and $(q^4 + 1)|H : S|$ are square-free, or equivalently, $q^2 - 1$, $q^4 + 1$, and $|H : S|$ are square-free, $(q^2 - 1, |H : S|) = 1$, and $(q^4 + 1, |H : S|) = 1$.

Going beyond Huppert's question for simple groups, we consider this question for nonsolvable groups. Although a complete classification of nonsolvable odd-square-free groups does not seem feasible, we are able to show that if G is a nonsolvable odd-square-free group, then we have some control on the structure of G .

Theorem B. *If G is a nonsolvable odd-square-free group, then G has normal subgroups N and M so that M is a solvable odd-square-free group and either*

- 1. $N/M = S$ is a simple odd-square-free group and $S \leq G/M \leq \text{Aut}(S)$, or
- 2. $N/M = S \times T$ where S and T are simple odd-square-free groups with $\rho(S) \cap \rho(T) = \{2\}$ and $S \times T \leq G/M \leq \text{Aut}(S) \times \text{Aut}(T)$.

In particular, G has at most two nonabelian chief factors and these chief factors are simple.

We note that we can have odd-square-free groups with two nonabelian simple chief factors. Using the classification of finite simple groups, it is not difficult to show that every nonabelian simple group has character degrees that are divisible by either 3 or 5. In fact, only the Suzuki groups ${}^2B_2(q^2)$ have no degrees divisible by 3, and these groups do have degrees divisible by 5. If $q \equiv \pm 2 \pmod{5}$, then the groups $\text{PSL}_2(q)$, $\text{PSL}_3(q)$, $\text{PSU}_3(q^2)$, $G_2(q)$, and ${}^3D_4(q^3)$ have no degrees divisible by 5; nor does ${}^2G_2(3^{2m+1})$ when $3^{2m+1} \equiv \pm 2 \pmod{5}$.

If G has a simple factor whose degrees are divisible by both 3 and 5, then G will have a unique simple factor. On the other hand, if G has two simple factors S and T , then one must not have degrees divisible by 3 and the other must not have degrees divisible by 5. This implies that one of S or T is a Suzuki group and the other is in the list above. For example, $G = \text{PSL}_2(23) \times {}^2B_2(8)$ is odd-square-free.

The next result is somewhat incompatible with the goal of classifying all nonsolvable odd-square-free groups. It shows that if a group has a normal subgroup whose index is odd-square-free and 2 is the only prime dividing both the index and the degrees of the characters of the normal subgroup, then the group will be odd-square-free. It will therefore be difficult to classify such groups since a wreath product of any 2-group or any abelian group with one of the simple groups having odd-square-free order will satisfy these hypotheses.

Theorem C. *If G is a group with an odd-square-free normal subgroup N such that $|G:N|$ is odd-square-free and $\pi(G:N) \cap \rho(N) \subseteq \{2\}$, then G is odd-square-free.*

Note that there is a gap between Theorem C and the conclusion of Theorem B. However, if A_7 is involved in G , then we obtain a result that is similar to the theorem of Huppert and Manz. We note that A_7 and $3.A_7$ (the central extension of A_7 by a cyclic group of order 3) are both odd-square-free, as is S_7 . Both A_7 and S_7 arise in the second part of the following theorem when S is the trivial group.

Theorem D. *If G is an odd-square-free group and A_7 is involved in G , then there is an odd-square-free solvable normal subgroup S of G such that $\rho(S) \cap \{3, 5, 7\} = \emptyset$ and either*

1. $G = NS$, with $N \cong 3.A_7$ and $|N \cap S| \leq 3$, or
2. G has a subgroup H of index dividing 2 such that $H = N \times S$ with $N \cong A_7$.

2 Almost Simple Groups

Our goal here is to determine the groups H that are odd-square-free, where $S \leq H \leq \text{Aut}(S)$ and S is a (nonabelian) finite simple group, and thus prove Theorem A. We first consider the simple groups S themselves. By the Classification of Finite Simple Groups, S is a sporadic group, an alternating group, a group of exceptional Lie type, or a group of classical Lie type.

2.1 Sporadic and Alternating Groups

Lemma 2.1. *A sporadic simple group S is odd-square-free if and only if $S \cong J_1$.*

Proof. The character degree set of J_1 is

$$\text{cd}(J_1) = \{1, 2^3 \cdot 7, 2^2 \cdot 19, 7 \cdot 11, 2^3 \cdot 3 \cdot 5, 7 \cdot 19, 11 \cdot 19\},$$

and so the character degrees of J_1 are all OSF as claimed. Table 1 lists a character degree that is *not* OSF for each of the sporadic simple groups other than J_1 . The notation for the characters is from the Atlas [4]. □

Table 1: Non-OSF Degrees of Sporadic Groups

Group	Char.	Degree
M_{11}	χ_9	$45 = 3^2 \cdot 5$
M_{12}	χ_6	$45 = 3^2 \cdot 5$
M_{22}	χ_3	$45 = 3^2 \cdot 5$
J_2	χ_6	$36 = 2^2 \cdot 3^2$
M_{23}	χ_3	$45 = 3^2 \cdot 5$
HS	χ_7	$175 = 5^2 \cdot 7$
J_3	χ_6	$324 = 2^2 \cdot 3^4$
M_{24}	χ_3	$45 = 3^2 \cdot 5$
M^cL	χ_4	$252 = 2^2 \cdot 3^2 \cdot 7$
He	χ_4	$153 = 3^2 \cdot 17$
Ru	χ_2	$378 = 2 \cdot 3^3 \cdot 7$
Suz	χ_9	$5940 = 2^2 \cdot 3^3 \cdot 5 \cdot 11$
$O'N$	χ_2	$10944 = 2^6 \cdot 3^2 \cdot 19$
Co_3	χ_5	$275 = 5^2 \cdot 11$
Co_2	χ_4	$275 = 5^2 \cdot 11$
Fi_{22}	χ_8	$10725 = 3 \cdot 5^2 \cdot 11 \cdot 13$
HN	χ_8	$8910 = 2 \cdot 3^4 \cdot 5 \cdot 11$
Ly	χ_{11}	$1534500 = 2^2 \cdot 3^2 \cdot 5^3 \cdot 11 \cdot 31$
Th	χ_4	$27000 = 2^3 \cdot 3^3 \cdot 5^3$
Fi_{23}	χ_6	$30888 = 2^3 \cdot 3^3 \cdot 11 \cdot 13$
Co_1	χ_6	$17250 = 2 \cdot 3 \cdot 5^3 \cdot 23$
J_4	χ_4	$299367 = 3^2 \cdot 29 \cdot 31 \cdot 37$
Fi'_{24}	χ_3	$57477 = 3 \cdot 7^2 \cdot 17 \cdot 23$
B	χ_3	$96255 = 3^3 \cdot 5 \cdot 23 \cdot 31$
M	χ_4	$842609326 = 2 \cdot 13^2 \cdot 29 \cdot 31 \cdot 47 \cdot 59$

The alternating group A_n is simple for $n \geq 5$.

Lemma 2.2. *The alternating group A_n , $n \geq 5$, is odd-square-free if and only if $n = 5$ or $n = 7$.*

Proof. The character degree set of A_5 is $\text{cd}(A_5) = \{1, 3, 2^2, 5\}$ and the character degree set of A_7 is

$$\text{cd}(A_7) = \{1, 2 \cdot 3, 2 \cdot 5, 2 \cdot 7, 3 \cdot 5, 3 \cdot 7, 5 \cdot 7\},$$

and so A_5 and A_7 are odd-square-free as claimed. In Atlas [4] notation, the group A_6 has character degree $\chi_6(1) = 9$, while A_8 has $\chi_{10}(1) = 45$, A_9 has $\chi_5(1) = 27$, and A_{10} has $\chi_2(1) = 9$. These degrees are all divisible by $9 = 3^2$, hence are not OSF. Therefore, the result holds for $n \leq 10$.

We now assume $S = A_n$, where $n \geq 11$. We consider the character $\chi_{r,s}$ of the symmetric group S_n corresponding to the partition $\lambda_{r,s} = (n - s - r, s + 1, 1^{r-1})$ of n . (See §3 of [13] for notation.) This character of S_n exists whenever $r \geq 1$, $s \geq 0$, and $n \geq r + 2s + 1$, and has degree

$$\chi_{r,s}(1) = \binom{n}{s} \binom{n-s-1}{r-1} \frac{n-2s-r}{r+s}.$$

Moreover, the irreducible constituents of the restriction of $\chi_{r,s}$ to the alternating group A_n have degree $\chi_{r,s}(1)$ or $\chi_{r,s}(1)/2$, hence have the same odd part as $\chi_{r,s}(1)$.

In particular, we consider the characters $\chi_{6,1}$ and $\chi_{6,2}$. If $r = 6$ and $s = 1$ or 2 , then

$$r + 2s + 1 \leq 6 + 2(2) + 1 = 11 \leq n,$$

and so both characters of S_n exist. We have

$$\chi_{6,1}(1) = \binom{n}{1} \binom{n-2}{5} \frac{n-8}{7} = \frac{n(n-2)(n-3)(n-4)(n-5)(n-6)(n-8)}{2^3 \cdot 3 \cdot 5 \cdot 7}$$

and

$$\chi_{6,2}(1) = \binom{n}{2} \binom{n-3}{5} \frac{n-10}{8} = \frac{n(n-1)(n-3)(n-4)(n-5)(n-6)(n-7)(n-10)}{2^7 \cdot 3 \cdot 5}.$$

We claim that in any case the 3-part of one of these degrees is at least 3^2 . If $n \equiv 0 \pmod{3}$, then

$$(\chi_{6,1}(1))_3 = \left(\frac{n(n-3)(n-6)}{3} \right)_3 \geq 3^2.$$

If $n \equiv 1 \pmod{3}$, then

$$(\chi_{6,2}(1))_3 = \left(\frac{(n-1)(n-4)(n-7)(n-10)}{3} \right)_3 \geq 3^3.$$

If $n \equiv 2 \pmod{3}$, then

$$(\chi_{6,1}(1))_3 = \left(\frac{(n-2)(n-5)(n-8)}{3} \right)_3 \geq 3^2.$$

Therefore, if $n \geq 11$, then there is a character degree of S_n , hence also a degree of A_n , that is divisible by 3^2 , and so A_n is not odd-square-free. $\square$

2.2 Groups of Exceptional Lie Type

We consider next the groups of exceptional Lie type, that is, those of types:

$$\begin{aligned}
&G_2(q) \text{ for } q \neq 2, \\
&F_4(q), E_6(q), E_7(q), E_8(q), {}^2E_6(q^2), {}^3D_4(q^3), \\
&{}^2B_2(q^2) \text{ and } {}^2F_4(q^2) \text{ for } q^2 \neq 2, \\
&{}^2F_4(2)', \\
&{}^2G_2(q^2) \text{ for } q^2 \neq 3.
\end{aligned}$$

Lemma 2.3. *Let S be a simple group of exceptional Lie type. The group S is odd-square-free if and only if $S \cong {}^2B_2(q^2)$, where $q^2 = 2^{2m+1}$, $m \geq 1$, and $(q^2 - 1)(q^4 + 1)$ is square-free.*

Proof. If $S = {}^2F_4(2)'$, then by the Atlas [4], S has degree $\chi_4(1) = 27 = 3^3$, hence S is not OSF. Table 2 lists degrees that are not OSF for the remaining groups of exceptional type other than the Suzuki groups ${}^2B_2(q^2)$ (see §13.9 of [3]). In the table, Φ_k denotes the k -th cyclotomic polynomial evaluated at q . (Note that the degree of ${}^2F_4(q^2)$ with label ρ_2 is given incorrectly in [3]. The correct expression for the degree is given in [12].)

Table 2: Non-OSF Degrees of Groups of Exceptional Lie Type

Group	Labels	Degrees
$G_2(q)$	$\phi_{1,6}$	q^6
	$G_2[\theta]$	$\frac{1}{3}q\Phi_1^2\Phi_2^2$
$F_4(q)$	$\phi'_{8,3}$	$q^3\Phi_4^2\Phi_8\Phi_{12}$
$E_6(q)$	$\phi_{24,6}$	$q^6\Phi_4^2\Phi_8\Phi_9\Phi_{12}$
$E_7(q)$	$\phi_{27,2}$	$q^2\Phi_3^2\Phi_6^2\Phi_9\Phi_{12}\Phi_{18}$
$E_8(q)$	$\phi_{8,91}$	$q^{91}\Phi_4^2\Phi_8\Phi_{12}\Phi_{20}\Phi_{24}$
${}^3D_4(q^3)$	$\phi_{2,2}$	$\frac{1}{2}q^3\Phi_2^2\Phi_{12}$
${}^2E_6(q^2)$	$\phi''_{8,3}$	$q^6\Phi_4^2\Phi_8\Phi_{12}\Phi_{18}$
${}^2G_2(q^2)$	ε	q^6
${}^2F_4(q^2)$	ρ_2	$\frac{1}{2}q^4\Phi_8^2\Phi_{24}$

In each case, the given degree is divisible by q^2 (using the Steinberg character labelled $\phi_{1,6}$ of degree q^6 in the case of $G_2(q)$). Hence, if q is odd, then the degree is not OSF.

If q is even, then $\Phi_2 = q + 1$, $\Phi_3 = q^2 + q + 1$, $\Phi_4 = q^2 + 1$, and $\Phi_8 = q^4 + 1$ are all odd and greater than 1. Each degree is divisible by the square of one of these odd numbers, hence is not OSF (using the character with label $G_2[\theta]$ for $G_2(q)$ and noting that q is odd for ${}^2G_2(q^2)$).

We have shown that no group of exceptional type is OSF except possibly the Suzuki groups, which we now consider. Let $S \cong {}^2B_2(q^2)$, where $q^2 = 2^{2m+1}$ and $m \geq 1$. By [15], we have

$$\text{cd}(S) = \{1, q^4, \frac{1}{\sqrt{2}}q\Phi_1\Phi_2, \Phi_8, \Phi_1\Phi_2\Phi'_8, \Phi_1\Phi_2\Phi''_8\},$$

where $\Phi'_8 = q^2 + \sqrt{2}q + 1$ and $\Phi''_8 = q^2 - \sqrt{2}q + 1$. Thus $\Phi_1\Phi_2 = q^2 - 1$, Φ'_8 , and Φ''_8 are odd integers with $\Phi'_8\Phi''_8 = \Phi_8 = q^4 + 1$.

If S is OSF, then clearly $\Phi_1\Phi_2 = q^2 - 1$ and $\Phi_8 = q^4 + 1$ must be OSF, and since they are odd and relatively prime, $(q^2 - 1)(q^4 + 1)$ must be square-free. Conversely, suppose $(q^2 - 1)(q^4 + 1)$ is square-free. As q^2 is even, it follows that all degrees other than $\Phi_1\Phi_2\Phi'_8$ and $\Phi_1\Phi_2\Phi''_8$ are OSF. Finally, we have that $\Phi_1\Phi_2 = q^2 - 1$ and $\Phi_8 = q^4 + 1$ are relatively prime, hence also $(\Phi_1\Phi_2, \Phi'_8) = (\Phi_1\Phi_2, \Phi''_8) = 1$. Since $\Phi'_8\Phi''_8 = \Phi_8$ is square-free, so are Φ'_8 and Φ''_8 , and it follows that $\Phi_1\Phi_2\Phi'_8$ and $\Phi_1\Phi_2\Phi''_8$ are OSF. Hence $S \cong {}^2B_2(q^2)$ is odd-square-free if and only if $(q^2 - 1)(q^4 + 1)$ is square-free, as claimed. $\square$

Although we do not know of necessary and sufficient conditions on m so that

$$(2^{2m+1} - 1)(2^{2(2m+1)} + 1) = (q^2 - 1)(q^4 + 1)$$

will be square-free, we do have the following partial result.

Corollary 2.4. *Let $S \cong {}^2B_2(q^2)$, where $q^2 = 2^{2m+1}$ and $m \geq 1$. If $m \equiv 2 \pmod{5}$, then S is not odd-square-free.*

Proof. If $m \equiv 2 \pmod{5}$, then $2m + 1$ is divisible by 5. Thus $2m + 1 = 5k$ for some odd integer k and $q^4 = 2^{10k}$. Since $2^{10} + 1 = 1025$, we have $2^{10} \equiv -1 \pmod{5^2}$, and so

$$q^4 + 1 \equiv (2^{10})^k + 1 \equiv (-1)^k + 1 \equiv -1 + 1 \equiv 0 \pmod{5^2}.$$

Hence $q^4 + 1$ is not OSF and so S is not OSF. $\square$

2.3 Groups of Classical Lie Type

Finally, we consider the groups of classical Lie type, that is, those of types:

$$A_1(q) \cong \text{PSL}_2(q), \text{ for } q \neq 2, 3,$$

$$A_\ell(q) \cong \text{PSL}_{\ell+1}(q), {}^2A_\ell(q^2) \cong \text{PSU}_{\ell+1}(q^2), \text{ for } \ell \geq 2 \text{ (and } q \neq 2 \text{ if } \ell = 2),$$

$$B_\ell(q) \cong \Omega_{2\ell+1}(q), C_\ell(q) \cong \text{PSp}_{2\ell}(q), \text{ for } \ell \geq 2 \text{ (and } q \neq 2 \text{ if } \ell = 2),$$

$$D_\ell(q) \cong \text{P}\Omega_{2\ell}^+(q), {}^2D_\ell(q^2) \cong \text{P}\Omega_{2\ell}^-(q) \text{ for } \ell \geq 4.$$

The restrictions on ℓ and q are so that the groups will be simple and (generally) not isomorphic to others in the list. These restrictions will always be assumed in what follows. The only classical groups that can be odd-square-free are those of type $A_1(q) \cong \text{PSL}_2(q)$, and we first eliminate the others from consideration.

Lemma 2.5. *If S is a simple group of classical Lie type and is not isomorphic to $A_1(q) \cong \text{PSL}_2(q)$, then S is not odd-square-free.*

Proof. We assume S is one of $A_\ell(q)$ ($\ell \geq 2$), ${}^2A_\ell(q^2)$, $B_\ell(q)$, $C_\ell(q)$, $D_\ell(q)$, or ${}^2D_\ell(q^2)$. The Steinberg character of S has degree q^k , where $k = \ell(\ell+1)/2$ for types A_ℓ and ${}^2A_\ell$, $k = \ell^2$ for types B_ℓ and C_ℓ , and $k = \ell(\ell-1)$ for types D_ℓ and ${}^2D_\ell$. As $\ell \geq 2$ in all cases, it follows that $k \geq 2$. Therefore, if q is odd, then S is not OSF and we will now assume q is a power of 2.

Case 1: $S \cong A_\ell(q) \cong \text{PSL}_{\ell+1}(q)$

First let $\ell = 2$, so that $S \cong \text{PSL}_3(q)$. Since S is not isomorphic to $\text{PSL}_2(r)$ for any r and $\text{PSL}_3(2) \cong \text{PSL}_2(7)$, we have $q > 2$. By [14], S has an irreducible character of degree $(q-1)^2(q+1)$. Since $q-1$ is odd and greater than 1, this degree is not OSF.

If $\ell = 3$, then $S \cong \text{PSL}_4(q) \cong \text{PGL}_4(q)$, as q is even. By Lemma 2.2 of [16], S has irreducible character χ_2 of degree

$$\chi_2(1) = (q^2 - 1)(q^4 - 1),$$

which is divisible by the odd number $(q+1)^2$ and is therefore not OSF.

For $\ell \geq 4$, $S \cong \text{PSL}_{\ell+1}(q)$ is a normal subgroup of $\text{PGL}_{\ell+1}(q)$ of index $d = (\ell+1, q-1)$. Again by Lemma 2.2 of [16], $\text{PGL}_{\ell+1}(q)$ has irreducible character χ_1 of degree

$$\chi_1(1) = (q-1)(q^2-1)(q^3-1)(q^4-1) \cdots (q^\ell-1).$$

Hence by Lemma 11.29 of [10], an irreducible constituent $\mu \in \text{Irr}(S)$ of the restriction of χ_1 to S has degree divisible by $\chi_1(1)/d$. Therefore, $\mu(1)$ is divisible by $(q^2-1)(q^4-1)$, hence by $(q+1)^2$, and so μ is not OSF.

Case 2: $S \cong {}^2A_\ell(q^2) \cong \text{PSU}_{\ell+1}(q^2)$

First let $\ell = 2$, so that $S \cong \text{PSU}_3(q^2)$ and $q > 2$. By [14], S has an irreducible character of degree $(q-1)(q+1)^2$. Since $q+1$ is odd and greater than 1, this degree is not OSF.

For $\ell \geq 3$, $S \cong \text{PSU}_{\ell+1}(q^2)$ is a normal subgroup of $\text{PU}_{\ell+1}(q^2)$ of index $d = (\ell+1, q+1)$. By Lemma 2.3 of [16], $\text{PU}_{\ell+1}(q^2)$ has irreducible character χ_1 of degree

$$\chi_1(1) = (q+1)(q^2-1)(q^3+1) \cdots (q^{\ell-1} - (-1)^{\ell-1})(q^\ell - (-1)^\ell).$$

Again using Lemma 11.29 of [10], we have that an irreducible constituent $\mu \in \text{Irr}(S)$ of the restriction $(\chi_1)_S$ has degree divisible by $\chi_1(1)/d$. Therefore, since $\ell \geq 3$, $\mu(1)$ is divisible by $(q^2-1)(q^3+1)$, hence by $(q+1)^2$, and so μ is not OSF.

Case 3: $S \cong B_\ell(q) \cong C_\ell(q) \cong \text{PSp}_{2\ell}(q)$

Since q is a power of 2, the simple groups of types B_ℓ and C_ℓ are isomorphic. If $\ell = 2$, then $S \cong \text{PSp}_4(q)$ and $q > 2$. By [6], S has character degree $\chi_5(1) = (q-1)^2(q+1)^2$, and since $q+1$ is odd, this degree is not OSF. If $\ell \geq 3$, then by Lemma 2.3 of [17], S has an irreducible character of degree

$$\chi_c(1) = (q^2-1)(q^4-1) \cdots (q^{2(\ell-1)}-1)(q^\ell-1),$$

which is divisible by $(q^2-1)(q^4-1)$. Hence $\chi_c(1)$ is divisible by $(q+1)^2$ and so is not OSF.

Case 4: $S \cong D_\ell(q) \cong \text{P}\Omega_{2\ell}^+(q)$ or $S \cong {}^2D_\ell(q^2) \cong \text{P}\Omega_{2\ell}^-(q)$

In this case, $\ell \geq 4$. By Lemmas 2.4 and 2.5 of [17], S has an irreducible character denoted χ_c . If $S \cong D_\ell(q)$, then χ_c has degree

$$\chi_c(1) = (q-1)(q^{\ell-1}-1)(q^\ell-1)(q^4-1) \prod_{i=3}^{\ell-2} (q^{2i}-1).$$

Since one of ℓ or $\ell - 1$ is even, $(q^{\ell-1} - 1)(q^\ell - 1)(q^4 - 1)$ is divisible by $(q + 1)^2$, and so $\chi_c(1)$ is not OSF. If $S \cong {}^2D_\ell(q^2)$, then χ_c has degree

$$\chi_c(1) = (q^2 - 1)(q^4 - 1)(q^6 - 1) \cdots (q^{2(\ell-2)} - 1)(q^{2(\ell-1)} - 1),$$

which is divisible by $(q + 1)^2$, and so again $\chi_c(1)$ is not OSF. $\square$

Lemma 2.6. *The group $S \cong A_1(q) \cong \text{PSL}_2(q)$, $q = p^a \geq 4$, is odd-square-free if and only if one of the following holds:*

- i. $q = 2^a$, $a \geq 2$, and $q^2 - 1$ is square-free.
- ii. $q = p \geq 5$ is an odd prime and $q^2 - 1$ is odd-square-free.

Proof. Let $S \cong \text{PSL}_2(q)$. If $q = 2^a$ and $a \geq 2$, then

$$\text{cd}(S) = \{1, 2^a - 1, 2^a, 2^a + 1\}.$$

Hence S is OSF if and only if $2^a - 1$ and $2^a + 1$ are square-free. Since $(2^a - 1, 2^a + 1) = 1$, this holds if and only if $(2^a - 1)(2^a + 1) = q^2 - 1$ is square-free.

Now let $q = p^a \geq 5$, where p is an odd prime. Since $\text{PSL}_2(5) \cong \text{PSL}_2(4)$, which was considered above, we may assume $q > 5$. In this case, we have

$$\text{cd}(S) = \{1, q - 1, q, q + 1, (q + \epsilon)/2\},$$

where $\epsilon = (-1)^{(q-1)/2}$. If $a > 1$, then since p is odd and $q = p^a$ is a character degree, S is not OSF. Thus we may assume $a = 1$ and $q = p$ is prime. We then have that S is OSF if and only if $q - 1$, $q + 1$, and $(q + \epsilon)/2$ are OSF. Since $(q + \epsilon)/2$ divides one of $q - 1$ or $q + 1$, this holds if and only if $q - 1$ and $q + 1$ are OSF. Finally, since $(q - 1, q + 1) = 2$, this holds if and only if $(q - 1)(q + 1) = q^2 - 1$ is OSF. $\square$

Lemmas 2.1–2.6 show that a simple group S is odd-square-free if and only if S is one of the simple groups listed in Theorem A.

Although we do not know of necessary and sufficient conditions on a so that $2^{2a} - 1 = q^2 - 1$ will be square-free, we do have the following partial result.

Corollary 2.7. *Let $S \cong \text{PSL}_2(2^a)$ with $a \geq 2$. If there is an odd prime p such that $p(p - 1)/2$ divides a , then S is not odd-square-free.*

Proof. Suppose $p(p - 1)/2 \mid a$ for some odd prime p , so that $p(p - 1) \mid 2a$. By Euler's Theorem,

$$2^{\varphi(p^2)} \equiv 2^{p(p-1)} \equiv 1 \pmod{p^2}.$$

Hence $p^2 \mid 2^{p(p-1)} - 1$, and since $p(p - 1) \mid 2a$, we have $2^{p(p-1)} - 1 \mid 2^{2a} - 1$. Therefore, $p^2 \mid 2^{2a} - 1$ and S is not OSF. $\square$

2.4 Automorphism Groups

In this section, we determine the groups H that are odd-square-free, where $S \leq H \leq \text{Aut}(S)$ for a finite simple group S . The following lemma allows us to restrict our attention to the simple groups S that are odd-square-free; that is, by the results of the previous section, the simple groups S listed in Theorem A.

Lemma 2.8. *Let S be a finite simple group and $S \leq H \leq \text{Aut}(S)$.*

- i. *If S is not odd-square-free, then H is not odd-square-free.*
- ii. *If $|H : S|$ is a power of 2, then H is odd-square-free if and only if S is odd-square-free.*

Proof. (i) Suppose $\chi \in \text{Irr}(S)$ and $\chi(1)$ is not OSF. Since $S \trianglelefteq H$, we have $\chi(1) \mid \psi(1)$ for every irreducible constituent $\psi \in \text{Irr}(H)$ of χ^H . Hence H is not OSF.

(ii) If H is OSF, then S is OSF in any case by (i). Suppose H is not OSF and $|H : S|$ is a power of 2. Let $\psi \in \text{Irr}(H)$ with $\psi(1)$ divisible by the square of an odd prime r , and let $\chi \in \text{Irr}(S)$ be a constituent of the restriction of ψ to S . By Lemma 11.29 of [10], $\psi(1)/\chi(1)$ divides $|H : S|$, a power of 2. Hence r^2 divides $\chi(1)$ as well, and so S is not OSF. $\square$

Lemma 2.9. *Let S be one of J_1 , A_5 , A_7 , or $\text{PSL}_2(p)$ with $p \geq 5$ an odd prime and $p^2 - 1$ odd-square-free. If $S \leq H \leq \text{Aut}(S)$, then H is odd-square-free.*

Proof. By Lemmas 2.1, 2.2, and 2.6, S is OSF in each case. As noted previously, $\text{Aut}(J_1) = J_1$. In each of the remaining cases, $|\text{Aut}(S) : S| = 2$, and so the result follows from Lemma 2.8. $\square$

It remains to determine when H is odd-square-free if $S \leq H \leq \text{Aut}(S)$ and S is either $\text{PSL}_2(2^a)$ or ${}^2B_2(2^{2m+1})$. Observe that in both cases, the only outer automorphisms are field automorphisms, and $\text{Aut}(S)/S$ is cyclic of order a for $S \cong \text{PSL}_2(2^a)$ and order $2m + 1$ for $S \cong {}^2B_2(2^{2m+1})$.

Lemma 2.10. *Let $S \cong \text{PSL}_2(q)$, where $q = 2^a$ and $a \geq 2$. A group H with $S \leq H \leq \text{Aut}(S)$ is odd-square-free if and only if $(q^2 - 1)|H : S|$ is odd-square-free.*

Proof. Suppose first that $(q^2 - 1)|H : S| = (2^a - 1)(2^a + 1)|H : S|$ is OSF. If $\psi \in \text{Irr}(H)$, then by Lemma 11.29 of [10], $\psi(1)/\chi(1)$ divides $|H : S|$ for any irreducible constituent χ of the restriction of ψ to S . Thus for every $\psi \in \text{Irr}(H)$, $\psi(1)$ divides $\chi(1)|H : S|$ for some $\chi \in \text{Irr}(S)$.

We have $\text{cd}(S) = \{1, 2^a - 1, 2^a, 2^a + 1\}$, and so each degree $\psi(1)$ of H divides one of $|H : S|$, $(2^a - 1)|H : S|$, $2^a|H : S|$, or $(2^a + 1)|H : S|$. Since $(2^a - 1)(2^a + 1)|H : S|$ is OSF, certainly each of these integers is OSF, and therefore $\psi(1)$ is OSF for every $\psi \in \text{Irr}(H)$.

Conversely, suppose H is OSF. By Lemma 2.8, S is OSF, and so $q^2 - 1$ is square-free by Lemma 2.6. If $a = 2$, then $|H : S|$ divides 2 and so $(q^2 - 1)|H : S|$ is also square-free. Hence we may assume $a \geq 3$.

The outer automorphism group of S is cyclic of order a , consisting of field automorphisms induced on S by $\text{Gal}(\mathbb{F}_{2^a}/\mathbb{F}_2)$. Using the character table of S in [5], for example, it is not difficult to show that there are characters $\chi_1, \theta_1 \in \text{Irr}(S)$ of degrees $\chi_1(1) = 2^a + 1$ and $\theta_1(1) = 2^a - 1$, each of whose stabilizer in $\text{Aut}(S)$ (hence in H) is S . Therefore, by Theorem 6.11 of [10], the induced characters χ_1^H and θ_1^H are irreducible characters of H . Since H is OSF, the degrees $\chi_1^H(1) = (2^a + 1)|H : S|$ and $\theta_1^H(1) = (2^a - 1)|H : S|$ are both OSF. Finally, $(2^a + 1, 2^a - 1) = 1$ implies $(2^a - 1)(2^a + 1)|H : S| = (q^2 - 1)|H : S|$ is OSF. $\square$

For the Suzuki groups ${}^2B_2(q^2)$, we will use the character table and notation of [15], except that what we have denoted as q^2 is called q in [15].

Lemma 2.11. *Let $S \cong {}^2B_2(q^2)$, where $q^2 = 2^{2m+1}$ and $m \geq 1$. A group H with $S \leq H \leq \text{Aut}(S)$ is odd-square-free if and only if $(q^2 - 1)(q^4 + 1)|H : S|$ is square-free.*

Proof. Recall that

$$\text{cd}(S) = \{1, q^4, \frac{1}{\sqrt{2}}q\Phi_1\Phi_2, \Phi_8, \Phi_1\Phi_2\Phi'_8, \Phi_1\Phi_2\Phi''_8\},$$

where $\Phi'_8 = q^2 + \sqrt{2}q + 1$ and $\Phi''_8 = q^2 - \sqrt{2}q + 1$ are odd integers with $\Phi'_8\Phi''_8 = \Phi_8 = q^4 + 1$. Also, $|\text{Aut}(S) : S| = 2m + 1$ is odd and $|H : S|$ divides $2m + 1$.

Suppose first that $(q^2 - 1)(q^4 + 1)|H : S| = \Phi_1\Phi_2\Phi_8|H : S|$ is square-free. As in the proof of the previous lemma, if $\psi \in \text{Irr}(H)$, then $\psi(1)$ divides $\chi(1)|H : S|$ for some $\chi \in \text{Irr}(S)$. Hence the odd part of $\psi(1)$ must divide one of $|H : S|$, $\Phi_1\Phi_2|H : S|$, $\Phi_8|H : S|$, $\Phi_1\Phi_2\Phi'_8|H : S|$, or $\Phi_1\Phi_2\Phi''_8|H : S|$. As $\Phi_1\Phi_2\Phi_8|H : S|$ is square-free and Φ'_8, Φ''_8 are factors of Φ_8 , each of these integers is square-free, and so $\psi(1)$ is OSF for every $\psi \in \text{Irr}(H)$.

Conversely, suppose H is OSF. Our aim is to show that there is an irreducible character of each of the degrees Φ_8 and $\Phi_1\Phi_2\Phi''_8$ whose stabilizer in $\text{Aut}(S)$ is S . This will imply that H has irreducible characters of degrees $\Phi_8|H : S|$ and $\Phi_1\Phi_2\Phi''_8|H : S|$, and the result will follow.

The outer automorphism group of S is cyclic of order $2m + 1$, consisting of field automorphisms induced on S by $\mathcal{G} = \text{Gal}(\mathbb{F}_{2^{2m+1}}/\mathbb{F}_2)$. The group $\mathcal{G}$ is generated by the automorphism $\bar{\sigma}$, where $\bar{\sigma}(\gamma) = \gamma^2$ for all $\gamma \in \mathbb{F}_{2^{2m+1}}$. Let σ be the outer automorphism of S induced by $\bar{\sigma}$. If x is a semisimple element of S , then x^σ is conjugate in S to x^2 and x^{σ^k} is conjugate in S to x^{2^k} for any integer k .

Consider first the character $X_1 \in \text{Irr}(S)$ of degree $\Phi_8 = q^4 + 1$ (see Theorem 13 of [15], replacing q with q^2). As in §17 of [15], let ξ_0 be a generator of the subgroup A_0 of S of order $q^2 - 1$, and let ε_0 be a complex primitive $(q^2 - 1)$ -th root of unity. We have $X_1(\xi_0^j) = \varepsilon_0^j + \varepsilon_0^{-j}$.

Suppose σ^{-k} is an element of the stabilizer T of X_1 in H . In particular, this implies

$$\varepsilon_0 + \varepsilon_0^{-1} = X_1(\xi_0) = X_1^{\sigma^{-k}}(\xi_0) = X_1(\xi_0^{\sigma^k}) = X_1(\xi_0^{2^k}) = \varepsilon_0^{2^k} + \varepsilon_0^{-2^k},$$

and so

$$\varepsilon_0^{2^k} + \varepsilon_0^{-2^k} = \varepsilon_0 + \varepsilon_0^{-1}.$$

It is not difficult to see that this implies $2^k \equiv 1 \pmod{q^2 - 1}$ or $2^k \equiv -1 \pmod{q^2 - 1}$.

Recall that σ is of order $2m + 1$, so that $0 \leq k \leq 2m$, and $q^2 = 2^{2m+1}$. Hence $2^k - 1 < q^2 - 1$, and so $2^k \equiv 1 \pmod{q^2 - 1}$ implies $2^k - 1 = 0$. Thus $k = 0$ and σ^{-k} is the identity automorphism. If $2^k \equiv -1 \pmod{q^2 - 1}$, then we must have

$$2^{2m+1} - 1 \leq 2^k + 1 \leq 2^{2m} + 1.$$

But since $m \geq 1$, this is impossible. Hence the only power of σ in T is the identity automorphism, and so $T = S$.

Since $S \trianglelefteq H$ and the stabilizer of X_1 in H is S , we have that the induced character X_1^H is irreducible of degree

$$X_1^H(1) = \Phi_8|H : S| = (q^4 + 1)|H : S|.$$

Therefore, as H is OSF, we have that $\Phi_8|H : S|$ is OSF.

Next, we consider the character $Y_1 \in \text{Irr}(S)$ of degree $\Phi_1\Phi_2\Phi_8''$. As in §17 of [15], let ξ_1 be a generator of the subgroup A_1 of S of order Φ_8' , and let ε_1 be a complex primitive Φ_8' -th root of unity. We have

$$Y_1(\xi_1^j) = \varepsilon_1^j + \varepsilon_1^{-j} + \varepsilon_1^{q^2j} + \varepsilon_1^{-q^2j}.$$

Suppose σ^{-k} is an element of the stabilizer T of Y_1 in H . In particular, this implies

$$\varepsilon_1 + \varepsilon_1^{-1} + \varepsilon_1^{q^2} + \varepsilon_1^{-q^2} = Y_1(\xi_1) = Y_1^{\sigma^{-k}}(\xi_1) = Y_1(\xi_1^{\sigma^k}) = Y_1(\xi_1^{2^k}) = \varepsilon_1^{2^k} + \varepsilon_1^{-2^k} + \varepsilon_1^{q^2 2^k} + \varepsilon_1^{-q^2 2^k},$$

and so

$$\varepsilon_1^{2^k} + \varepsilon_1^{-2^k} + \varepsilon_1^{q^2 2^k} + \varepsilon_1^{-q^2 2^k} = \varepsilon_1 + \varepsilon_1^{-1} + \varepsilon_1^{q^2} + \varepsilon_1^{-q^2}.$$

Observe that

$$Y_i(\xi_1) = \varepsilon_1^i + \varepsilon_1^{-i} + \varepsilon_1^{q^2 i} + \varepsilon_1^{-q^2 i} = \varepsilon_1^j + \varepsilon_1^{-j} + \varepsilon_1^{q^2 j} + \varepsilon_1^{-q^2 j} = Y_j(\xi_1)$$

if and only if $j \equiv \pm i \pmod{\Phi_8'}$ or $j \equiv \pm q^2 i \pmod{\Phi_8'}$. (Indeed, it is clear that if one of these congruences holds, then the character values are equal, and in fact $Y_i = Y_j$. Hence there are at most $(\Phi_8' - 1)/4$ distinct characters of type Y_i . But as shown in [15], there are *exactly* $(\Phi_8' - 1)/4$ of these characters, and so the converse holds as well.) Hence if σ^{-k} is in T , we must have $2^k \equiv \pm 1 \pmod{\Phi_8'}$ or $2^k \equiv \pm q^2 \pmod{\Phi_8'}$.

Recall again that $0 \leq k \leq 2m$ and $q^2 = 2^{2m+1}$. In particular, $2^k < q^2$ and both $2^k - 1$ and $2^k + 1$ are strictly less than $q^2 + \sqrt{2}q + 1 = \Phi_8'$. Hence $2^k \equiv 1 \pmod{\Phi_8'}$ implies $2^k - 1 = 0$ and so $k = 0$, and since $2^k + 1 \neq 0$, $2^k \equiv -1 \pmod{\Phi_8'}$ is not possible. Similarly,

$$0 < q^2 - 2^k < q^2 + \sqrt{2}q + 1 = \Phi_8',$$

so $2^k \equiv q^2 \pmod{\Phi_8'}$ is impossible. Finally, we know $2^k \mid q^2$ and Φ_8' is odd, so if $\Phi_8' \mid q^2 + 2^k$, then Φ_8' must divide $(q^2/2^k) + 1$. But

$$0 < \frac{q^2}{2^k} + 1 < q^2 + \sqrt{2}q + 1 = \Phi_8',$$

so this is not possible. Hence $2^k \not\equiv -q^2 \pmod{\Phi_8'}$.

We therefore have that if σ^{-k} is in T , then $k = 0$ and σ^{-k} is the identity automorphism. Hence $T = S$ and Y_1^H is an irreducible character of H of degree

$$Y_1^H(1) = \Phi_1\Phi_2\Phi_8''|H : S| = (q^2 - 1)(q^2 - \sqrt{2}q + 1)|H : S|.$$

Therefore, as H is OSF, we have that $\Phi_1\Phi_2\Phi_8''|H : S|$ is OSF.

We have shown that if H is OSF, then $\Phi_8|H : S| = (q^4 + 1)|H : S|$ and

$$\Phi_1\Phi_2\Phi_8''|H : S| = (q^2 - 1)(q^2 - \sqrt{2}q + 1)|H : S|$$

are both OSF, hence square-free. Since $(q^4 + 1, q^2 - 1) = 1$, this implies $(q^2 - 1)(q^4 + 1)|H : S|$ is square-free. $\square$

Theorem A now follows from Lemmas 2.1, 2.2, 2.3, 2.5, 2.6, 2.9, 2.10, and 2.11.

3 Proofs of Theorems B and C

We now work to prove our general results. We begin with a lemma showing that any nonabelian chief factor of an odd-square-free group must be simple.

Lemma 3.1. *Let G be an odd-square-free group. If $M < N$ are normal subgroups of G so that N/M is a nonabelian chief factor, then N/M is a simple odd-square-free group.*

Proof. Since every degree in $\text{cd}(N/M)$ divides a degree in $\text{cd}(G)$, it follows that N/M is an odd-square-free group. There is a nonabelian simple group S so that $N/M \cong S_1 \times \cdots \times S_n$ and $S_i \cong S$ for each i . We know that there is a degree a in $\text{cd}(S)$ that is divisible by an odd prime. If $n \geq 2$, then a^2 divides some degree in $\text{cd}(N/M)$. Thus, we must have $n = 1$, and the lemma is proved. $\square$

We next prove that a nonsolvable odd-square-free group has at most two nonabelian simple factors.

Lemma 3.2. *If G is an odd-square-free group such that all minimal normal subgroups of G are nonabelian, then one of the following occurs:*

1. *G has a unique minimal normal subgroup S and S is a simple odd-square-free group. In particular, $S \leq G \leq \text{Aut}(S)$.*
2. *G has exactly two minimal normal subgroups S and T and these are simple odd-square-free groups that satisfy $\rho(S) \cap \rho(T) = \{2\}$. In particular, $S \times T \leq G \leq \text{Aut}(S) \times \text{Aut}(T)$.*

Proof. Let S be a minimal normal subgroup of G . Note that S is a chief factor of G . Thus, we may apply Lemma 3.1 to see that S is a simple odd-square-free group. Let $C = \mathbf{C}_G(S)$. We know that $C \cap S = 1$ and C is normal in G . Also, G/C is isomorphic to a subgroup of $\text{Aut}(S)$. Any other minimal normal subgroup of G will be contained in C . Thus, if $C = 1$, then S is the unique minimal normal subgroup of G , and $S \leq G \leq \text{Aut}(S)$.

Suppose now that $C > 1$. Then G will have a minimal normal subgroup T that is contained in C . By Lemma 3.1, we have that T is a simple odd-square-free group. Let $D = \mathbf{C}_G(T)$. We know $D \cap T = 1$ and D is normal in G . Also, G/D is isomorphic to a subgroup of $\text{Aut}(T)$. Notice that $S \leq D$. We see that $ST = S \times T$ is a normal subgroup of G . Every degree in $\text{cd}(ST)$ divides a degree in $\text{cd}(G)$, so ST is odd-square-free. Hence, if an odd prime divides a degree in $\text{cd}(S)$, then it can divide no degree in $\text{cd}(T)$. This implies $\rho(S) \cap \rho(T) \subseteq \{2\}$. We know that $G/(C \cap D)$ is isomorphic to a subgroup of $G/C \times G/D$, which in turn is isomorphic to a subgroup of $\text{Aut}(S) \times \text{Aut}(T)$. Hence, if $C \cap D = 1$, then S and T are the only minimal normal subgroups of G and $G \subseteq \text{Aut}(S) \times \text{Aut}(T)$, and the lemma is proved.

Suppose that $C \cap D > 1$. It follows that $C \cap D$ contains a minimal normal subgroup U of G . By Lemma 3.1, U is a simple subgroup. We have that $STU = S \times T \times U$ is a normal subgroup of G , and hence, it is odd-square-free. On the other hand, we know that every simple group must have either 3 or 5 dividing a character degree. Hence we will be able to find a degree of STU that is divisible by either 3^2 or 5^2 , which contradicts the fact that STU is odd-square-free. $\square$

We now prove Theorem B.

Proof of Theorem B. Take M to be maximal among solvable normal subgroups of G . Since every degree in $\text{cd}(M)$ divides a degree in $\text{cd}(G)$, we see that M is odd-square-free. Notice that G/M is an odd-square-free group and every minimal normal subgroup of G/M is nonabelian. We may apply Lemma 3.2 to see that either G/M has a unique minimal normal subgroup S that is a simple odd-square-free group or G/M has exactly two minimal normal subgroups S and T that are simple odd-square-free groups with $\rho(S) \cap \rho(T) = \{2\}$. We also have either $S \leq G/M \leq \text{Aut}(S)$ or $S \times T \leq G/M \leq \text{Aut}(S) \times \text{Aut}(T)$. In the first case, we take $N/M = S$ and in the second case, we take $N/M = S \times T$. Observe that in the first case, G/N is isomorphic to a subgroup of the outer automorphism group of S , and in the second case, G/N is contained in the direct product of the outer automorphism groups of S and T . $\square$

Next, we prove Theorem C.

Proof of Theorem C. Let $\chi \in \text{Irr}(G)$ and let $\nu \in \text{Irr}(N)$ be a constituent of χ_N . By Corollary 11.29 of [10], we know that $\chi(1)/\nu(1)$ divides $|G : N|$. Hence, $\chi(1)/\nu(1)$ is OSF. Since N is an odd-square-free group, we know that $\nu(1)$ is OSF. Finally, no odd prime divides both $\chi(1)/\nu(1)$ and $\nu(1)$ as $\pi(G : N) \cap \rho(N) \subseteq \{2\}$. Thus $\chi(1)$ is OSF, and hence G is an odd-square-free group. $\square$

4 Proof of Theorem D

We now turn to odd-square-free groups that involve A_7 . Our proof is similar to the proof employed in [11]. The key to our work is the following lemma, which appeared in [11]. This lemma was discovered by Isaacs.

Lemma 4.1. *Let S be a group and let V be an irreducible module for S with associated character χ . If $H < S$ is maximal satisfying $[\chi_H, 1_H] > 0$, then there is an element $0 \neq v \in V$ so that $\mathbf{C}_S(v) = H$.*

The following lemma should be compared with Lemma 2.3 of [11]. We use the same data as was produced for that note.

Lemma 4.2. *If $S = A_7$ acts via automorphisms on a nontrivial elementary abelian p -group V , where $p \geq 11$ is a prime and $\mathbf{C}_S(V) = 1$, then there exists an element $1 \neq v \in V$ and a degree $a \in \text{cd}(\mathbf{C}_S(v))$ so that 9 divides $a|S : \mathbf{C}_S(v)|$.*

Proof. We can view V as a module for S . Since p does not divide $|S|$, we know via Maschke's theorem that V is completely reducible as an S -module. Without loss of generality, we may assume that V is an irreducible S -module. Let $F = \mathbb{F}_p$ be the field of p elements. Let K be the ring of FS -endomorphisms of V . Note that by Schur's lemma, since V is a simple FS -module, K is a division ring, and since K is finite, we have that K is a field. We may also view V as a K -vector space. Then V is a simple KS -module, and as such is absolutely irreducible. In particular, V corresponds to an irreducible complex character of S , and so V corresponds to an ordinary character $\chi \in \text{Irr}(S)$.

The stabilizers in S of elements of V are not affected by viewing V as vector space over K rather than over F . Therefore, it suffices to show that for each nonprincipal character $\psi \in \text{Irr}(S)$, there is a subgroup H with the desired properties that is maximal subject to $[\psi_H, 1_H] > 0$. To do this, we use the computer algebra system Magma [2] (or GAP [7]) to compute the character table of A_7 , which is shown in Table 3, and the subgroup lattice of A_7 , which is shown in Table 4.

Table 3: The Character Table for A_7

	1	2	3 ₁	3 ₂	4	5	6	7 ₁	7 ₂
χ_1	1	1	1	1	1	1	1	1	1
χ_2	6	2	3	0	0	1	-1	-1	-1
χ_3	10	-2	1	1	0	0	1	α	$\bar{\alpha}$
χ_4	10	-2	1	1	0	0	1	$\bar{\alpha}$	α
χ_5	14	2	-1	2	0	-1	-1	0	0
χ_6	14	2	2	-1	0	-1	2	0	0
χ_7	15	-1	3	0	-1	0	-1	1	1
χ_8	21	1	-3	0	-1	1	1	0	0
χ_9	35	-1	-1	-1	1	0	-1	0	0

where $\alpha = \epsilon + \epsilon^2 + \epsilon^4$, $\epsilon = e^{2\pi i/7}$.

Using Magma [2], we computed $[\chi_i, 1_{H_j}]$ for each nonprincipal character $\chi_i \in \text{Irr}(A_7)$ and each H_j in the subgroup lattice of A_7 . In Table 5, we present for each i the values j so that $[\chi_i, 1_{H_j}] \neq 0$.

For each χ_i , a subgroup H_j that satisfies Lemma 4.1 is indicated by * in Table 5. In each case, one can check that this subgroup has the property that there is a character degree a so that 9 divides $a|G : H_j|$. Since there is such a subgroup for each χ_i , the lemma is proved. $\square$

We should note that a list of subgroups of A_7 with square-free index is included in [9]. However, comparing that list with the list we obtained from Magma, we see that several groups are missing (in particular, 17, 19, 23, 26, and 30). This does not affect the argument in [9], since these subgroups can also be handled using the methods in [9].

The next lemma will be applied to the case where we have a faithful irreducible module over $3.A_7$. We would not be surprised if this result is known.

Lemma 4.3. *If V is a faithful irreducible module for $S = 3.A_7$ and v is a nonzero element of V , then 9 divides $|S : \mathbf{C}_S(v)|$.*

Proof. Let Z be the center of S . We begin with the observation that a Sylow 3-subgroup P of S is nonabelian of order 27. To see this, let $\lambda \in \text{Irr}(Z)$ be faithful. If P is abelian, then λ extends to P , and so λ extends to S by Theorem 6.26 of [10]. This contradicts $S = S'$, and thus P is not abelian.

Note that $\mathbf{C}_V(Z)$ is an S -submodule of V . Since V is irreducible and S acts faithfully, we conclude that $\mathbf{C}_V(Z) = 0$. This implies $\mathbf{C}_S(v) \cap Z = 1$ for all $v \in V \setminus \{0\}$. Let Q be a Sylow 3-subgroup of $\mathbf{C}_S(v)$. Without loss of generality, we may assume that $Q \leq P$. We have $ZQ \leq P$ and $Q \cap Z = 1$. If $|Q| = 9$, then $P = QZ$. Since Z is central, this would imply Q is normal in P , which is a contradiction since $Q \cap Z = 1$ and $Z = \mathbf{Z}(P)$. Hence we have $|Q| \leq 3$, and it follows that 9 divides $|S : \mathbf{C}_S(v)|$. $\square$

If F is any field, then there is a natural permutation module V for A_7 of dimension 7 over F . We can view V as the set $\{(a_1, \dots, a_7) \mid a_i \in F\}$, where the action by A_7 permutes the coordinates. The module V is not irreducible. In fact, the principal module is a submodule of V . Another submodule of the permutation module is the module $W = \{(a_1, \dots, a_7) \in V \mid a_1 + \dots + a_7 = 0\}$, which we call the *zero-sum diminished permutation module* for A_7 over F . When F does not have

Table 4: Lattice of Conjugacy Classes of Subgroups of A_7

j	$ A_7 : H_j $	H_j	Maximal Subgps
1	2520	1	
2	$1260 = 2^2 \cdot 3^2 \cdot 5 \cdot 7$	Z_2	1
3	$840 = 2^3 \cdot 3 \cdot 5 \cdot 7$	Z_3	1
4	$840 = 2^3 \cdot 3 \cdot 5 \cdot 7$	Z_3	1
5	$504 = 2^3 \cdot 3^2 \cdot 7$	Z_5	1
6	$360 = 2^3 \cdot 3^2 \cdot 5$	Z_7	1
7	$630 = 2 \cdot 3^2 \cdot 5 \cdot 7$	$Z_2 \times Z_2$	2
8	$630 = 2 \cdot 3^2 \cdot 5 \cdot 7$	$Z_2 \times Z_2$	2
9	$630 = 2 \cdot 3^2 \cdot 5 \cdot 7$	Z_4	2
10	$420 = 2^2 \cdot 3 \cdot 5 \cdot 7$	Z_6	2, 3
11	$420 = 2^2 \cdot 3 \cdot 5 \cdot 7$	S_3	2, 3
12	$420 = 2^2 \cdot 3 \cdot 5 \cdot 7$	S_3	2, 4
13	$280 = 2^3 \cdot 5 \cdot 7$	$Z_3 \times Z_3$	3, 4
14	$252 = 2^2 \cdot 3^2 \cdot 7$	$Z_5 \rtimes Z_2$	2, 5
15	$120 = 2^3 \cdot 3 \cdot 5$	$Z_7 \rtimes Z_3$	4, 6
16	$315 = 3^2 \cdot 5 \cdot 7$	D_8	7, 8, 9
17	$210 = 2 \cdot 3 \cdot 5 \cdot 7$	A_4	4, 7
18	$210 = 2 \cdot 3 \cdot 5 \cdot 7$	A_4	3, 7
19	$210 = 2 \cdot 3 \cdot 5 \cdot 7$	A_4	4, 7
20	$210 = 2 \cdot 3 \cdot 5 \cdot 7$	$Z_2 \times Z_2 \times Z_3$	7, 10
21	$210 = 2 \cdot 3 \cdot 5 \cdot 7$	$Z_3 \rtimes Z_4$	9, 10
22	$210 = 2 \cdot 3 \cdot 5 \cdot 7$	A_4	4, 8
23	$210 = 2 \cdot 3 \cdot 5 \cdot 7$	$S_3 \times Z_2$	8, 10, 11
24	$140 = 2^2 \cdot 5 \cdot 7$	$(Z_3 \times Z_3) \rtimes Z_2$	11, 12, 13
25	$126 = 2 \cdot 3^2 \cdot 7$	$Z_5 \rtimes Z_4$	9, 14
26	$105 = 3 \cdot 5 \cdot 7$	S_4	12, 16, 17
27	$105 = 3 \cdot 5 \cdot 7$	S_4	11, 16, 18
28	$105 = 3 \cdot 5 \cdot 7$	$(Z_2 \times Z_2 \times Z_3) \rtimes Z_2$	16, 20, 21, 23
29	$105 = 3 \cdot 5 \cdot 7$	S_4	12, 16, 22
30	$105 = 3 \cdot 5 \cdot 7$	S_4	12, 16, 19
31	$70 = 2 \cdot 5 \cdot 7$	$A_4 \times Z_3$	13, 17, 18, 19, 20
32	$70 = 2 \cdot 5 \cdot 7$	$(Z_3 \times Z_3) \rtimes Z_4$	9, 24
33	$42 = 2 \cdot 3 \cdot 7$	A_5	11, 14, 18
34	$42 = 2 \cdot 3 \cdot 7$	A_5	12, 14, 22
35	$35 = 5 \cdot 7$	$(A_4 \times Z_3) \rtimes Z_2$	24, 26, 27, 28, 30, 31
36	$21 = 3 \cdot 7$	S_5	23, 25, 27, 33
37	$15 = 3 \cdot 5$	$\text{PSL}_2(7)$	15, 26, 29
38	$15 = 3 \cdot 5$	$\text{PSL}_2(7)$	15, 29, 30
39	7	A_6	27, 29, 32, 33, 34
40	1	A_7	35, 36, 37, 38, 39

Table 5: $[\chi_i, 1_{H_j}] \neq 0$

χ_i	j
χ_2	1, 2, 3, 4, 5, 7, 8, 9, 10, 11, 12, 13, 14, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 39*
χ_3	1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 15, 17, 18, 19, 20, 21, 22*, 31
χ_4	1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 15, 17, 18, 19, 20, 21, 22*, 31
χ_5	1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 34, 35, 37, 38*
χ_6	1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 35, 36*
χ_7	1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 27, 31, 33*
χ_8	1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 19, 20, 21, 22, 23, 24, 25, 26, 28, 29, 30, 34*
χ_9	1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30*, 32

characteristic 7, the zero-sum diminished permutation module complements the principal module. We are now ready to prove the main theorem. This proof should be compared to Lemmas 2.5, 2.6, and 2.7 of [9].

Proof of Theorem D. Let N be minimal so that N is normal in G and G/N is solvable. Take M to be normal in G so that N/M is a chief factor for G . Observe that G/M is not solvable, so N/M is not abelian. By Lemma 3.1, we see that N/M is simple. Since A_7 is involved in G and 3 and 5 both divide degrees in $\text{cd}(A_7)$, we know via Theorem B that A_7 is the only simple factor involved in G , so $N/M \cong A_7$ and M is a solvable group. Write $C/M = \mathbf{C}_{G/M}(N/M)$. Observe that $C \cap N = M$ and $NC/C \cong N/M \cong A_7$. Also, G/C is isomorphic to a subgroup of $\text{Aut}(A_7)$, so $|G : CN|$ divides 2.

Let K be the intersection of the kernels of all the N -invariant linear characters in M . Observe that K is normal in N . Observe also that $[M, N]$ is contained in the kernel of every linear N -invariant character of M , so $[M, N] \leq K$. This implies M/K is central in N/K . Note that the minimality of N implies that $N = N'$. It follows that M/K is isomorphic to a subgroup of the Schur multiplier of A_7 and N/K is a central extension of A_7 . Thus, M/K has order dividing 6. If $|M : K| = 2$ or 6, then we use the Atlas [4] to see that $\text{cd}(N/K)$ contains 36, a contradiction. Hence we have that N/K is isomorphic to either A_7 or $3.A_7$.

We now work to prove that $K = 1$, so we suppose that $K > 1$. Take L to be normal in N so that K/L is a chief factor for N . Since M is solvable, we know that K/L is abelian. If $\text{Irr}(K/L)$ has a nonprincipal N -invariant character ϕ , then ϕ extends to $\mu \in \text{Irr}(M)$ since M/K is cyclic. Now, N acts on the extensions of ϕ to M , so the stabilizer of μ in N has index at most 3. Since A_7 has no proper subgroup of index at most 3, it follows that μ is N -invariant. This, however, contradicts the definition of K since K is not contained in the kernel of μ .

We conclude that 1 is the only N -invariant character in $\text{Irr}(K/L)$. We may view K/L as an

irreducible module for N/K . If $M/K > 1$ and N/K acts faithfully on K/L , then we may use Lemma 4.3 to see that 9 will divide the index of the stabilizer of any nonprincipal $\lambda \in \text{Irr}(K/L)$. Thus 9 divides some degree in $\text{cd}(G)$. We may therefore assume that M/K centralizes K/L and view K/L as an A_7 -module.

We know that K/L is an elementary abelian p -group for some prime p . We first suppose that $p \geq 11$. By Lemma 4.2, we can find $\lambda \in \text{Irr}(K/L)$ so that the stabilizer T of λ in N has a character degree $a \in \text{cd}(T/M)$ so that 9 divides $a|N : T|$. Let q be a prime and let Q/M be a Sylow q -subgroup of T/M . We know that λ extends to Q by Corollary 6.28 of [10]. Applying Corollary 11.31 of [10], we conclude that λ extends to T . By Gallagher's theorem, $a = a\lambda(1)$ is the degree of some character in $\text{Irr}(T)$ lying over λ , and by Clifford's theorem, $a|N : T| \in \text{cd}(N)$. Since 9 divides $a|N : T|$, this forces 9 to divide a degree in $\text{cd}(N)$, which is a contradiction.

We must have $p \leq 11$, which implies that p is 2, 3, 5, or 7. Consider $1 \neq \phi \in \text{Irr}(K/L)$ and let T be the stabilizer of ϕ in N . We now look at the possible subgroups of A_7 that can arise as TM/M . We first show that $\text{PSL}_2(7)$ cannot arise. To find a contradiction, we suppose that $TM/M \cong \text{PSL}_2(7)$. Let μ be an irreducible constituent of ϕ^M . Since $\text{PSL}_2(7)$ has no proper subgroup of index at most 3, we see that μ is invariant in T . Using the Atlas [4], we see that $6 \in \text{cd}(T \mid \mu)$ whether or not μ extends to T . Since $|N : T| = 15$, this implies that 90 divides some degree in $\text{cd}(N)$, a contradiction.

Next, we show that we cannot have $Z_7 \rtimes Z_3$ as TM/M . Since M/K is cyclic, ϕ extends to $\mu \in \text{Irr}(M)$. If μ is not T -invariant, then its stabilizer in T must have index 3 in T , and we conclude that 9 divides a degree in $\text{cd}(N \mid \mu)$. Thus, μ is T -invariant. All Sylow subgroups of TM/M are cyclic, so μ extends to $\text{Irr}(T)$. By Gallagher's theorem, we have $3 \in \text{cd}(T \mid \mu)$, and so 9 divides a degree in $\text{cd}(N \mid \mu)$. We conclude that TM/M cannot be $Z_7 \rtimes Z_3$.

Looking at the lattice of subgroups of A_7 , we see that these are the only subgroups of A_7 with index divisible by neither 7 nor 9. Since 9 does not divide $|N : T|$, we conclude that 7 divides $|N : T|$, and hence 7 divides the size of every orbit of the action of N/K on $\text{Irr}(K/L)$. If $|K : L| = p^n$, then 7 divides $p^n - 1$. This implies $p \neq 7$.

Using the Atlas of Modular Representations [1], we see that in characteristic 2, A_7 has irreducible modules of sizes 2^4 , 2^6 , 2^{14} , and 2^{20} (over $\mathbb{F}_2$). Of these, only $2^6 - 1$ is divisible by 7. It is not difficult to see that this is the zero-sum diminished permutation module for A_7 over $\mathbb{F}_2$. Hence, we can assume that

$$\text{Irr}(M/L) \cong \{(k_1, \dots, k_7) \mid k_i \in \mathbb{F}_2, k_1 + \dots + k_7 = 0\}.$$

If we let $\nu \in \text{Irr}(M/L)$ correspond to $(1, 1, 0, 0, 0, 0, 0)$, it is not difficult to show that the stabilizer T of ν satisfies $TM/M \cong S_5$. Using the Atlas [4], we see that $6 \in \text{cd}(T \mid \mu)$ whether or not μ extends to T . Also, $|N : T| = 21$. By Clifford's theorem, $6|N : T|$ divides a degree in $\text{cd}(G)$, so 9 divides a degree in $\text{cd}(G)$.

Again using [1], we see that in characteristic 3, A_7 has irreducible modules of sizes 3^6 , 9^{10} , 3^{13} , and 3^{15} (where q^ℓ denotes the size of a module over the field $\mathbb{F}_q$). Of these, only $3^6 - 1$ is divisible by 7. Again, it is not difficult to see that this is the zero-sum diminished permutation module for A_7 over $\mathbb{F}_3$. If we let $\nu \in \text{Irr}(M/L)$ correspond to $(1, -1, 0, 0, 0, 0, 0)$, it is not difficult to show that the stabilizer T of ν satisfies $TM/M \cong A_5$. Since the Sylow 3-subgroups of A_5 are cyclic, ν extends to A_5 , and hence $3 \in \text{cd}(T \mid \mu)$. By Clifford's theorem, we have $3|N : T| \in \text{cd}(N)$, and since $|N : T| = 42$, this yields a contradiction.

One final time, we use [1], to see that in characteristic 5, A_7 has irreducible modules of sizes 5^6 , 5^8 , 25^{10} , 5^{13} , 5^{15} , 5^{20} , and 5^{35} . Of these, only $5^6 - 1$ is divisible by 7. Again, this is the module

afforded by the zero-sum diminished permutation module for A_7 over $\mathbb{F}_5$. From here, the proof is essentially identical to the proof in the $p = 3$ case.

We conclude that $K = 1$. If $N = A_7$, then $C \cap N = 1$ and G/C is isomorphic to a subgroup of $\text{Aut}(A_7)$. We let $S = C$ and $H = NC = N \times S$, and we have that $|G : H| = 2$.

Finally, suppose $N = 3.A_7$ and let $S = \mathbf{C}_G(N)$. We know that $S \cap N$ has order 3, S is normal in G , and G/S is isomorphic to a subgroup of $\text{Aut}(3.A_7)$ which is A_7 . Thus $G = NS$. $\square$

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