

MATH 10024

Core Mathematics IV

This book is available as an e-book at
<http://www.math.kent.edu/ebooks/10024/Contents.htm>

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CHAPTER 1

Advanced GCF
Factoring Techniques;
Absolute Value
Equations and Inequalities

1.1 Greatest Common Factor

Prerequisite knowledge and skills:

- a working knowledge and understanding of the prime factorization of a composite number
- a working knowledge and understanding of the greatest common factor of two numbers
- a working knowledge and understanding of the rules of exponents
- a working knowledge and understanding of negative and fractional exponents

Terms to know:

- Factor
- Factoring
- Integers
- Greatest Common Factor
- Term
- Product

Skill Prep: Finding the GCF

I. Finding the GCF

In the prep exercise, we reviewed how to find the greatest common factor of a set of polynomial terms:

When given any number of terms with the same variable to different powers, the GCF is the term with the **lowest** exponent.

We can extend this notion of the GCF to include expressions with negative and fractional exponents. When given expressions with such exponents, we follow the same procedure as we did when dealing with positive integer exponents: we factor out the term with the lower exponent.

Section 1.1

Example 1. Find the term to be factored out: $x^{-5} + x^{-3}$.

The term to be factored out is the term with the **lower power**: x^{-5} .

Example 2. Find the term to be factored out: $x^{1/2} + x^{1/3}$

The term to be factored out

is the term with the lower power: Since $\frac{1}{3} < \frac{1}{2}$, the term to be factored out is $x^{1/3}$.



Checkpoint GCF 2

II. Factoring out the GCF

Now we look at factoring out the GCF, once we've found it.

Example 3. Write in factored form: $x^2 + x^3$.

The GCF is the term with the lower power: x^2 . So $x^2 + x^3 = x^2 \cdot (\text{other factor})$

$$\frac{x^2 + x^3}{x^2} = \frac{x^2}{x^2} + \frac{x^3}{x^2} = 1 + x$$

Divide each term by the GCF to obtain the other factor:

Multiply the two factors together: $x^2(1 + x)$

Check 1: We can use a table of values to support our answer.

Check 2: We can use graphs to support our answer.

Section 1.1

Example 4. Write in factored form: $x^{-5} + x^{-3}$.

The term with the lower power is x^{-5} .

So $x^{-5} + x^{-3} = x^{-5} \cdot (\text{other factor})$

Divide each term by the term with the lower power

to obtain the other factor:

$$\frac{x^{-5} + x^{-3}}{x^{-5}} = \frac{x^{-5}}{x^{-5}} + \frac{x^{-3}}{x^{-5}} = 1 + x^2$$

Multiply the two factors together:

$$x^{-5}(1 + x^2) \text{ or } \frac{1}{x^5}(1 + x^2)$$

Check 1: Are the tables of values equivalent?

Check 2: Do the graphical representations support our answer?

Example 5. Write in factored form: $x^{1/2} + x^{1/3}$.

The term with the lower power:

Since $\frac{1}{3} < \frac{1}{2}$, the term with the lower power is $x^{1/3}$.

$$\text{So } x^{1/2} + x^{1/3} = x^{1/3} \cdot (\text{other factor})$$

Divide all the terms by the term with the lower power to obtain the other factor:

$$\frac{x^{1/2} + x^{1/3}}{x^{1/3}} = \frac{x^{1/2}}{x^{1/3}} + \frac{x^{1/3}}{x^{1/3}} = x^{1/2-1/3} + 1 = x^{1/6} + 1$$

Multiply the two factors together:

$$x^{1/3} \left(x^{1/6} + 1 \right)$$

Check 1: Are the tables of values equivalent?

Check 2: Are the graphs equivalent?

 Checkpoint GCF 3

To factor an expression with the same variable to different powers:

- 1. Factor out the term with the lowest power;*
- 2. Divide all terms by this factor to obtain the other factor;*
- 3. Multiply the two factors together.*

[More worked examples](#)

1.1 Factoring out the GCF: Negative and Fractional Exponents: Homework Exercises

For each of the following expressions: Factor out the common term with the lesser power and write in factored form. Write your answers with positive exponents only.

1. $x^{-5} + x^{-6}$

2. $x^{-7} + x^{-8}$

3. $x^{-8} + x^{-11}$

4. $x^{-7} + x^{-15}$

5. $\frac{1}{x^5} + \frac{1}{x^6}$

6. $\frac{1}{x^7} + \frac{1}{x^8}$

7. $\frac{1}{x^4} + \frac{1}{x^3}$

8. $\frac{1}{x^5} + \frac{1}{x^7}$

9. $x^{-2} + x^{-5} - x^2$

10. $x^{-4} + x^{-7} - x^2$

11. $x^{-3} + x^{-7} - x^3$

12. $x^{-5} + x^{-7} - x^4$

13. $x^{2/3} + x$

14. $x^{4/5} + x^2$

15. $x^{2/3} - x^{3/4}$

16. $x^{4/5} - x^{5/6}$

17. $x^{2/5} - x^{3/4}$

18. $x^{4/7} - x^{5/6}$

19. $x^{-2/3} + x^{-1/3}$

20. $x^{-4/5} + x^{-3/5}$

21. $x^{-5/8} + x^{-1/2}$

22. $x^{-2/3} + x^{-1/6}$

23. $x^{-2} - x^{-5/3}$

24. $x^{-2} - x^{-7/4}$

25. $x^{-3} + x^{-7/2} - x^{-1/3}$

26. $x^{-4} + x^{-9/2} - x^{-1/5}$

27. $\frac{1}{x^{1/5}} + x^{-1/3}$

28. $\frac{1}{x^{1/4}} + x^{-1/3}$

29. $\frac{1}{x^{1/5}} + x^{-1/6}$

30. $\frac{1}{x^{1/3}} + x^{-1/4}$

31. $\frac{1}{\sqrt{x}} + \frac{1}{\sqrt[3]{x}}$

32. $\frac{1}{\sqrt{x}} + \frac{1}{\sqrt[4]{x}}$

1.2. When the GCF is a binomial.

Prerequisite knowledge and skills:

- a working knowledge and understanding of the greatest common factor of two terms in a polynomial
- a working knowledge and understanding of the rules of exponents
- a working knowledge and understanding of negative and fractional exponents

Terms to know:

- Binomial
- Factor
- Factoring
- Greatest Common Factor
- Monomial
- Term

Sometimes the GCF is a binomial.

Example 1. Factor out the GCF: $5(x-2)^3 + 2x(x-2)^2$

Think of this expression as having two big terms:

$$\underbrace{5(x-2)^3}_{\text{one term}} + \underbrace{2x(x-2)^2}_{\text{other term}}$$

The binomial $(x-2)$ is a factor in both terms. How many times is it **in common** to both?

The GCF is the binomial with the lower power: $(x-2)^2$

Divide all the terms by the GCF:

$$\begin{aligned} & \frac{5(x-2)^3}{(x-2)^2} + \frac{2x\cancel{(x-2)^2}}{\cancel{(x-2)^2}} \\ & \quad \text{Subtract exponents here} \\ & = 5(x-2) + 2x \\ & = 5x - 10 + 2x \\ & = 7x - 10 \end{aligned}$$

Multiply the two factors together:

$$(x-2)^2(7x-10)$$

Section 1.2

Check 1: Are the tables of values equivalent?

Check 2: Are the graphical representations identical?

Example 2. Factor out the GCF: $(2x - 7)(9x - 5)^2 + (2x - 7)^2(9x - 5)$

This expression, though longer, still has only two terms:

$$\underbrace{(2x - 7)(9x - 5)^2}_{\text{one term}} + \underbrace{(2x - 7)^2(9x - 5)}_{\text{other term}}$$

The binomials $(2x - 7)$ and $(9x - 5)$ are factors in both terms. How many times are they **common** to both?

The GCF consists of **each** common factor to the **lower** power:

The GCF is $(2x - 7)(9x - 5)$

Divide each term by the GCF and cancel fractions = to 1:

$$\frac{\cancel{(2x - 7)}(9x - 5)^2}{\cancel{(2x - 7)}(9x - 5)} + \frac{(2x - 7)^2 \cancel{(9x - 5)}}{(2x - 7)\cancel{(9x - 5)}}$$

↑
subtract exponents here

$$= (9x - 5) + (2x - 7)$$

$$= 11x - 12$$

Multiply the factors together: $(2x - 7)(9x - 5)(11x - 12)$

Check 1: Are the tables of values equivalent?

Check 2: Are the graphical representations identical?

✓ Checkpoint Binomial GCF 1

We can extend this idea to include binomials taken to negative or fractional powers as we did in section 1.1 with monomial common factors.

Example 3. Factor out the term with the lower power: $3(x+6)^{-6} + 4x(x+6)^{-7}$

As before, we have two terms:

$$\underbrace{3(x+6)^{-6}}_{\text{one term}} + \underbrace{4x(x+6)^{-7}}_{\text{other term}}$$

The factor $(x+6)$ is common to both. How many times is it in common?

We take out the binomial with the lower power:

$$(x+6)^{-7}$$

Divide each term by this factor:

$$\begin{aligned} & \frac{3(x+6)^{-6}}{(x+6)^{-7}} + \frac{4x(x+6)^{-7}}{(x+6)^{-7}} \\ & \text{subtract exponents here} \\ & = 3(x+6)^{-6-(-7)} + 4x \\ & = 3(x+6) + 4x \\ & = 3x + 18 + 4x \\ & = 7x + 18 \end{aligned}$$

Multiply the factors together: $(x+6)^{-7}(7x+18)$ or $\frac{1}{(x+6)^7}(7x+18)$

Check 1: Are the tables of values equivalent?

Check 2: Are the graphical representations identical? Where is the expression undefined?
Where is the x -intercept?

✓ Checkpoint Binomial GCF 2

Example 4. Factor out the term with the lowest power: $3(x+4)^{\frac{3}{2}} + 8x(x+4)^{\frac{1}{2}}$

one term
other term

The factor $(x+4)$ is common to both terms. How many times is it in common?

We take out the binomial with the lower power: $(x+4)^{\frac{1}{2}}$

Divide both terms by this factor:

$$\begin{aligned}
 & \frac{3(x+4)^{\frac{3}{2}}}{(x+4)^{\frac{1}{2}}} + \frac{8x(x+4)^{\frac{1}{2}}}{\cancel{(x+4)^{\frac{1}{2}}}} \\
 & \text{subtract exponents here} \\
 & = 3(x+4)^{\frac{3}{2} - \frac{1}{2}} + 8x \\
 & = 3(x+4) + 8x \\
 & = 11x + 12
 \end{aligned}$$

Multiply the factors together: $(x+4)^{\frac{1}{2}}(11x+12)$

Check 1: Are the tables of values equivalent?

Check 2: Are the graphical representations identical? Where is the expression undefined? Where is the x -intercept?

✓ Checkpoint Binomial GCF 3

Section 1.2

Example 5. Factor out the GCF:

$$\underbrace{8x^7(3x-8)^4}_{\text{one term}} + \underbrace{12x^8(3x-8)^3}_{\text{other term}}$$

The factors 4 , x , and $(3x-8)$ are common to both terms. How many times are they in common to both terms?

The GCF consists of each common factor to the lower power: $4x^7(3x-8)^3$

Divide each term by this GCF:

$$\begin{aligned} & \frac{8x^7(3x-8)^4}{4x^7(3x-8)^3} + \frac{12x^8(3x-8)^3}{4x^7(3x-8)^3} \\ & \quad \quad \quad \uparrow \quad \quad \quad \uparrow \\ & \quad \quad \quad \text{subtract exponents here} \\ & = 2(3x-8) + 3x \\ & = 6x - 16 + 3x \\ & = 9x - 16 \end{aligned}$$

Multiply the factors together: $4x^7(3x-8)^3(9x-16)$

The checks are left for you.

 Checkpoint Binomial GCF 4

More worked examples

1.2 Binomial Common Factors: Homework Exercises

For each of the following expressions: Factor out the GCF or any common factors including binomial factors to the lesser power. Write your answers with positive exponents only.

1. $x(2x+1)^3 - 4(2x+1)^2$

2. $x(3x-2)^4 - 5(3x-2)^5$

3. $4x(x+3)^2 - 2(x+3)$

4. $10x(x+8)^3 - 5(x+8)$

5. $x(x-4)^4(x+2)^2 + 2(x-4)^3(x+2)^3$

6. $x(2x-9)^5(x-7)^3 + 2(2x-9)^4(x-7)^6$

7. $4(x+1)^3(x-3)^2 + 2(x+1)^2(x-3)^3$

8. $6(x+4)^4(x-1)^4 + 3(x+4)^3(x-1)^5$

9. $5(x+2)^4 - 3(x+2)^5$

10. $4(x+3)^{-6} - 7(x+3)^{-5}$

11. $6(x+5)^{-7} - 11(x+5)^{-4}$

12. $9(x-11)^{-8} - 3(x-11)^{-7}$

13. $7(x+1)^{-2} + \frac{2}{x+1}$

14. $8(x+5)^{-3} + \frac{3x}{(x+5)^2}$

15. $6x^2(x-1)^{-2} + \frac{2x}{x-1}$

16. $8x^2(x+2)^{-4} + \frac{4x}{(x+2)^3}$

17. $x(x-2)^{\frac{3}{2}} + 2(x-2)^{\frac{1}{2}}$

18. $x(x-11)^{\frac{4}{3}} + 5(x-11)^{\frac{1}{3}}$

19. $x(x+4)^{\frac{1}{5}} + 3(x+4)^{\frac{1}{6}}$

20. $x(x-15)^{\frac{1}{8}} - 3(x-15)^{\frac{1}{6}}$

21. $9x^2(2x+3)^{\frac{1}{2}} + 3x(2x+3)^{\frac{1}{6}}$

22. $12x^2(3x-1)^{\frac{1}{4}} + 4x(3x-1)^{\frac{1}{8}}$

23. $15x^2(8x+3)^{\frac{1}{3}} + 3x(8x+3)^{\frac{1}{5}}$

24. $10x(4x-1)^{\frac{1}{4}} + 2x^2(4x-1)^{\frac{1}{5}}$

25. $x(x-1)^{-\frac{3}{2}} + 4(x-1)^{-\frac{1}{2}}$

26. $x(x+5)^{-\frac{4}{3}} - 3(x+5)^{-\frac{1}{3}}$

27. $x(x+6)^{-\frac{1}{5}} - 14(x+6)^{-\frac{1}{6}}$

28. $x(x-10)^{-\frac{1}{8}} + 4(x-10)^{-\frac{1}{6}}$

29. $2x(x+5)^{-\frac{3}{2}} - x(x+5)^{-2}$

30. $3x(x-4)^{-\frac{9}{4}} - x(x-4)^{-2}$

31. $4x^2(x-2)^2(x+3)^3 - 2x(x-2)^3(x+3)^2$

32. $10x^2(x-1)^4(x+2)^6 - 5x(x-1)^5(x+2)^5$

1.3 Absolute Value Equations, Inequalities, and Functions

Prerequisite knowledge and skills:

- a working knowledge and understanding of absolute value of a number

Terms to know:

- Absolute value
- Distance on number line

Skill Prep 1: Distances on the number line

Skill Prep 2: Factoring out the leading coefficient

Concept Prep: Absolute value and graphs

Introduction

Try It!

Suppose the "average" score on a test is 100 points and 68% of the population is within 15 points of that score either way. What is the range of scores of 68% of the population?

Write a mathematical expression using symbols such as $>$, $<$, \geq , \leq to represent this situation if s is the score on the test.

Write a mathematical expression using absolute value notation ($| \quad |$) to represent this scenario if s is the score on the test.

Try It!

Suppose that a number line represents Main Street in Kent with units measured in "city blocks." Suppose also that the origin represents KSU. If you are at $x = 3$ or $x = -3$, how many blocks are you away from KSU?

Write a mathematical expression using absolute value notation ($| \quad |$) to represent this scenario.

Section 1.3

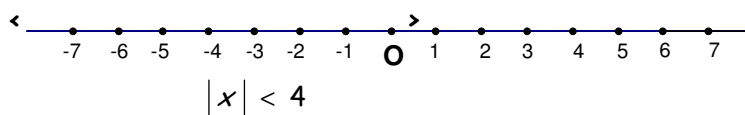
Clearly you are 3 blocks away from KSU; in both cases your distance from the origin is 3. We use absolute value to represent distance from the origin. Here $|3| = |-3| = 3$. Whether you are three blocks east or west, you still need to walk 3 blocks to get to KSU. Distance is always positive or zero, never negative.

If you were 4 blocks from KSU, your position x satisfies $|x| = 4$. What are the possible values of x ?

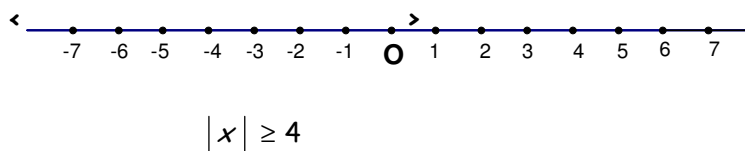
For $c > 0$,

If $|x| = c$ then $x = c$ OR $x = -c$.

Where might you be if your position, x , is described as $|x| < 4$ blocks? Sketch your solution on the number line below.



What if $|x| \geq 4$? Sketch your solution on the number line below.



Your graphs should look like [this](#).

Notice that for $|x| < 4$, the numbers fall in the interval $(-4, 4)$.

For $|x| \geq 4$ the numbers fall in either of the intervals $(-\infty, -4]$ or $[4, \infty)$.

Section 1.3

Experiment with the examples in the following checkpoint exercise, then try to generalize your answer. What patterns did you find?

 Checkpoint 1

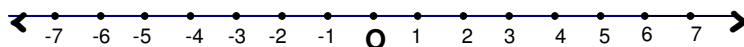
For $c > 0$:

If $|x| < c$, then $-c < x < c$.

If $|x| > c$ then $x < -c$ OR $x > c$

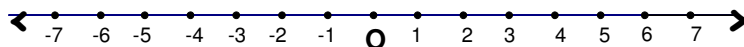
We can think of the absolute value of a number x as its distance from the origin on the number line. We write $|x|$, but can also think of it as $|x - 0|$.

Suppose you were a distance of four blocks from a point two blocks east of the origin. Where could you be? Mark your possible positions on the number line below.



Mathematicians would write $|x - 2| = 4$ to represent this situation.

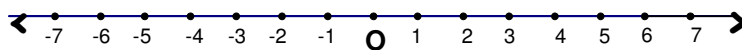
Suppose you were less than four blocks from a point two blocks east of the origin. Where could you be? Mark your possible positions on the number line below.



Mathematicians would write $|x - 2| < 4$ to represent this situation.

Section 1.3

Finally, if you were more than four blocks from a point two blocks east of the origin, where would you be? Mark your possible positions on the number line below.



Mathematicians would write $|x - 2| > 4$ to represent this situation.

Just as $|x|$ or $|x - 0|$ is the distance of x from the origin on the number line, $|x - 2|$ is the distance from the number 2 on the number line. In general, $|x - a|$ is the distance from a on the number line.

✓ Checkpoint 2

For $c > 0$ and X is some mathematical expression,

If $|X| = c$ then $X = c$ OR $X = -c$.

If $|X| < c$, then $-c < X < c$.

If $|X| > c$ then $X < -c$ OR $X > c$


We can use the above summary to help us solve equations and inequalities involving absolute values.

Example 1. Solve $|x - 5| = 8$

Write an equivalent statement: $x - 5 = 8$ or $x - 5 = -8$

Solve for x : $x - 5 + 5 = 8 + 5$ or $x - 5 + 5 = -8 + 5$

$x = 13$ or $x = -3$

Check: $|13 - 5| = |8| = 8$ or $|-3 - 5| = |-8| = 8$ 

Check by graphing

Example 2. Solve $|x - 5| < 8$

Write an equivalent statement: $-8 < x - 5 < 8$

Think: we want numbers whose distance from 5 is LESS THAN 8.

Solve for x : $-8 + 5 < x - 5 + 5 < 8 + 5$
 $-3 < x < 13$

Write as an interval: $(-3, 13)$

Check on the number line: Which numbers are less than 8 units away from 5 on the number line?

Check by graphing

Example 3 $|x - 5| > 8$

Write an equivalent statement: $x - 5 < -8$ OR $x - 5 > 8$

Think: we want numbers whose distance from 5 is GREATER THAN 8.

Solve for x : $x - 5 < -8$ OR $x - 5 > 8$
 $x < -3$ OR $x > 13$

Write using intervals: $(-\infty, -3) \cup (13, \infty)$

Section 1.3

Check on the number line: Which numbers are more than 8 units away from 5 on the number line?

Check by graphing

Example 4 $|2x - 3| > 8$

Method 1

Write an equivalent statement: $2x - 3 < -8$ OR $2x - 3 > 8$

Solve for x :

$$2x - 3 + 3 < -8 + 3 \quad \text{OR} \quad 2x - 3 + 3 > 8 + 3$$

$$2x < -5 \quad \text{OR} \quad 2x > 11$$

$$x < -\frac{5}{2} \quad \text{OR} \quad x > \frac{11}{2}$$

Write using intervals: $\left(-\infty, -\frac{5}{2}\right) \cup \left(\frac{11}{2}, \infty\right)$

Check by graphing

Method 2

If we want to think about our number line model, we could factor out the leading coefficient, like so:

$$|2x - 3| > 8 \text{ is equivalent to}$$

$$2\left|x - \frac{3}{2}\right| > 8$$

Divide through by this leading coefficient: $\left|x - \frac{3}{2}\right| > 4.$

Now we can use our number line model:

Think: we want numbers whose distance from $\frac{3}{2}$ is GREATER THAN 4. You might want to sketch a number line to help you think it through.

$$x - \frac{3}{2} < -4 \quad \text{OR} \quad x - \frac{3}{2} > 4$$

Solve for x :

$$x - \frac{3}{2} + \frac{3}{2} < -4 + \frac{3}{2} \quad \text{OR} \quad x - \frac{3}{2} + \frac{3}{2} > 4 + \frac{3}{2}$$

$$x < -\frac{5}{2} \quad \text{OR} \quad x > \frac{11}{2}$$

Write using intervals:

$$\left(-\infty, -\frac{5}{2}\right) \cup \left(\frac{11}{2}, \infty\right)$$

Which is the same answer we obtained previously.

Example 5 $|3x - 4| < -2$

If we're really on the ball here and paying close attention, we can save ourselves some work!! (That's always a good thing.)

Notice that this inequality is looking for values that make an expression in absolute value LESS THAN 0. This can never be! So we're done! There is **NO SOLUTION** to this inequality.

[More Worked Examples: Equations](#)

[More Worked Examples: Inequalities](#)

1.3 Solving Absolute Value Equations and Inequalities: Homework Problems

A. Solve each of these equations. Be sure to check your solution either algebraically or graphically.

1. $|x + 2| = 1$

2. $|3x - 5| = 12$

3. $|3x - 2| = 5$

4. $|1 - x| = 6$

5. $|x + 6| = 5$

6. $|20x - 7| = 6$

7. $|8x + 16| = 32$

8. $|2x + 5| = 3$

9. $|6x - 13| = 5$

10. $|-x - 6| = 7$

11. $|x - 5| - 4 = 3$

12. $|x + 1| + 2 = 7$

13. $|5x + 3| + 7 = 15$

14. $|4x - 2| - 3 = 6$

15. $3 + |x + 1| = 5$

16. $7 + |x - 2| = 11$

17. $8 - |2x - 1| = 6$

18. $4 - |3x + 6| = 1$

19. $20 + |5x - 4| = 15$

20. $9 + |4x + 3| = 1$

21. $\left|\frac{1}{2}x + \frac{1}{3}\right| + 3 = 5$

22. $\left|\frac{2}{5}x - \frac{1}{2}\right| - 4 = 2$

23. $\left|\frac{1}{3}x - 5\right| - 4 = \frac{2}{3}$

24. $\left|\frac{3}{5}x + 2\right| - \frac{1}{2} = 4$

25. $3|x - 7| + 5 = 14$

26. $2|x + 5| + 6 = 20$

27. $4|x + 3| - 2 = 6$

28. $3|x - 2| - 7 = 5$

29. $6 - 2|3x + 1| = 4$

30. $7 - 5|2x - 4| = -3$

31. $8 + 2|x - 5| = 2$

32. $6 + 3|4x - 1| = 3$

33. $|x^2 + 3x + 1| = 1$

34. $|x^2 - 5x + 2| = 2$

35. $|x^2 - 5x - 7| = 7$

36. $|x^2 + 2x - 4| = 4$

B. Solve each of these inequalities and write your answer using interval notation. Be sure to check your answer either algebraically or graphically.

1. $|x + 1| > 1$

2. $|-5x - 12| > 3$

3. $|2x - 5| < 3$

4. $|x - 6| > 1$

5. $|x - 6| \geq 20$

6. $|6x + 5| < 20$

7. $|5x - 3| > 2$

8. $|6x + 7| \leq 10$

9. $|16x - 32| \leq 8$

10. $|13x - 5| \geq 6$

11. $|x - 5| - 4 > 3$

12. $|x + 1| + 2 < 7$

13. $|5x + 3| + 7 \leq 15$

14. $|4x - 2| - 3 \geq 6$

15. $3 + |x + 1| \geq 5$

16. $7 + |x - 2| \leq 11$

17. $8 - |2x + 4| < 6$

18. $5 - |2x + 4| > 1$

19. $4|x + 2| - 3 \geq 13$

20. $3|x - 2| - 7 \leq 5$

21. $6 - 2|3x + 1| > 4$

22. $7 - 5|2x - 4| < -3$

23. $|x^2 - x - 3| \leq 3$

24. $|x^2 + 3x + 1| \geq 1$

1.4 Radical Equations and Functions

Prerequisite knowledge and skills:

- a working knowledge and understanding of simplifying radicals

Terms to know:

- Cube root
- Equivalent equations
- Extraneous solution
- Index
- Perfect cubes
- Perfect squares
- Quadratic equation
- Radical
- Radical sign
- Radicand
- Rational number
- Square root
- Term

Skill prep: Solving quadratic equations; Radicals to rational exponents; Rational exponent to radicals

Concept prep: Solving simple radical equations

*An equation in which variables appear in one or more radicands is called a **radical equation**.*

The following are examples of radical equations.

$$\sqrt[3]{2x-1} = 3$$

$$\sqrt{x+5} = \sqrt{3x+2}$$

The equation

$$x\sqrt{3} + 2 = \frac{x}{\sqrt{3}}$$

is NOT a radical equation, even though it contains radicals because the variable is NOT under the radical sign.

How did you solve #3a - 3j in the concept prep exercises ?

One way is to raise the entire equation (both sides) to an exponent, thus eliminating the radical sign. Remember, exponents and radicals are inverse operations, meaning one undoes the other. To eliminate a square root, square the equation; to eliminate a cube root, cube the equation, etc.

[More detail](#)

To solve a radical equation we use the following property of powers:

For any positive integer n ,
If $a = b$ is true, then $a^n = b^n$ is also true.

Be sure to isolate the radical term first.

When the equation is raised to an exponent, each **SIDE** must be raised to the exponent **NOT** each term individually.

[Common Errors](#)

Example 1. Solve $\sqrt{x+1} = 2$

Square both sides:

$$(\sqrt{x+1})^2 = 2^2$$

Solve for x :

$$\begin{aligned} x+1 &= 4 \\ x &= 3 \end{aligned}$$

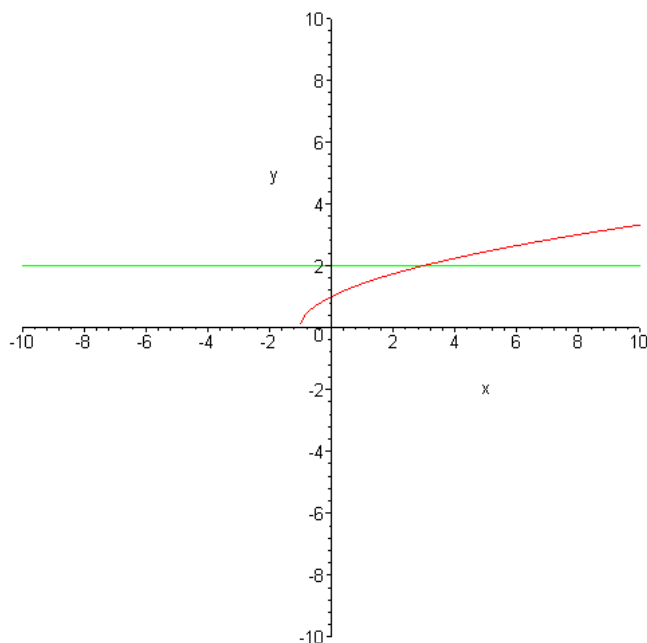
Check:

$$\sqrt{3+1} = \sqrt{4} = 2$$



Check by graphing:

Note that if we use our graphing calculator and set $Y1 = \sqrt{x+1}$ and $Y2 = 2$, we obtain a graph like the following: (The graph below was created with a software product called MAPLE. Your calculator graph may not have as good resolution.)



Section 1.4

Notice that the point of intersection occurs at $x = 3$. What does this tell us about the solution?

Example 2. Solve $3 + \sqrt{x-2} = 9$

Isolate the radical term:

$$3 + \sqrt{x-2} - 3 = 9 - 3$$

$$\sqrt{x-2} = 6$$

Square both sides:

$$(\sqrt{x-2})^2 = 6^2$$

$$x - 2 = 36$$

Solve for x :

$$x = 38$$

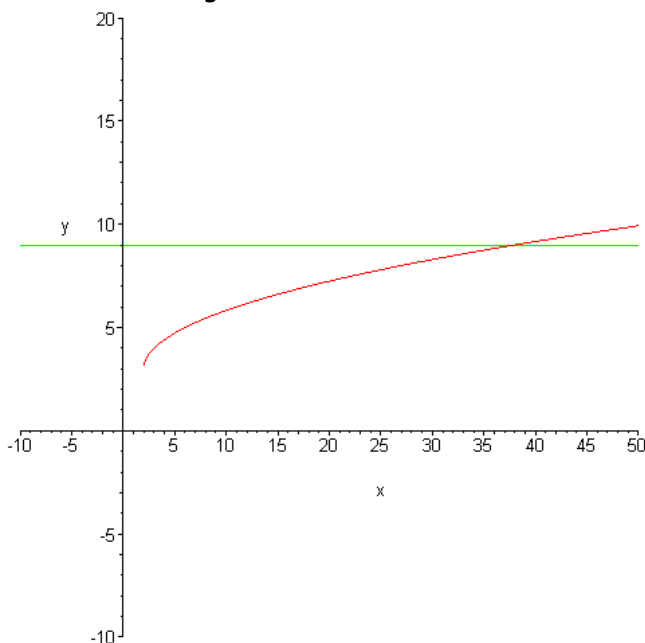
Check:

$$3 + \sqrt{38-2} = 3 + \sqrt{36} = 3 + 6 = 9$$



Check by graphing:

On your graphing calculator, set $Y1 = 3 + \sqrt{x-2}$ and $Y2 = 9$ and your viewing window to be $[-10, 50]$ by $[-10, 20]$ to obtain a graph like the following.



Note that the point of intersection has an x -coordinate of 38. What does this tell us about our algebraic solution?

[More detail](#)

SOLVING RADICAL EQUATIONS CONTAINING TWO RADICALS

The following example has two radicals and is thus more involved. The basic procedure is the same, though we need to square both sides of the equation twice.

We still need to isolate (either) one of the radicals first.

Example 3. Solve $\sqrt{4m+5} + \sqrt{m+5} = 3$

Isolate one radical term:

$$\sqrt{4m+5} = 3 - \sqrt{m+5}$$

Square both sides:

$$(\sqrt{4m+5})^2 = (3 - \sqrt{m+5})^2$$

Notice that the right side is a binomial, so we need to be careful to square it correctly, using binomial multiplication, or FOIL:

$$\begin{aligned} (\sqrt{4m+5})^2 &= (3 - \sqrt{m+5})^2 \\ &= (3 - \sqrt{m+5})(3 - \sqrt{m+5}) \\ &= 9 - 3\sqrt{m+5} - 3\sqrt{m+5} + (\sqrt{m+5})^2 \end{aligned}$$

Combine like terms here

$$\text{So, } 4m+5 = 9 - 6\sqrt{m+5} + m+5$$

Combine like terms:

$$4m+5 = 14 - 6\sqrt{m+5} + m$$

Now we're going to repeat the entire procedure by isolating the radical, then squaring both sides again.

Isolate the radical that's left by

subtracting 14 and m from both sides :

$$4m+5 - 14 - m = 14 - 6\sqrt{m+5} + m - 14 - m$$

$$3m - 9 = -6\sqrt{m+5}$$

Square both sides again:

$$(3m - 9)^2 = (-6\sqrt{m+5})^2$$

↑

We don't need to FOIL here, just use rules of exponents.

Section 1.4

$$(3m - 9)^2 = (-6)^2 \sqrt{m + 5}^2$$

$$9m^2 - 54m + 81 = 36(m + 5)$$

$$9m^2 - 54m + 81 = 36m + 180$$

Get one side = 0:

$$9m^2 - 54m - 36m + 81 - 180 = 0$$

$$9m^2 - 90m - 99 = 0$$

To make factoring easier, we divide by 9:

$$m^2 - 10m - 11 = 0$$

Factor:

$$(m - 11)(m + 1) = 0$$

Set each factor to 0 and solve:

$$m - 11 = 0 \text{ or } m + 1 = 0$$

$$m = 11 \text{ or } m = -1$$

Check in the original equation:

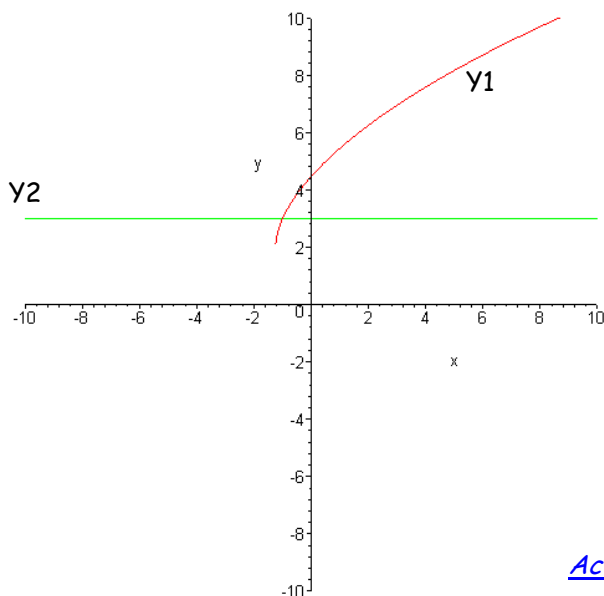
For $m = 11$: $\sqrt{4(11) + 5} + \sqrt{11 + 5} = \sqrt{49} + \sqrt{16} = 7 + 4 = 11 \neq 3$ X

So $m = 11$ is NOT a solution. This happened because we squared both sides of the equation.

For $m = -1$: $\sqrt{4(-1) + 5} + \sqrt{-1 + 5} = \sqrt{1} + \sqrt{4} = 1 + 2 = 3$ ✓

So $m = -1$ is the only solution.

Note that the following graph with $Y1 = \sqrt{4x + 5} + \sqrt{x + 5}$ and $Y2 = 3$ supports our algebraic solution:



[More Worked Examples](#)

[Activity with graphs](#)

1.4 Solving Radical Equations - Homework Exercises

Solve each of these

1. $\sqrt{x-2} = 5$

2. $\sqrt{x+5} = 2$

3. $\sqrt{x} - 2 = 5$

4. $\sqrt{x} - 7 = 6$

5. $\sqrt{x} + 6 = 2$

6. $\sqrt{x} + 9 = 8$

7. $\sqrt{2x-4} + 5 = 3$

8. $\sqrt{3x+2} + 11 = 4$

9. $\sqrt{9x^2+4} = 3x+2$

10. $\sqrt{25x^2+16} = 5x-4$

11. $\sqrt{2x^2+6x+9} = 3$

12. $\sqrt{3x^2+6x+4} = 2$

13. $\sqrt{9x^2-2x+8} = 3x$

14. $\sqrt{4x^2+3x-2} = 2x$

15. $\sqrt{7x-4} + 2 = 5$

16. $\sqrt{6x+5} - 3 = 2$

17. $3\sqrt{x-5} = 2$

18. $4\sqrt{x-1} = 3$

19. $\sqrt{2x+3} = x$

20. $\sqrt{10x-21} = x$

21. $2\sqrt{x+8} = 3\sqrt{x-2}$

22. $5\sqrt{x-3} = 2\sqrt{x+1}$

23. $\sqrt{x-3} + \sqrt{x} = 3$

24. $\sqrt{x-15} + \sqrt{x} = 5$

25. $\sqrt{2x+6} - \sqrt{x+4} = 1$

26. $\sqrt{3x+7} - \sqrt{x+3} = 1$

27. $x + 6\sqrt{x} + 5 = 0$

28. $x - 5\sqrt{x} + 6 = 0$

29. $4x = \sqrt{4-6x} + 1$

30. $3x = \sqrt{3-x} - 1$

31. $x = \sqrt{5x-1} - 1$

32. $x = \sqrt{3x+12} + 2$

33. $\sqrt{2x+5} - 1 = x$

34. $\sqrt{3x+7} - 3 = x$

35. $\sqrt{1-8x} - x = 4$

36. $\sqrt{4x-7} - x = -1$

37. $\sqrt{2x+1}+7=x$

38. $\sqrt{x+3}+3=x$

39. $x-3=\sqrt{30-2x}$

40. $x+5=\sqrt{20-3x}$

ANSWERS TO ODDS

DETAILED SOLUTIONS TO ODDS

CHAPTER 2

Exponential Functions

2.1 Exponential Growth: Compound Interest

Goals:

- Understand the nature of exponential growth
- Write exponential model given initial value and growth factor or growth rate
- Understand the difference between growth factor and growth rate.
- Recognize data that has either a linear or exponential growth pattern

Terms to know:

- compound interest
- effective annual yield
- interest
- nominal interest rate
- simple interest

Concept Prep: Real world linear vs. exponential

Dynagraph Lab Activity

Let's take a closer look at the first scenario in the concept prep exercises.

a) After the first year, your new salary

$$\begin{aligned}
 &= \text{Original salary} + 3\% \text{ of original salary} \\
 &= 40,000 + .03(40,000) \\
 &= 40,000 + 1200 \\
 &= 41,200
 \end{aligned}$$

Note that your raise would be \$1200.

After the second year, your new salary would be

$$\begin{aligned}
 &= \text{Former salary} + 3\% \text{ of former salary} \\
 &= 41,200 + .03(41,200) \\
 &= 41,200 + 1236 \\
 &= 42,436
 \end{aligned}$$

Note that your raise this time would be \$1236.

Section 2.1

After the third year, your new salary would be

$$\begin{aligned}
 &= \text{Former salary} + 3\% \text{ of former salary} \\
 &= 42,436 + .03(42,436) \\
 &= 42,436 + 1273.08 \\
 &= \mathbf{43,709.08}
 \end{aligned}$$

Note that your raise this time would be \$1273.08.

Notice also that because we are taking 3% of a larger number each successive year, the size of the raise was greater than that of the previous year. A very different situation occurs when the raise is the **same** each year, as in the situation below.

b) With a starting salary of \$40,000 and a \$1200 raise each year, at the end of the third year your salary would be

$$\begin{aligned}
 &= \text{Original salary} + 3 \times 1200 \\
 &= 40,000 + 3600 \\
 &= \mathbf{43,600}
 \end{aligned}$$

* * *

Year	Salary with 3% annual raise	Year	Salary with \$1200 annual raise
0	\$40,000	0	\$40,000
1	\$41,200	1	\$41,200
2	\$42,436	2	\$42,400
3	\$43,709	3	\$43,600
4	\$45,020	4	\$44,800
5	\$46,371	5	\$46,000

TABLE 1. A Comparison of salaries

Section 2.1

Suppose we wanted to generalize the first scenario above, i.e. we are earning 3% annual raises.

After the **first** year, your salary is $40,000 + .03(40,000)$

Factor out the GCF, 40,000: $40,000(1 + .03) =$
 $40,000(1.03)$

After the **second** year, your salary is $40,000(1.03) + .03(40,000)(1.03)$

Factor out the GCF, $40,000(1.03)$: $40,000(1.03)[1 + .03] =$
 $40,000(1.03)^2$

After the **third** year, your salary is $40,000(1.03)^2 + .03(40,000)(1.03)^2$

Factor out the GCF, $40,000(1.03)^2$: $40,000(1.03)^2[1 + .03] =$
 $40,000(1.03)^3$

Are you noticing a pattern? What would the salary be after the 10th year?

If we let t = number of years since the beginning of the contract, and continue in the above fashion, we find that the amount of money, A , depends upon t :

$$\text{Amount}(t) = 40000(1.03)^t$$

Notice that the variable, t , is in the exponent.

A function of the form $A(t) = Ca^t$ where $a > 0$ and $a \neq 1$ is an *exponential function*.

The number C gives the initial value of the function (when $t = 0$) and the number a is the growth (or decay) factor.

Note that if the growth **rate** is r , the growth **factor** is $1 + r$.

Example 1. Write a formula for an exponential function with initial value of 3,000 and a growth factor of 1.06.

SOLUTION. The general exponential function is of the form $A(t) = Ca^t$, with C as the initial value and a as the growth factor, so the function is given by

$$A(t) = 3,000(1.06)^t$$

↑
↑
initial amount
growth factor

Example 2. Write a formula for an exponential function with initial value of 10 and growing 3.5% every time period.

SOLUTION. Since the initial value is growing at a *rate* of 3.5%, the growth *factor* is $1 + .035$ or 1.035. The function is given by

$$A(t) = 10(1.035)^t$$

↑
↑
initial amount
growth factor

Example 3. How much money would be in an account after 10 years if you deposited \$2500 in a mutual fund which compounds interests 4% annually?

Section 2.1

SOLUTION. Since the initial value is growing at a rate of 4%, the growth *factor* is

$1 + .04$ or 1.04 . The function is given by

$$A(t) = 2500(1.04)^t.$$

After 10 years, we would have

$$\begin{aligned} & 2500(1.04)^{10} \\ & \quad \quad \quad \uparrow \\ & \quad \quad \quad t = 10 \\ & \approx \$3700.61 \end{aligned}$$

Not bad! Notice that you **added nothing** to the initial amount. The increase is due solely to interest earned on the original amount.

Example 4. Okay, suppose now you plan to stash away this \$2500 for 40 years (i.e. you're saving for retirement already - smart!). Because you don't need the money, you can invest in a riskier stock and it has been earning 12% per year, compounded once every year. Assuming that it continues to earn 12% compounded annually, how much money would you have at the end of 40 years?

SOLUTION. You're not going to believe this. Take a guess at how much you would have, before reading on.

$$2500(1.12)^{40} = 232,627.43 \quad (!!!)$$

When figuring compound interest, we use an exponential function, usually written:

$$A(t) = P(1 + r)^t$$

Where r is the interest rate, P is the principle, and t is the time in years. Note that this formula is just a slight adaptation of the general exponential formula.

✓ Checkpoint Exponential Functions 1

More on Compound Interest

In practice, most banks, savings and loans, stocks don't figure compound interest annually. Find an advertisement either in the newspaper or online and report to the class at least 3 different compounding periods that you found.

Some investment firms compound interest *quarterly*. Let's take that \$2500 again and invest it in a CD that advertises 12% interest compounded quarterly.

Think about it. The bank is not going to give you 12% four times a year, because the annual rate would then be much higher than 12%. They will take that 12% and divide it by 4 and give you that rate 4 times a year.

So, our formula now becomes:

$$2500 \left(1 + \frac{.12}{4} \right)^{22}$$

↑
12% interest divided by 4

How many times will interest be compounded ?

If compounding occurs 4 times a year, then after 1 year, there would be 4 compoundings, after 2 years there would be $4 \times 2 = 8$ compoundings after 3 years $4 \times 3 = 12$, etc. In general, there would be $4t$ compoundings, where t is the number of years of the investment.

After 40 years, then, our \$2500 would earn

$$2500 \left(1 + \frac{.12}{4} \right)^{(4 \times 40)}$$

↑
4 compoundings per year for 40 years

$$= 2500(1 + .03)^{160}$$

$$\approx 283,071.38$$

about \$50,000 more than the final amount with annual compounding!

Section 2.1

It's quite interesting to note the effect of different interest rates on the amount earned.

Example 5. Let's take that \$2500 and figure the amount you would have after 40 years at each of the following interest rates. Write each answer as a dollar amount rounded to the nearest cent.

- | | |
|--------|-------|
| a) 10% | b) 8% |
| c) 6% | d) 4% |

SOLUTION

$$\text{a) } 2500 \left(1 + \frac{.10}{4} \right)^{(4 \times 40)} \approx \$129,944.67$$

$$\text{b) } 2500 \left(1 + \frac{.08}{4} \right)^{(4 \times 40)} \approx \$ 59,424.77$$

$$\text{c) } 2500 \left(1 + \frac{.06}{4} \right)^{(4 \times 40)} \approx \$ 27,071.15$$

$$\text{d) } 2500 \left(1 + \frac{.04}{4} \right)^{(4 \times 40)} \approx \$ 12,284.57$$

Example 6. Now, keep the interest rate steady at 8% and figure how much you would have if interest were compounded:

- annually
- semi-annually
- quarterly
- monthly
- daily

SOLUTION

$$\text{a) } 2500 \left(1 + \frac{.08}{1} \right)^{(1 \times 40)} = 2500(1.08)^{(40)} \approx \$ 54,311.30$$

$$\text{b) } 2500 \left(1 + \frac{.08}{2} \right)^{(2 \times 40)} = 2500(1.04)^{(80)} \approx \$ 57,624.50$$

$$\text{c) } 2500 \left(1 + \frac{.08}{4} \right)^{(4 \times 40)} = 2500(1.02)^{(160)} \approx \$ 59,424.77$$

$$d) 2500 \left(1 + \frac{.08}{12}\right)^{(12 \times 40)} = 2500 \left(1 + \frac{.08}{12}\right)^{(480)} \approx \$60,683.46$$

$$e) 2500 \left(1 + \frac{.08}{365}\right)^{(365 \times 40)} = 2500 \left(1 + \frac{.08}{365}\right)^{(14600)} \approx \$61,309.82$$

MORAL OF THE STORY: **SAVE YOUR MONEY!!!**

When figuring compound interest that is compounded more than once a year, we use the function given by:

$$A = P \left(1 + \frac{r}{n}\right)^{nt} \text{ where}$$

P is the principle or present value
 r is the advertised interest rate,
 n is the number of compoundings per year, and
 t is the time in years.

Checkpoint Exponential Functions 2

Effective vs Nominal Interest Rate

Let's take another look at the situation where \$2500 is invested at 8% interest compounded quarterly. At the end of 1 year we would have

$$2500 \left(1 + \frac{.08}{4}\right)^{(4)},$$

which simplifies to

$$2500(1.02)^{(4)} = 2500(1.0824).$$

Notice that if we take $\left(1 + \frac{.08}{4}\right)^{(4)}$ we get 1.0824, which is equivalent to having 8.24% interest figured only once. So when we take 8% compounded quarterly, we are actually earning 8.24% a year! This higher rate is called the *effective annual rate* or effective annual yield in the business world. The original or advertised rate of 8% is called the nominal rate, because it is the rate "in name" only.

Example 7. Find the effective annual yield for an account that gives 5.5% nominal interest compounded monthly.

SOLUTION.

Use the compound interest formula for $t = 1$ and $P = 1$: $\left(1 + \frac{.055}{12}\right)^{(12)}$

Use your calculator to round the answer to 4 decimal places: ≈ 1.0564

Subtract 1 and change the decimal to a percent: 5.64%

The effective annual yield is 5.64 %

[More worked examples](#)

2.1 Exponential Growth: Compound Interest Homework

Solve each of these

- Write a formula for an exponential function with initial value of 2,500 and a growth factor of
 - 1.25
 - 1.05
 - 1.18

Section 2.1

2. Write a formula for an exponential function with initial value of 4,000 and a growth factor of
 - a) 2.5
 - b) 3.05
 - c) 4.1
3. Write a formula for an exponential function with initial value of 100 and growth rate (every time period) of
 - a) 2.5%
 - b) 3.1%
 - c) 100%
4. Write a formula for an exponential function with initial value of 5,000 and growth rate (every time period) of
 - a) 5%
 - b) 8%
 - c) 200%
5. How much money would be in an account after 20 years if you deposited \$15,000 at each of the following interest rates compounded annually?
 - a) 1.2%
 - b) 28%
 - c) 6.56%
 - d) 4.75%
6. How much money would be in an account after 15 years if you deposited \$2,500 at each of the following interest rates compounded quarterly?
 - a) 2.1%
 - b) 8.2%
 - c) 6.5%
 - d) 5.74%
7. How much money would be in an account after 20 years if you deposited \$3,000 in a mutual fund which compounds interests 2.2%
 - a) annually
 - b) semi-annually
 - c) quarterly
 - d) monthly
 - e) daily
8. How much money would be in an account after 2 and a half years if you deposited \$25,000 in a mutual fund which compounds interests 1.1%
 - a) annually
 - b) semi-annually
 - c) quarterly
 - d) monthly
 - e) daily

Section 2.1

9. Zach is beginning high school and has \$5000 in his savings account. How much more does he need to add to the \$5000 so that a CD paying 5% compounded monthly returns \$10,000 when he finishes college in 8 years?
10. Find the effective annual yield for an account that gives 2.75% nominal interest compounded semi-annually.
11. Find the effective annual yield for an account that gives 9% nominal interest compounded daily.
12. Suppose you deposit \$1000 into an account that compounds interest 4% every 6 months. Write a formula for an exponential function representing this scenario if
 - a) t represents time in years of the investment.
 - b) t represents the number of 6 month periods of time of the investment.
13. The black squirrel population in Kent has increased from 1000 to 3000 in the last 10 years and is growing exponentially. What is the yearly rate of increase?
14. Suppose you deposit \$5000 into an account that compounds interest 6% every year. Write a formula for an exponential function representing this scenario if
 - a) t represents time in years of the investment.
 - b) t represents the number of 6 month periods of time of the investment.

2.2 Exponential Decay

Goals:

- Understand the nature of exponential decay
- Write exponential model given initial value and decay factor or decay rate
- Understand the difference between decay factor and decay rate

Terms to know:

- decay
- decay factor
- decay rate
- exponential function

Example 1

Suppose now we were considering the population of a certain community. Suppose also that 120,000 people lived there in 2005 and that 2% of the population leave every year. How many people would be living there in 2008?

After the **first year**, the population

$$\begin{aligned}
 &= \text{Original population} - 2\% \text{ of original population} \\
 &= 120,000 - .02(120,000) \\
 &= 120,000 - 2400 \\
 &= 117,600.
 \end{aligned}$$

Note that 2400 people left town during the first year.

After the **second year**, the population

$$\begin{aligned}
 &= \text{Former population} - 2\% \text{ of former population} \\
 &= 117,600 - .02(117,600) \\
 &= 117,600 - 2352 \\
 &= 115,248.
 \end{aligned}$$

Note that 2352 people left town during the second year, which is less than the number that left during the first year.

Section 2.2

After the **third year**, the population

$$\begin{aligned}
 &= \text{Former population} - 2\% \text{ of former population} \\
 &= 115,248 - .02(115,248) \\
 &= 115,248 - 2304.96 \\
 &\approx 112,943 \text{ (rounding to the nearest person)}
 \end{aligned}$$

Note that 2305 people left town during the third year, which is less than the number that left during the second year.

Example 2

Now, to generalize as we did for the salary example,

After the **first year**, the population is $120,000 - .02(120,000)$

Factor out the GCF, 120,000:

$$\begin{aligned}
 &120,000(1 - .02) = \\
 &120,000(.98)
 \end{aligned}$$

After the **second year**, the population is $120,000(.98) - .02(120,000(.98))$

Factor out the GCF, 120,000(.98):

$$\begin{aligned}
 &120,000(.98)[1 - .02] = \\
 &120,000(.98)^2
 \end{aligned}$$

After the **third year**, the population is $120,000(.98)^2 - .02(120,000(.98)^2)$

Factor out the GCF, 120,000(.98)²:

$$\begin{aligned}
 &120,000(.98)^2 [1 - .02] = \\
 &120,000(.98)^3
 \end{aligned}$$

Section 2.2

If we let n = number of years since 2005, and continue in the above fashion, we find that the population, P , depends upon n :

$$\text{Population}(n) = 120,000(.98)^n$$

↑

Notice that since the population is decreasing the factor, or base of the exponential, is less than one.

A function of the form $A(n) = Ca^n$ where $a > 0$ and $a \neq 1$ is an **exponential function**.

The number C gives the initial value of the function (when $n = 0$) and the number a is the growth (or decay) factor.

If $a > 1$, the function represents *growth*;

If $0 < a < 1$, the function represents *decay*.

Note that if the decay **rate** is r , the decay **factor** is $1 - r$



Checkpoint Exponential Decay 1

Example 3. Identify each of the following as a growth or decay exponential function. Identify the growth or decay factor, the growth or decay rate, and the initial value.

a) $y = 100(1.03)^n$

d) $y = (0.4)^n$

b) $y = 20\left(\frac{3}{4}\right)^n$

e) $y = 10(5)^{-n}$

c) $y = 0.1(3)^n$

f) $y = -2(5)^n$

SOLUTION

$$\text{a) } y = 100(1.03)^n$$

↑
factor

This function represents **growth** since the factor is greater than 1.

The **growth factor** is 1.03. The **growth rate** is 3%.

The **initial value** is 100.

$$\text{b) } y = 20\left(\frac{3}{4}\right)^n$$

↑
factor

This function represents **decay** since the factor is less than 1.

The **decay factor** is $\frac{3}{4}$. The **decay rate** is 25%.

The **initial value** is 20

$$\text{c) } y = 0.1(3)^n$$

↑
factor

This function represents **growth** since the factor is greater than 1.

The **growth factor** is 3. The **growth rate** is 200%.

The **initial value** is 0.1

$$\text{d) } y = (0.4)^n$$

↑
factor

This function represents **decay** since the factor is less than 1.

The **decay factor** is 0.4. The **decay rate** is 60%.

The **initial value** is 1.

Section 2.2

e) $y = 10(5)^{-n}$

We can rewrite the expression on the right using the definition of negative

exponents: $y = 10(5)^{-n} = 10\left(\frac{1}{5}\right)^n$

↑
factor

This function represents **decay** since the factor is less than 1.

The **decay factor** is $\frac{1}{5}$. The **decay rate** is 80%.

The **initial value** is 10.

f) $y = -2(5)^n$

This one is tricky. The factor is bigger than one, so one would assume that this function represents growth. The negative sign out front, however, means that all values are getting bigger in the negative direction. We say this function represents **negative growth** with a **factor** of 5.

The **rate** is 400%.

The initial value is -2



Checkpoint Exponential Decay 2

2.2 Exponential Decay: Homework Exercises

Solve each of these.

1. Identify each of the following as a growth or decay exponential function. Identify the growth or decay factor, the growth or decay rate, and the initial value.

a) $A(n) = 50\left(\frac{1}{4}\right)^{-n}$

b) $A(n) = 75(0.06)^n$

c) $A(n) = -10(2.5)^n$

d) $A(n) = 40(1.1)^n$

Section 2.2

2. Write a formula for an exponential function with initial value of 4,000 and a decay factor of
 - a) 0.75
 - b) 0.80
 - c) 0.90

3. Write a formula for an exponential function with initial value of 100 and decay rate of
 - a) 2.5%
 - b) 10%
 - c) 33%

4. Suppose we were considering the population of a certain community. Suppose also that 150,000 people lived there in 2000 and that 3% of the population leave every year. How many people would be living there in 2007?

5. Suppose we were considering the population of a certain community. Suppose also that 500,000 people lived there in 2002 and that 7% of the population leave every year. How many people would be living there in 2008?

6.
 - a) In Example 5 in the "More Worked Examples" of this section, the world oil reserve was assumed to have a rate of 3% per year. If we assume that the rate of depletion rises slightly to 5%, how much oil will be left in 50 years?
 - b) At the depletion rate of 5%, how much oil will be left in 75 years?

7. The population of Duel is declining exponentially. If there were 500 residents 4 years ago and half as many now, what is the yearly rate of decrease?

8. River City has been losing 2% of its population yearly over the last 15 years. What was the population 10 years ago if the current population of River City is 50,000?

2.3 Graphs of Exponential Functions

Goals: The student will

- Graph by hand exponential functions (growth and decay) with different bases
- Be able to explain why exponential functions have a common vertical intercept
- Be able to discuss the properties of graphs of exponential functions, including domain, range, asymptote
- Understand the effect of changing the growth factor on the graph of an exponential function.
- Understand the effect of changing the initial value on the graph of an exponential function

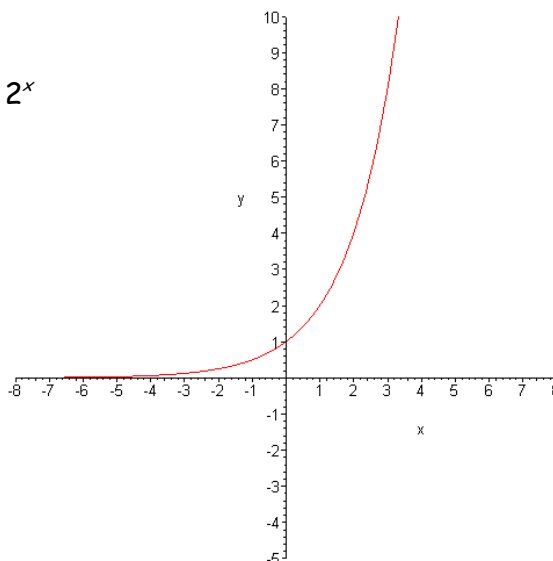
Terms to know:

- Concave down
- Concave up
- Domain
- Exponential function
- Growth (or decay) factor
- Growth (or decay) rate
- Horizontal asymptote
- Linear function
- Range
- Slope

Concept Prep: Graphs of Exponential Functions

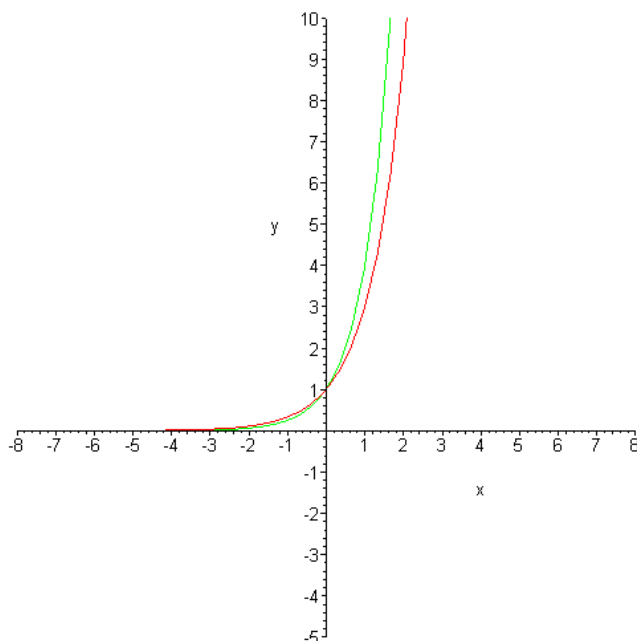
In this section we will look at the graphs of exponential functions. Graphs of all exponential functions are similar in some ways, yet different in others. The goal of this section is understand these similarities and differences.

In the concept prep assignment, your graph of $y = 2^x$ should look something like this:



The Effect of the Factor, a

- I. When the factor, a is greater than 1, we have an exponential growth function. Your graphs of $y = 3^x$ and $y = 4^x$ should look like this:



Note the following characteristics of graphs of functions of the form $y = a^x$ where $a > 1$:

- a) They have a common y-intercept which is 1. This means all these graphs go through the point (0,1).

Please understand WHY this is true. Talk it over with your colleague. (HINT: Substitute 0 in for x in the formula.)

- b) Any real number can be the input.

We call the set of all possible inputs the domain of a function. The domain of these exponential functions is, therefore, $(-\infty, \infty)$.

- c) The output is never negative, nor is it 0.

We call the set of all outputs of a function the range. The range, then, of these exponential functions is $(0, \infty)$.

Why is the output never negative? Will this always be the case?

Section 2.3

- d) Though the output is never negative or 0, as $|x|$ gets bigger and bigger, but x is NEGATIVE (we write $x \rightarrow -\infty$), the value of $y = a^x$ gets very, very close to 0. Take a look at this table of values for $y = 2^x$ to convince yourself:

x	2^x
0	1
-1	0.5
-2	0.25
-3	0.125
-4	0.0625
-5	0.03125
-10	0.000976563
-20	9.5367E-07

x	2^x
-30	9.3132E-10
-40	9.0949E-13
-50	8.8818E-16
-60	8.6736E-19
-70	8.4703E-22
-80	8.2718E-25
-90	8.0779E-28
-100	7.8886E-31

Note that $2^{-20} = 9.5367E-07$, which is scientific notation for 0.0000009536, a mighty small number. Symbolically, mathematicians would write:

$$\text{As } x \rightarrow -\infty, y \rightarrow 0$$

Since y approaches 0 as x gets large in the negative direction, we say the line $y = 0$ is a [horizontal asymptote](#) for the graph of the function.

- e) The output values are increasing over the entire domain. This means that as the x 's get larger, as we move from left to right on the horizontal axis, the y 's also get larger.

II. When the [factor](#), a , is less than 1 but greater than 0, we have an [exponential decay function](#).

Suppose that we have a scenario where after each time period, we have $\frac{1}{2}$ of the previous value left. If the initial amount was 1, the following table shows the output values for various values of the input.

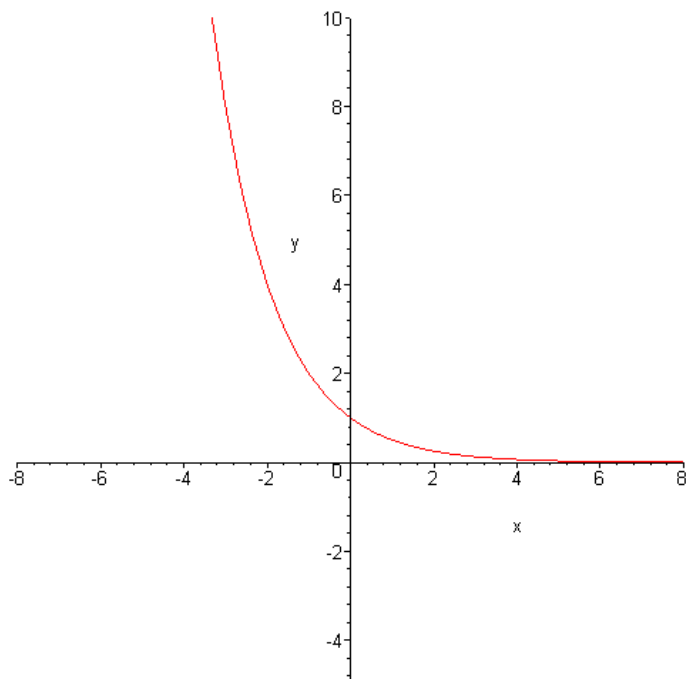
Section 2.3

x	$\left(\frac{1}{2}\right)^x$
0	1
1	1/2
2	1/4
3	1/8
4	1/16
5	1/32

If we look at the negative values of the input we obtain the following table of values:

x	$\left(\frac{1}{2}\right)^x$
-5	32
-4	16
-3	8
-2	4
-1	2

Plotting these values on a graph gives us the following.



Section 2.3

Note that the characteristics of the graph of the function are similar to those mentioned for functions where $a > 1$:

- a) It has a y-intercept which is 1. This means that the graph goes through the point (0,1).
- b) Any real number can be the input. This means that the domain is $(-\infty, \infty)$.
- c) The output is never negative, nor is it 0. This means the range is $(0, \infty)$.
- d) The line $y = 0$ is a horizontal asymptote of the graph. This time, though, the outputs approach 0 as the inputs approach *positive* infinity:

$$\text{As } x \rightarrow +\infty, y \rightarrow 0.$$

e) The output values are *decreasing* over the entire domain. This means that as the xs get larger, as we move from left to right on the horizontal axis, the ys get smaller.

Checkpoint: Graphs of Exponential Functions 1

The Effect of the Initial Value, C

Class Activity:

Plot each of the following sets of graphs on one grid. Comment on the effect of the initial value, C , on the shape of the graph of $y = Ca^x$. Be sure to choose enough points so that you are certain that your graph is accurate.

1. $y = 2^x$

$y = 3 \cdot 2^x$

$y = 10 \cdot 2^x$

2. $y = 2^x$

$y = -2^x$

$y = -3 \cdot 2^x$

3. $y = \left(\frac{1}{2}\right)^x$

$y = 3 \cdot \left(\frac{1}{2}\right)^x$

$y = 10 \cdot \left(\frac{1}{2}\right)^x$

The Effect of Adding a Constant, k

Class Activity:

Plot each of the following sets of graphs on one grid. How does the constant affect the location of the horizontal asymptote? Be sure to choose enough points so that you are certain that your graph is accurate.

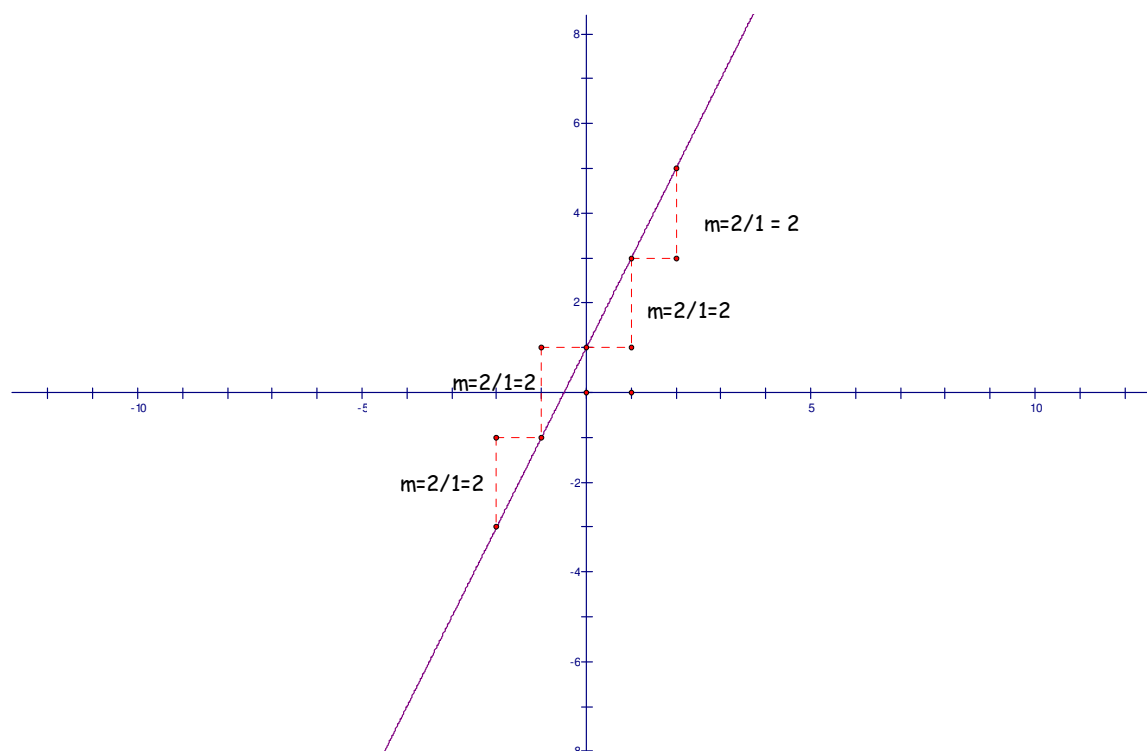
Section 2.3

$$\begin{array}{lll}
 1. & y = 2^x + 1 & 2. & y = \left(\frac{1}{2}\right)^x + 4 & 3. & y = -2^x + 1 \\
 & y = 2^x + 3 & & y = \left(\frac{1}{2}\right)^x - 3 & & y = -2^x + 3 \\
 & y = 2^x - 1 & & y = \left(\frac{1}{2}\right)^x - 1 & & y = -2^x - 5
 \end{array}$$

Exponential Graphs and Concavity

As we have noticed, the graphs of exponential functions curve upward. These are very unlike the graphs of [linear functions](#) which are straight lines.

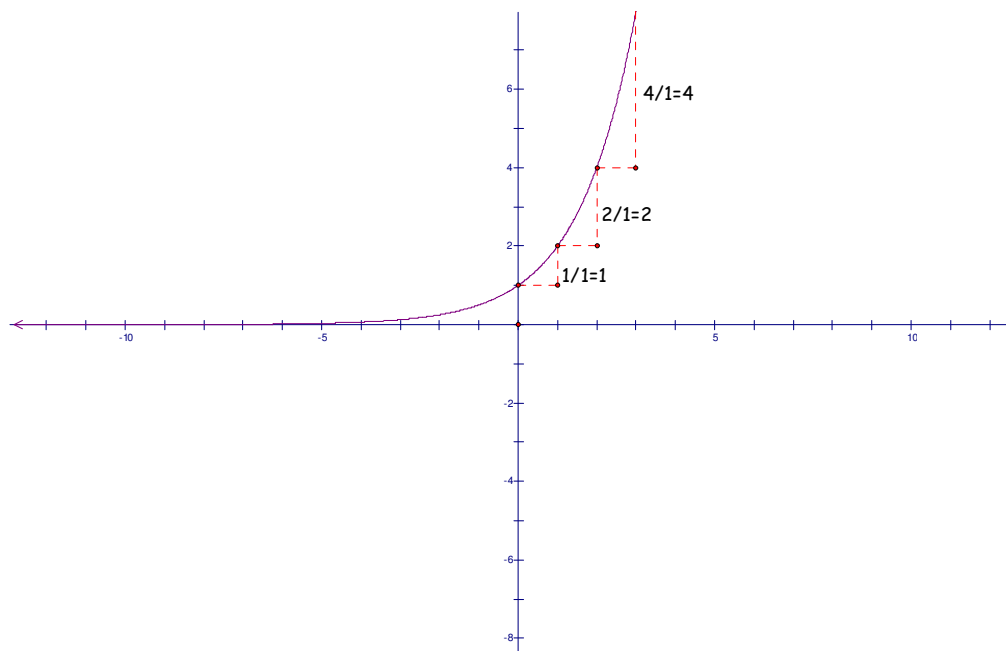
Recall also that the average rate of change of a linear function is constant. We call this average rate of change [slope](#). Here's a sketch to refresh your memory.



Note that for any two points we choose on the line, the slope (average rate of change) is the same.

Section 2.3

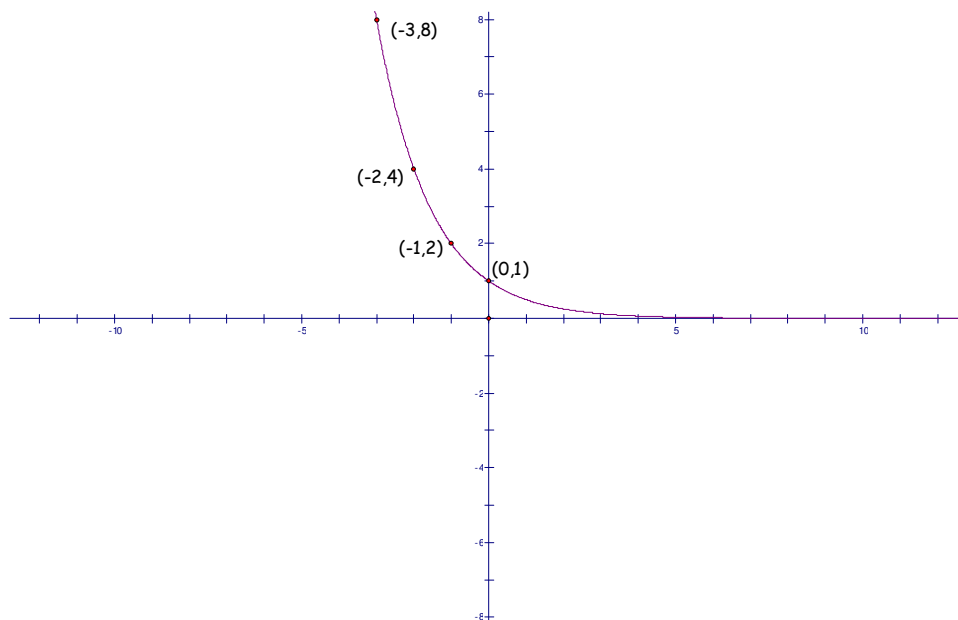
A quite different situation occurs when we examine the graph of an exponential function. Let's again look at the graph of $y = 2^x$ and note the rates of change between points.



Notice that if we take the $\frac{\text{change in } y}{\text{change in } x}$, we do NOT obtain the same value given different sets of

points. This graph is [concave up](#). A graph is concave up if the graph turns left (counterclockwise) as you travel along the curve from left to right. Such graphs have the property that if you find the average rate of change over two intervals, with one interval lying (entirely) to the right of the other, the average rate of change over the interval to the right will be greater.

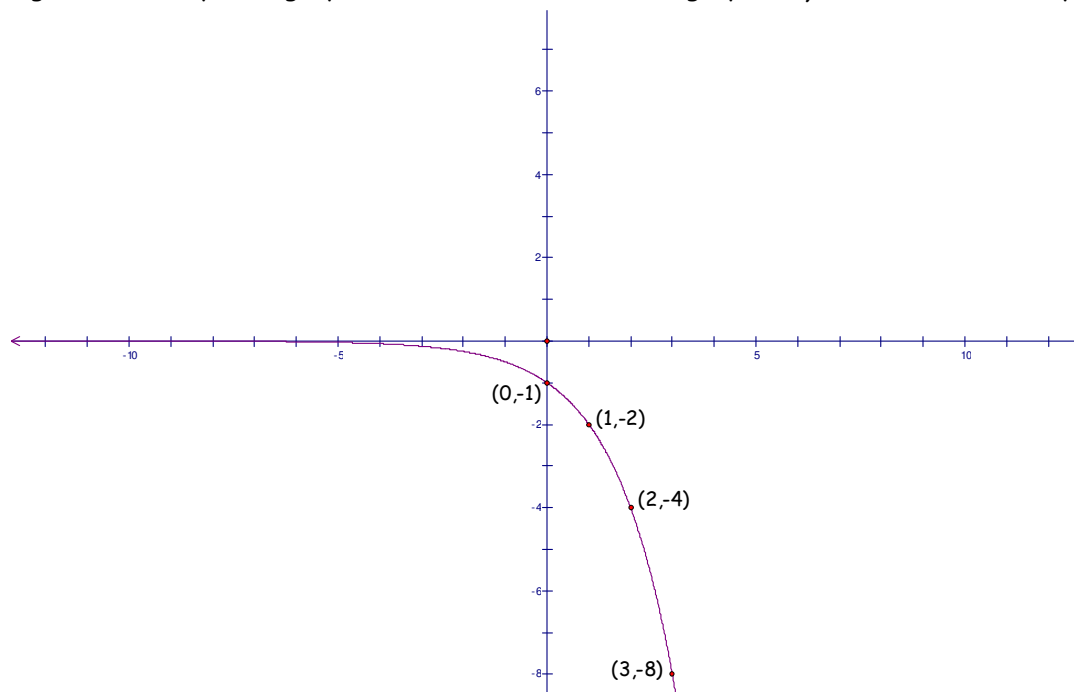
Below is the graph of $y = \left(\frac{1}{2}\right)^x$. Would you expect the average rate of change to be increasing, decreasing, or constant as the value of x increase?



Section 2.3

Find the average rate of change for the points listed in the graph and see if your conjecture was correct.

Just as a side note, not all graphs are concave up, even though the exponential graphs we looked at above are. Graphs can turn to the right (clockwise) as you travel along the curve from left to right and we say such graphs are concave down. The graph of $y = -2^x$ is one example.



Find the average rate of change between each of the points listed. What do you notice about it as the value of x increases?

For a function whose graph is concave up, the **average rate of change increases** as you move from left to right (using non-overlapping intervals).

For a function whose graph is concave down, the **average rate of change decreases** as you move from left to right (using non-overlapping intervals).

For a function whose graph is a straight line (neither concave up nor concave down), the average rate of change is constant.



[Checkpoint: Graphs of Exponential Functions 2](#)

[More worked examples](#)

2.3 Graphs of Exponential Functions: Homework Exercises

Characterize the graph of each of the following, naming the domain, range, y-intercept, end behavior, and horizontal asymptote. Use this information to sketch the graph *by hand*, finding one additional point algebraically if necessary. Use your graphing calculator as a check only

1. $y = 3^x$

2. $y = 4^x$

3. $y = 3^x + 2$

4. $y = 4^x + 2$

5. $y = 3^x + 5$

6. $y = 4^x + 3$

7. $y = 3^x - 2$

8. $y = 4^x - 3$

9. $y = 3^x - 5$

10. $y = 4^x - 5$

11. $y = 5 \cdot 3^x$

12. $y = 2 \cdot 4^x$

13. $y = 2 \cdot 3^x$

14. $y = 3 \cdot 4^x$

15. $y = \frac{1}{2} \cdot 3^x$

16. $y = \frac{1}{2} \cdot 4^x$

17. $y = 5 \cdot 3^x + 2$

18. $y = 2 \cdot 4^x + 3$

19. $y = 5 \cdot 3^x - 2$

20. $y = 2 \cdot 4^x - 3$

21. $y = -3^x$

22. $y = -4^x$

23. $y = -3^x + 2$

24. $y = -4^x + 3$

25. $y = -3^x - 1$

26. $y = -4^x - 5$

27. $y = -5 \cdot 3^x - 2$

28. $y = -3 \cdot 4^x - 1$

29. $y = \left(\frac{1}{3}\right)^x$

30. $y = \left(\frac{1}{4}\right)^x$

31. $y = -\left(\frac{1}{3}\right)^x$

32. $y = -\left(\frac{1}{4}\right)^x$

Section 2.3

33. $y = \left(\frac{1}{3}\right)^x - 2$

34. $y = \left(\frac{1}{4}\right)^x - 3$

35. $y = 2\left(\frac{1}{3}\right)^x$

36. $y = 3\left(\frac{1}{4}\right)^x$

Section 2.4

2.4 Continuous Growth and Decay

Goals:

- Develop a working knowledge and understanding of the number e .
- Correctly use the compound interest formula for continuous compounding.
- Correctly use a given model for continuous decay.

Terms to know:

- compound interest
- continuous compounding
- effective annual yield

- exponential function
- natural exponential function

Let's consider another scenario - a somewhat unlikely one - but instructive none-the-less. Suppose you have \$1.00 to invest and you have a friend who will give you 100% interest. How much money would you have after 1 year if you vary the number of compoundings?

Applying the compound interest formula and letting $P = 1$, and $r = 1.00$ and $t = 1$, we obtain:

$$1\left(1 + \frac{1}{n}\right)^n$$

Complete the following table for increasingly large values of n :

Section 2.4

n	$\left(1 + \frac{1}{n}\right)^n$
1	
2	
3	
5	
10	
25	
50	
100	
500	
1000	

Do the values of the output appear to be “leveling off?” If so, to what number? (You’ll have to approximate here.)

Since the set of counting numbers has infinitely many elements, we can continue the above process as long as we like. The increase in the output changes very little, though, as n gets very large.

We define the number e to be the number that this sequence of numbers is getting arbitrarily close to, even though we don’t know exactly what this number is. Seem strange?

Mathematicians use the term *limit* to refer to such a number and think of the number e as a limit. In particular, they would write:

$$e = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n$$

You’ll study limits in more detail when you take calculus. For now, what you need to know about the number e is the following:

$$e \approx 2.7$$

e is an irrational number

When used as a base for an exponential function, the function is called the *natural exponential function*.

This natural exponential function is used when modeling **continuous growth or decay**.

Most historians believe that Leonhard Euler, a great 18th century Swiss mathematician, named this number e .

Section 2.4

Play around with the e button on your calculator. What is e^2 ? e^3 ? What about $\frac{1}{e}$?

How could you find these approximate values without using the e button?

If interest on an initial deposit of P dollars is *compounded continuously* at an annual rate r , the amount A , t years later can be calculated using the formula

$$A = Pe^{rt}$$

where r is the nominal interest rate.

Example 1. You plan to deposit money in an account giving 6% interest compounded continuously. How much money would you have in the account after 5 years if you deposit \$1000?

For continuous compounding, use $A = Pe^{rt}$:

$$A = 1000e^{(.06 \times 5)}$$

↑
Be sure to use () on your calculator here!

$$= 1000e^{(.3)} = 1349.86$$

You would have \$1349.86.

Example 2. Which is the better deal: An account that pays 7 % interest compounded quarterly or one that pays 6.95% compounded continuously?

SOLUTION

For the account that pays 7 % compounded quarterly,

use the formula $A = P \left(1 + \frac{r}{n}\right)^{nt}$:

$$A = P \left(1 + \frac{.07}{4}\right)^{4t}$$

$$= P(1.0175)^{4t}$$

Take (1.0175) to the fourth power $= P(1.07186)^t$

The effective annual yield is 7.186%

For the account that pays 6.95 % compounded continuously,

Use the formula $A = Pe^{rt}$ $A = Pe^{0.0695t}$

Take e to the 0.0695 power : $= P(1.07197)^t$

The effective annual yield is 7.197%.

The account paying 6.95 % interest compounded continuously is the better deal since it has a higher effective annual yield.

You can also use the table feature of your calculator to compare outputs for specific values of t . Suppose you had \$1000 to invest. Using the variable x instead of t , type in

$$Y1 = 1000 \left(1 + \frac{.07}{4}\right)^{4x} \quad \text{and} \quad Y2 = 1000e^{.07x} \quad \text{then look at the values in the table.}$$

Example 3. How much money would you need to invest now in an account that gives 6% interest compounded continuously if you want to have \$1000 in this account in 4 years?

SOLUTION

Since interest is compounded continuously, we use the formula $A(t) = Pe^{rt}$. This time, we know the final amount, $A(t)$. We also know r and t . We plug these numbers into the formula, like so:

$$1000 = Pe^{0.06 \times 4}$$

The only unknown is P , so we solve the above equation for P :

$$1000 = Pe^{0.06 \times 4}$$

$e^{0.06 \times 4}$ is the coefficient of P , so we divide both sides of the equation by it.

$$\frac{1000}{e^{0.06 \times 4}} = P$$

$$\$786.63 = P$$

So you would need to deposit \$ 786.63 now into the account

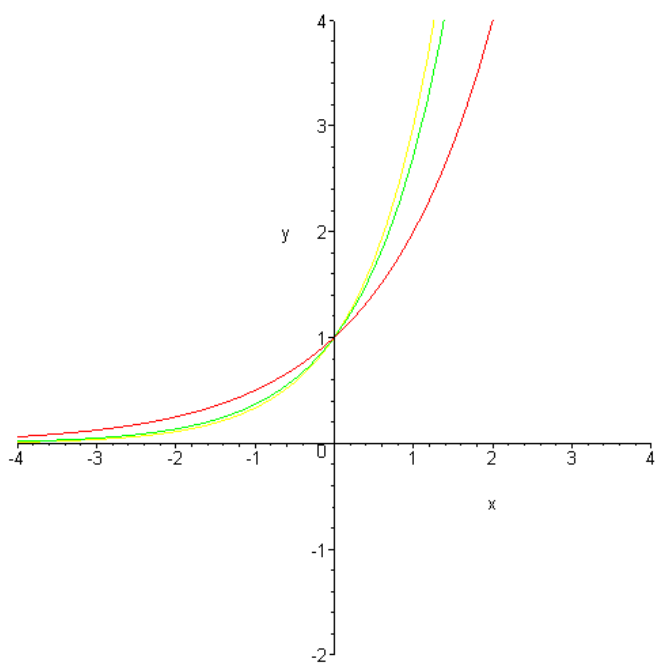
Example 4. Using your calculator, complete the following table, then graph the exponential function $y = e^x$.

Section 2.4

x	e^x
-4	
-3	
-2	
-1	
0	
1	
2	
3	
4	
5	

SOLUTION

As we might expect, the graph of $y = e^x$ falls between the graphs of $y = 2^x$ and $y = 3^x$.



The formula $A = Pe^{rt}$ is used whenever we wish to model ANY TYPE of continuous growth or decay. It is often used to model population or bacteria growth, or decaying matter (as in a compost pile) and many phenomena in both the physical and business world.

Section 2.4

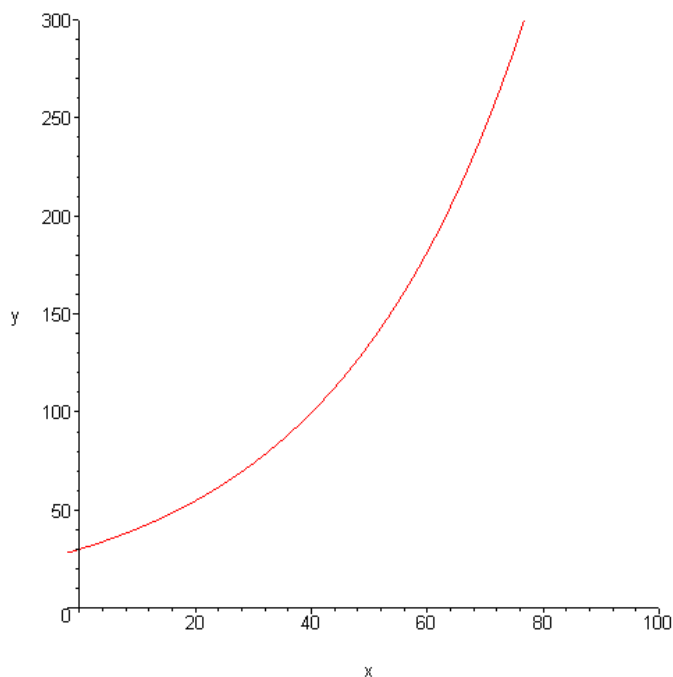
Example 5. The approximate number of fruit flies in an experimental population after t hours is given by

$$A(t) = 30e^{0.03t}, \quad t > 0$$

- What is the initial number of fruit flies?
- How large is the population of fruit flies after 48 hours?
- Sketch the graph of the function A .

SOLUTION

- Since the initial number occurs when $t = 0$, the initial number of fruit flies is $A(0) = 30e^{0.03 \cdot 0} = 30e^0 = 30 \cdot 1 = 30$.
- After 48 hours, the number of fruit flies is $A(48) = 30e^{0.03 \cdot 48} \approx 126$ fruit flies.
- The independent variable is t (*time*) and the dependent is A , *the number of fruit flies*. For this problem, it only makes sense to have positive values for t .



Example 6. Medical doctors often use radioactive iodine as a tracer when diagnosing some thyroid gland disorders. The iodine decays in such a way that after t days, the amount left is given by

$$A(t) = 6e^{-0.087t}$$

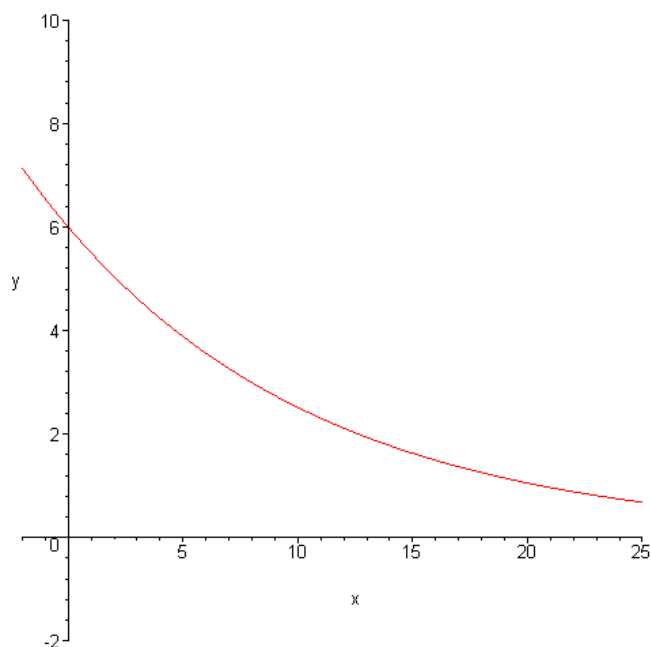
where A is measured in grams.

Section 2.4

- What is the initial amount of iodine ?
- How much iodine remains after 15 days?
- Sketch a graph of the function, A .

SOLUTION

- The initial amount occurs when $t = 0$, so the initial amount is $A(0) = 6e^0 = 6$ grams.
- After 15 days there is $A(15) = 6e^{-0.087 \cdot 15} \approx 1.63$ grams.
- The independent variable is t (*time*) and the dependent variable is A (*Amount*).



✓ Checkpoint Continuous Growth and Decay

More worked examples

2.4 Continuous Growth and Decay: Homework Exercises

Solve each of these

- Write a formula for continuous decay at 1.25% annually with initial value of
 - 2,500
 - 3,000
 - 2
- How much money would be in an account after 20 years if you deposited \$15,000 at each of the following interest annually rates compounded continuously?
 - 1.2%
 - 28%
 - 6.56%
 - 4.75%
- Which is the better deal: An account that pays 8% interest compounded daily or one that pays 7.95% compounded continuously?
- Which is the better deal: An account that pays 4% interest compounded daily or one that pays 3.95% compounded continuously?
- Find how much money you would need to invest in an account that gives 7.3% interest compounded continuously if you want to have \$50,000 in this account in 15 years.
- Find how much money you would need to invest in an account that gives 6.5% interest compounded continuously if you want to have \$100,000 when you retire in 40 years.
- Find how much money you would need to invest in an account that gives 3% interest compounded continuously if you want to have \$2500 for a car down payment in 3 years?
- The number of people in a small population after t years is given by

$$A(t) = 400e^{0.77t}$$

- Is the population growing or shrinking?
 - What is the initial number in the population?
 - How large is the population after 10 years?
- The number of people in a larger population after t years is given by

$$A(t) = 15,000e^{-0.417t}$$

- Is the population growing or shrinking?
- What is the initial number in the population?
- How large is the population after 10 years?

Section 2.4

10. Given that an amount (in grams) of radioactive material decays in such a way that after t years, the amount left is given by

$$A(t) = 100e^{-0.0017t}$$

- What is the initial amount of radioactive material?
- How much radioactive material remains after 700 years?

Section 2.5

2.5 Solving Exponential Equations

Goal: The student will be able to

- solve exponential equations of the type $a^x = a^n$, where $a \neq -1, 0, 1$.

Terms to know:

- base of the exponential
- exponential equation

The simplest type of exponential equation is one like

$$2^x = 8.$$

We can write the right hand side as a power of two, like this:

$$2^x = 2^3$$

Since both sides have the same base, we can simply set the exponents equal to each other, i.e.

$$x = 3.$$

We can check algebraically;

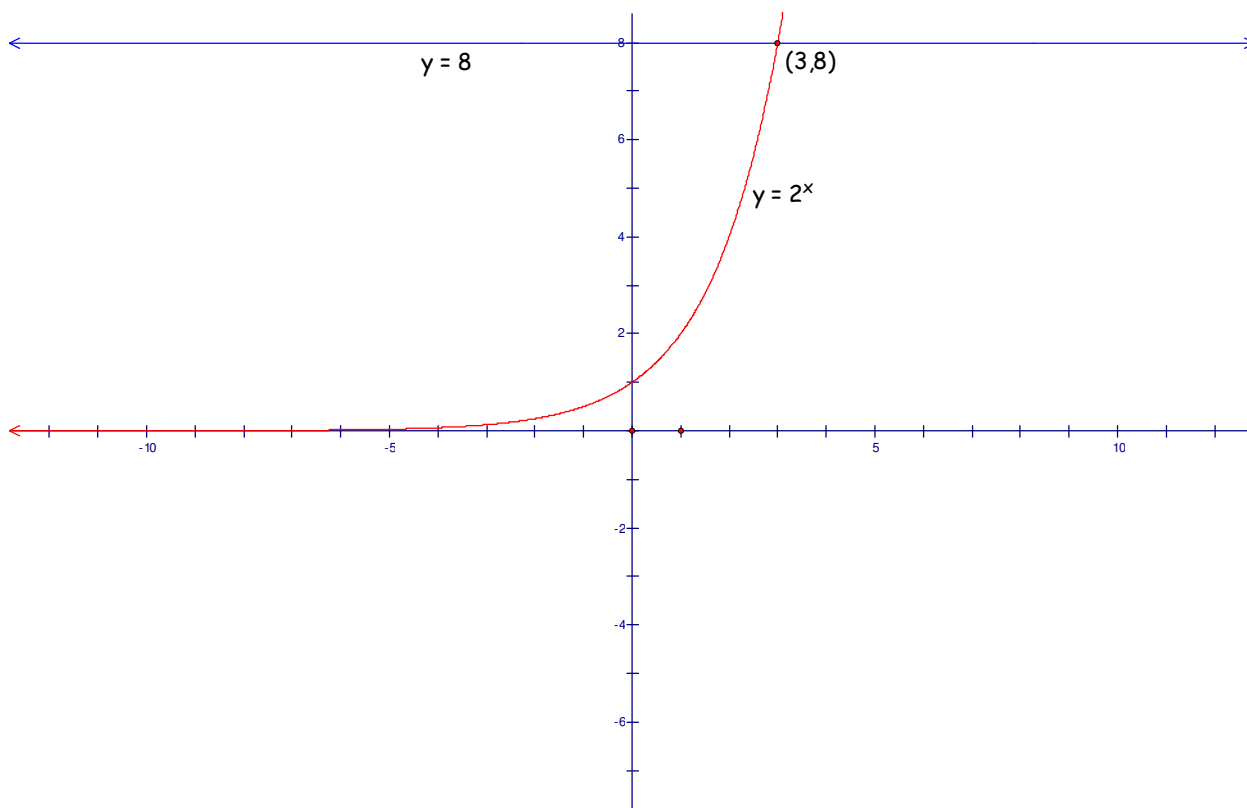
$$2^3 = 8$$

$$8 = 8.$$



Section 2.5

To solve graphically, we look for the point of intersection of the graphs of each side of the equation. We let $Y1 = 2^x$ and $Y2 = 8$.



The two graphs intersect at the point with $x = 3$. When $x = 3$, $y = 8$; and $y = 2^3 = 8$. So $x = 3$ is our solution.

We can also check by looking at a [table of values](#) listing the powers of two.

An equation that contains a variable in the exponent is called an **exponential equation**. The simplest form is $a^x = b$. If b can be expressed as a^n , then:

$$a^x = a^n \Rightarrow x = n, \text{ where } a \neq -1, 0, 1.$$

Section 2.5

Example 1. Solve each of the following exponential equations:

a) $3^x = \frac{1}{9}$

b) $7^{x^2-1} = 1$

c) $4^{2x+1} - 8^x = 0$

SOLUTIONS.

a) $3^x = \frac{1}{9}$

Note that $9 = 3^2$. Since the 9 is in the denominator, we use a negative exponent.

Rewrite both sides with the same base:

Set the exponents equal to each other.

Check:

$$3^x = \frac{1}{9} = \frac{1}{3^2} = 3^{-2}$$

$$3^x = 3^{-2}$$

$$x = -2$$

$$3^{-2} = \frac{1}{9}$$



[Graphical Solution](#)

[Check with Table of Values](#)

b) $7^{x^2-1} = 1$

Note that the right hand side can be written as a power of 7, since any number to the 0 power = 1:

Set the exponents equal to each other.

Solve for x.

Check:

$$7^{x^2-1} = 7^0$$

$$x^2 - 1 = 0$$

$$(x+1)(x-1) = 0$$

$$x = -1 \text{ or } x = 1.$$

$$7^{1^2-1} = 7^0 = 1$$

$$7^{(-1)^2-1} = 7^0 = 1$$



[Graphical Solution](#)

Section 2.5

c) $4^{2x+1} - 8^x = 0$

Note that both 4 and 8 can be expressed as powers of two.

To use our method, bring one term to the right:

$$4^{2x+1} = 8^x$$

Express both sides using the same base, which is 2:

$$(2^2)^{2x+1} = (2^3)^x$$

Use the properties of exponents to simplify:

Power to a power \Rightarrow MULTIPLY exponents

$$(2^2)^{2x+1} = (2^3)^x$$

$$2^{2(2x+1)} = 2^{3x}$$

$$2^{4x+2} = 2^{3x}$$

Since the bases are the same, set the exponents equal to each other.

$$4x + 2 = 3x$$

Solve:

$$-3x + 4x + 2 = 3x - 3x$$

$$x + 2 = 0$$

$$x = -2$$

Check:

$$4^{2(-2)+1} - 8^{-2} =$$

$$4^{-4+1} - \frac{1}{8^2} =$$

$$4^{-3} - \frac{1}{64} =$$

$$\frac{1}{64} - \frac{1}{64} = 0$$



Example 2. By using the substitution $y = 3^x$, find the value(s) of x such that $2(3^{x-1}) = 3^x - 9$.

SOLUTION

We want to use the substitution $y = 3^x$, but we have a term with a 3^{x-1} in it. We need to deal with that first.

Note by the rules of exponents, $3^{x-1} = 3^x \cdot 3^{-1}$, so we can write the original equation as

$$2(3^x \cdot 3^{-1}) = 3^x - 9$$

We know from the definition of negative exponents that

$3^{-1} = \frac{1}{3}$, so we can now write:

$$2\left(3^x \cdot \frac{1}{3}\right) = 3^x - 9$$

NOW, we can rewrite the equation using the substitution $u = 3^x$:

$$2\left(u \cdot \frac{1}{3}\right) = u - 9$$

Solve for u :

$$2\left(u \cdot \frac{1}{3}\right) = u - 9$$

$$\frac{2}{3}u = u - 9$$

$$\frac{2}{3}u - u = u - u - 9$$

$$-\frac{1}{3}u = -9$$

$$u = 27$$

Now, however, we need to switch back to find x .

We use the original substitution $y = 3^x$

$$y = 3^x$$

$$27 = 3^x$$

$$3 = x$$

The check is left for you.



Checkpoint Exponential Equations

More worked examples

2.5 Solving Exponential Equations: Homework

Solve the following exponential equations. Convince yourself of the correctness of your solution by solving graphically. Then check algebraically or numerically.

1. a) $4^{2x} = 64$

b) $3^{2x} = 27$

c) $5^{3x} = 25$

d) $4^x = 32$

e) $6^{4x} = 216$

f) $2^{7x} = 1024$

2. a) $4^{2x} = \frac{1}{64}$

b) $3^{2x} = \frac{1}{27}$

c) $5^{3x} = \frac{1}{25}$

d) $4^x = \frac{1}{32}$

e) $6^{4x} = \frac{1}{216}$

f) $2^{7x} = \frac{1}{1024}$

3. a) $\left(\frac{1}{4}\right)^{2x} = 64$

b) $\left(\frac{1}{3}\right)^{2x} = 27$

c) $\left(\frac{1}{5}\right)^{x+2} = 25$

d) $\left(\frac{1}{4}\right)^{x+1} = 32$

e) $\left(\frac{1}{6}\right)^{x-3} = 216$

f) $\left(\frac{1}{2}\right)^{7x} = 1024$

4. a) $\sqrt{2^{2x}} = 64$

b) $\sqrt{3^{2x}} = 27$

c) $\sqrt{5^{3x}} = 25$

d) $\sqrt[3]{4^x} = 32$

e) $\sqrt{6^{4x}} = 216$

f) $\sqrt[4]{2^{5x}} = 1024$

Section 2.5

5. a) $4^x(5^{2x}) = 10$ b) $3^{2x}(5^x) = 2025$

c) $5^x(2^{x+1}) = 2000$ d) $4^x(3^{x-5}) = \frac{16}{27}$

6. a) $5^x - 25 = 0$ b) $5^{x^2-9} - 1 = 0$

c) $3^{4x+2} - 27 = 0$ d) $2^{x^2} - 4^{50} = 0$

7. a) $8^x = 4^{x+1}$ b) $25^x = (\sqrt{5})^{x+1}$

c) $5^{x^2} = 25^{2(x-1)}$ d) $3^{x^2} = 27^{x+5}$

8. a) $3^{x^2-6} - 9^{x+1} = 0$ b) $7^{x^2} - 49^{(10-\frac{1}{2}x)} = 0$

c) $2^{x^2} = \frac{32^{(2x-3)}}{2^{4x-6}}$ d) $5^{x^2} = \frac{25^{x+1}}{125^x}$

9. By using the substitution $u = 3^x$, find the value(s) of x such that $3^{x+1} = 3^x + 18$.10. By using the substitution $u = 2^x$, find the value(s) of x such that $2^x = 3 - 2(2^{-x})$.

11. Solve the following equations:

a) $4^{2x} - 15 \cdot 4^x = 16$

b) $\left(\frac{1}{2}\right)^{2x} - 4 \cdot \left(\frac{1}{2}\right)^x - 32 = 0$

c) $3^{6x} - 10 \cdot 3^{3x} + 9 = 0$

d) $e^{4x+2} + 4e^{2x+1} = 5$

Hint: This equation can be written like

this: $e^{2(2x+1)} + 4e^{2x+1} = 5$

Section 2.5

12. Solve each of the following:

a) $2^{2x} - 3(2^x) - 4 = 0$

b) $3^{2x} - 7(3^x) - 18 = 0$

c) $24\left(\frac{1}{2}\right)^x - 2 = 2^x$

d) $45\left(\frac{1}{3}\right)^x + 4 = 3^x$

e) $\frac{5^{x+1} - 20}{10 + 5^x} = 3$

f) $9^{x+\frac{1}{2}} - 3^{x+2} = 9 + 17(3^x)$

13. Write a real world scenario that the following equation might model: $2^x = 512$

CHAPTER 3

Logarithms

3.1 Introduction to Logarithms

Goals:

- Figure out the meaning of logarithms
- Evaluate logarithms without the use of a calculator
- Write certain logarithmic equations as exponential equations
- Write certain exponential equations as logarithmic equations

Prerequisite skills and knowledge:

- a working knowledge of positive and negative integer exponents
- a working knowledge of positive and negative rational exponents

Terms to know:

- integer
- power

[Concept Prep: Introduction to logarithms](#)

[Dynagraph Lab 2 Activities](#)

The notation $\log x$, sometimes written $\log_{10} x$ means *the power you raise 10 to in order to get x .*

Example 1. Find $\log_{10}(100)$. This is sometimes written $\log(100)$.

What exponent do we need to raise 10 to to get 100? $10^{??} = 100$

Since $10^2 = 100$, the log is 2. $\log(100) = 2$

Example 2. Find $\log(1000)$

What exponent do we need to raise 10 to to get 1000? $10^{??} = 1000$

Since $10^3 = 1000$, the log is 3. $\log(1000) = 3$

Example 3. Find $\log(0.1)$

What exponent do we need to raise 10 to to get 0.1?

$$10^{??} = \frac{1}{10}$$

Since $10^{-1} = 0.1$, the log is -1.

$$\log\left(\frac{1}{10}\right) = -1.$$

Example 4. Find $\log(200)$

What exponent do we need to raise 10 to to get 200?

$$10^{??} = 200$$

Since $10^2 = 100$ and $10^3 = 1000$ the log lies between 2 and 3. We use the LOG key on our calculator to find the approximate value.

$$\log(200) \approx 2.3010$$

Check: $10^{2.3010} \approx 199.9862$

Example 5. Find two consecutive integers between which each of the following lies.

a) $\log(729)$

Between which two powers of 10 does 729 lie?

Since $100 < 729 < 1000$, we have

$$10^2 < 729 < 10^3, \text{ so}$$

Take the log of all three sides:

$$\log 10^2 < \log 729 < \log 10^3, \text{ or}$$

$$2 < \log 729 < 3$$

b) $\log(0.005)$

Between which two powers of 10 does 0.005 lie?

Since $0.001 < 0.005 < 0.01$, we have

$$10^{-3} < 0.005 < 10^{-2}, \text{ so}$$

Take the log of all three sides:

$$\log 10^{-3} < \log 0.005 < \log 10^{-2}, \text{ or}$$

$$-3 < \log(0.005) < -2$$

Example 6. Find each of the following.

a) $\log(625)$

Since $10^2 < 625 < 10^3$, we know that $\log(625)$ lies between 2 and 3.

We use the LOG key on our calculator and round to four decimal places:

$$\log(625) \approx 2.8028$$

Please note that this number is an APPROXIMATION for $\log(625)$.

b) $\log(3,921)$

Since $10^3 < 3,921 < 10^4$, we know that $\log(3921)$ lies between 3 and 4.

We use the LOG key on our calculator and round to four decimal places:

$$\log(3,921) \approx 3.5934.$$

Please note that this number is an APPROXIMATION for $\log(3921)$.

c) $\log(0.005)$

We know from example 6b) above that $-3 < \log(0.005) < -2$.

We use the LOG key on our calculator and round to four decimal places:

$$\log(0.005) \approx -2.3010.$$

Please note that this number is an APPROXIMATION for $\log(0.005)$.

Logarithms with a base of 10 are called **common logarithms** and are written either like this: $\log(x)$ or like this: $\log_{10} x$.

Logarithms to the base 10 were commonly used years ago to multiply and divide large numbers. Since logarithms are exponents, we can simply ADD them together rather than multiplying the original numbers. Of course, we would need a table or calculator to find the logarithms of the original numbers and then to switch back from the logarithms when we are done. [Click here for an example.](#)



[Checkpoint: Introduction to Logarithms](#)

[More worked examples](#)

3.1 Introduction: The Meaning of $\log(x)$: Homework Exercises

Evaluate each of the following without using your calculator.

1. $\log(1)$

2. $\log(10000)$

3. $\log(10)$

4. $\log(100)$

5. $\log(10^2)$

6. $\log(10^3)$

7. $\log(10^5)$

8. $\log(10^7)$

9. $\log\left(\frac{1}{100}\right)$

10. $\log\left(\frac{1}{1000}\right)$

11. $\log\left(\frac{1}{10}\right)$

12. $\log\left(\frac{1}{10000}\right)$

13. $\log(10^{-2})$

14. $\log(10^{-3})$

15. $\log(10^{-5})$

16. $\log(10^{-32})$

17. $\log(\sqrt{10})$

18. $\log(\sqrt[5]{10})$

Estimate each of the following by naming the two integers between which the indicated logarithm lies.

19. $\log(1,530)$

20. $\log(153)$

21. $\log(1.53)$

22. $\log(15.3)$

23. $\log(1,530,000)$

24. $\log(15300)$

25. $\log(0.05)$

26. $\log(0.5)$

27. $\log(0.003)$

28. $\log(0.0005)$

29. Find $\log(0.123)$

30. $\log(0.8765)$

3.2 Logarithms with Bases other than 10 and Basic Properties of Logs

Goals:

- Find logarithms to bases other than 10
- Develop the basic properties of logarithms

Prerequisite skills and knowledge:

- *a working knowledge of logarithms base 10*
- *a working knowledge of basic properties of exponents*

Terms to know:

- Base (of a common logarithm)
- Common logarithm
- Exponential form

We can consider logarithms with bases other than 10.

Example 1. Discuss (guess) in small groups the meaning of the following, then evaluate:

- | | |
|------------------|------------------|
| a) $\log_9(81)$ | b) $\log_7(49)$ |
| c) $\log_6(216)$ | d) $\log_2(128)$ |
| e) $\log_4(4)$ | f) $\log_4(1)$ |

SOLUTION

a) $\log_9 81 = 2$ since $9^2 = 81$

\swarrow \nearrow
 the base is 9

b) $\log_7 49 = 2$ since $7^2 = 49$

\swarrow \nearrow
 the base is 7

The notation $\log_a x$ for any base, $a > 0$, where $a \in \mathbb{R}$, means *the power you raise a to in order to get x*

Example 2. Write each of the following in exponential form:

a) $\log_5 25 = 2$ b) $\log_2 32 = 5$

c) $\log_3 \frac{1}{27} = -3$ d) $\log 1000 = 3$

a) Since the base is **5** and the exponent is **2**,
we have: $5^2 = 25$

b) Since the base is **2** and the exponent is **5**,
we have: $2^5 = 32$

c) Since the base is **3** and the exponent is **-3**,
we have: $3^{-3} = \frac{1}{27}$

d) Since the base is **10** and the exponent is **2**,
we have: $10^3 = 1000$

Example 3. Let's go in the reverse direction.: Write each of the following as a logarithmic equation:

a) $5^2 = 25$ b) $2^5 = 32$

c) $3^{-3} = \frac{1}{27}$ d) $10^3 = 100$

SOLUTION

a) Since the base is **5** and the exponent is **2**, we have: $\log_5 25 = 2$

b) Since the base is **2** and the exponent is **5**, we have: $\log_2 32 = 5$

c) Since the base is **3** and the exponent is **-3**, we have: $\log_3 \frac{1}{27} = -3$



Checkpoint Logarithms

You may have discovered some interesting patterns when working on the Checkpoint exercises.

Example 4. Look at problems 14 -17 from the Checkpoint exercise. Can you generalize your results?

14) $\log 1$

15) $\log_4 1$

16) $\log_3 1$

17) $\log_5 1$

SOLUTION

14) $\log 1$

Since a logarithm is an **exponent** and the base is 10,
we want the power to which we take 10 that will give us 1: $10^{??} = 1$

Since any number to the 0 power is 1, the logarithm is 0: $\log 1 = 0$

15) $\log_4 1$

Since a logarithm is an **exponent** and the base is 4,
we want the power to which we take 4 that will give us 1: $4^{??} = 1$

Since any number to the 0 power is 1, the logarithm is 0: $\log_4 1 = 0$

16) $\log_3 1$

Since a logarithm is an **exponent** and the base is 3,
we want the power to which we take 3 that will give us 1: $3^{??} = 1$

Since any number to the 0 power is 1, the logarithm is 0: $\log_3 1 = 0$

17) $\log_5 1$

Since a logarithm is an **exponent** and the base is 5,
we want the power to which we take 5 that will give us 1: $5^{??} = 1$

Since any number to the 0 power is 1, the logarithm is 0: $\log_5 1 = 0$

LOG PROPERTY 1

$$\text{Log}_a 1 = 0 \quad \text{for any base, } a > 0, \text{ where } a \in \mathbb{R}.$$

Now look at the following problems and note any pattern.

- Example 5.** a) $\log 10$ b) $\log_5 5$
 c) $\log_7 7$ d) $\log_4 4$

SOLUTION.

- a) $\log 10 = 1$ since $10^1 = 10$
 b) $\log_5 5 = 1$ since $5^1 = 5$
 c) $\log_7 7 = 1$ since $7^1 = 7$
 d) $\log_4 4 = 1$ since $4^1 = 4$

LOG PROPERTY 2:

$$\text{Log}_a a = 1 \quad \text{for any base, } a > 0, \text{ where } a \in \mathbb{R}.$$

Now try these.

Example 6.

- a) $\log_5 5^2$
 b) $\log_4 4^3$
 c) $\log_3 3^2$

SOLUTION.

- a) $\log_5 5^2$

Since the base is 5, we want the **exponent** to which we must take 5 in order to get 5^2 .
 This is almost a silly question because the answer is really right in the problem:

$$\log_5 5^2 \quad \begin{array}{l} \swarrow \text{exponent} \\ \nearrow \text{base} \end{array}$$

$$\log_5 5^2 = 2 \quad \text{since } 5^2 = 5^2$$

b) $\log_4 4^3$

Since the base is 4, we want the **exponent** to which we must take 4 in order to get 4^3 .

$$\log_4 4^3 = 3 \quad \text{since } 4^3 = 4^3$$

c) $\log_3 3^2$

Since the base is 3, we want the **exponent** to which we must take 3 in order to get 3^2 .

$$\log_3 3^2 = 2 \quad \text{since } 3^2 = 3^2$$

LOG PROPERTY 3

$$\log_a a^M = M \quad \text{for any base, } a > 0, \text{ where } a \in \mathbb{R}.$$

More worked examples

3.2 Evaluating Logs with Different Bases: Homework Exercises

Solve each of these

1. Find each of the following logarithms:

a) $\log_2(64)$

b) $\log_3(27)$

c) $\log_5(625)$

d) $\log_2(1024)$

e) $\log_6(216)$

f) $\log_7(343)$

2. Find each of the following logarithms.

a) $\log_7(7)$

b) $\log_9(9)$

c) $\log_{102}(102)$

Section 3.2

3. Find each of the following logarithms.

a) $\log_{125}(1)$

b) $\log_5(1)$

c) $\log_2(1)$

4. Write each of the following in exponential form:

a) $\log_5(125) = 3$

b) $\log_2(128) = 7$

c) $\log_2(32) = 5$

5. Write as a logarithmic equation:

a) $2^{-3} = \frac{1}{8}$

b) $5^{-2} = \frac{1}{25}$

c) $6^{-2} = \frac{1}{36}$

6. Find each of the following logarithms.

a) $\log_7\left(\frac{1}{49}\right)$

b) $\log_4\left(\frac{1}{64}\right)$

c) $\log_8\left(\frac{1}{512}\right)$

7. Find each of the following logarithms

a) $\log_{\sqrt{7}}(7)$

b) $\log_{\sqrt{5}}(5)$

c) $\log_{\sqrt[3]{2}}(8)$

d) $\log_{\sqrt[3]{3}}(3)$

e) $\log_{\sqrt{5}}(25)$

f) $\log_{\sqrt{4}}(16)$

8. Find each of the following logarithms:

a) $\log_6(6^4)$

b) $\log_3(3^{10})$

c) $\log_2(2^3)$

9. Write each of the following in exponential form:

a) $\log\left(\frac{1}{100}\right) = -2$

b) $\log\left(\frac{1}{1000}\right) = -3$

c) $\log\left(\frac{1}{10}\right) = -1$

10. Write each as a logarithmic equation:

a) $17^0 = 1$

b) $2^0 = 1$

c) $5^0 = 1$

3.3 The Natural Logarithmic Function

Goals:

- Apply knowledge of logarithms to base e

Prerequisite skills and knowledge:

- a working knowledge of logarithms to the base 10
- a working knowledge of basic properties of the natural exponential function

Terms to know:

- continuous growth or decay
- e
- exponential function
- natural exponential
- logarithm

Concept Prep: Review of natural exponential

Recall the number e from [our work with exponential functions](#). The summary box is reproduced below for your convenience.

$e \approx 2.7$
 e is an irrational number
 When used as a base for an exponential function, the function is called the *natural exponential function*.

This natural exponential function is used when modeling **continuous growth or decay**.

Recall from section 3.1 that in the equation , $A = 1000e^{0.06t}$, A gives us the amount of money we would have if we invest \$1000 in an account that compounds interest continuously at a nominal rate of 6%. Suppose now that we need \$3000 for a down payment on a car. To simplify our work even more, let's suppose also that some generous soul will give us 100% interest rather than 6%. Our equation becomes

$$3000 = 1000e^t$$

Section 3.3

How could we find the value of t ?

If we divide both sides by 1000, we now have

$$3 = e^t \quad (1)$$

We want the exponent to which we would take e to get 3.

You'll spend time in your Algebra for Calculus solving these types of equations. For now, we want you to understand that the number e can be used as a base for a logarithm. You would expect that such a logarithm would be written $\log_e x$, but we usually write it $\ln x$ refer to it as the "natural logarithm."

The notation $\ln x$, which means $\log_e x$ means *the power you raise e to in order to get x* . Logarithms to the base e are called *natural logarithms*.

Example 1. Express each equation in exponential form.

a) $\ln 7 = x$

b) $\ln y = 11$

c) $\ln(x+2) = 3$

d) $\ln(x-1) = 5$

SOLUTION

a) Since the base is e and the logarithm is an exponent, we write: $e^x = 7$

b) Since the base is e and the logarithm is an exponent, we write: $e^{11} = y$

c) Since the base is e and the logarithm is an exponent, we write: $e^3 = x+2$

d) Since the base is e and the logarithm is an exponent, we write: $e^5 = x-1$

Section 3.3

Example 2. Express each equation in logarithmic form:

a) $e^x = 10$

b) $e^4 = t$

c) $e^{x-1} = 102$

d) $e^{0.1} = y$

SOLUTION

- a) Since the base is e we use \ln notation. The logarithm is the exponent. We mean $\log_e 10 = x$, but we write: $\ln 10 = x$
- b) Since the base is e we use \ln notation. The logarithm is the exponent. We mean $\log_e t = 4$, but we write: $\ln t = 4$
- c) Since the base is e we use \ln notation. The logarithm is the exponent. We mean $\log_e 102 = x - 1$, but we write: $\ln 102 = x - 1$
- d) Since the base is e we use \ln notation. The logarithm is the exponent. We mean $\log_e y = 0.1$, but we write: $\ln y = 0.1$

Example 3. Evaluate each of the following without the use of your calculator.

a) $\ln 1$

b) $\ln e$

c) $\ln e^3$

d) $\ln \frac{1}{e^2}$

SOLUTION

- a) $\ln 1$
Remember that we can think of $\ln 1$ as meaning $\log_e 1$.
Since a logarithm is an exponent and the base is e , we want the power to which we take e that will give us 1: $e^{??} = 1$
Since any number to the 0 power is 1, the logarithm is 0: $\ln 1 = 0$
- b) $\ln e$
Remember that we can think of $\ln e$ as meaning $\log_e e$
Since a logarithm is an exponent and the base is e , we want the power to which we take e that will give us e : $e^{??} = e$
Since any number to the 1st power is itself the logarithm is 1: $\ln e = \log_e e = 1$

Section 3.3

c) $\ln e^3$

Remember that we can think of $\ln e^3$ as meaning $\log_e e^3$

Since a logarithm is an exponent and the base is e ,
we want the power to which we take e that will give us e^3 :

$$e^{??} = e^3$$

Silly question. The answer is given in the problem!

$$\ln e^3 = \log_e e^3 = 3$$

d) $\ln \frac{1}{e^2}$

Remember that we can think of $\ln \frac{1}{e^2}$ as meaning $\log_e \frac{1}{e^2}$.

Since a logarithm is an exponent and the base is e ,
we want the power to which we take e that will give us $\frac{1}{e^2}$.

$$e^{??} = \frac{1}{e^2}$$

Recall that $\frac{1}{e^2} = e^{-2}$, so we are really looking for

the power to which we take e to get e^{-2} .

$$\ln \frac{1}{e^2} = \log_e \frac{1}{e^2} = \log_e e^{-2} = -2$$



Checkpoint: The Natural Log Function

More worked examples

3.3 The Natural Logarithmic Function: Homework Exercises

1. Without using a calculator, find each of the following:

a) $\ln(e^7)$

b) $\ln(e^{10})$

c) $\ln\left(\frac{1}{e}\right)$

d) $\ln\left(\frac{1}{e^5}\right)$

e) $\ln\left(\frac{1}{e^{17}}\right)$

f) $\ln(e^{\sqrt{2}})$

g) $\ln\left(\frac{1}{e^{10}}\right)$

h) $\ln(e^{\sqrt{3}})$

2. Without using a calculator, find each of the following:

a) $\ln(\sqrt{e})$

b) $\ln(\sqrt[3]{e})$

c) $\ln(e^e)$

Section 3.3

3. Write each of the following in exponential form:

a) $\ln(5-x) = 2$

b) $\ln(x+8) = 3$

c) $\ln(x^2-1) = 4$

d) $\ln(x^2) = 5$

e) $\ln(x^2+10) = 2$

f) $\ln(2x-5) = 3$

4. Write each of the following as a logarithmic equation:

a) $e^2 = x$

b) $e^3 = x$

5. Write each of the following as a logarithmic equation, then solve for x :

a) $e^{20x-7} = 6$

b) $e^{2x^2} = 5$

c) $e^{x^2} = 3$

d) $e^{x+2} = 5$

6. Find

a) $\ln e^x$

b) $\ln e^{3x+2}$

c) $\ln e^{x^2-1}$

7. Find

a) $e^{\ln(x+5)}$

b) $e^{\ln 2x}$

c) $e^{\ln x^2}$

8. Evaluate

a) e^x as $x \rightarrow -\infty$

b) $\ln x$ as $x \rightarrow 0^+$

9. a) What is the domain and range of $y = e^x$? What is the asymptote of its graph?

b) What is the domain and range of $y = \ln x$? What is the asymptote of its graph?

10. Immediately following the gold medal performance of the US women's gymnastic team in the 1996 Olympic Games, and NBC commentator said of one of the team members: "Her confidence and performance have grown logarithmically." He clearly thought this was a wonderful compliment. Is it a compliment? *

*from Connally, Hughes-Hallett, Gleason et al. (2000). FUNCTIONS MODELING CHANGE. A Preparation for Calculus. New York: John Wiley & Sons, p. 138

3.4 Graphs and Domains of Logarithmic Functions

Goals:

- Graph by hand a logarithmic function
- Discuss the characteristics common to graphs of log functions with different bases
- Write a function rule for a given logarithmic graph
- Find domain of given logarithmic function

Prerequisite skills and knowledge:

- a working knowledge of logarithms to any base

Terms to know:

- domain of a function
- e
- exponential function
- logarithm
- natural logarithm
- sign chart
- vertical asymptote

Concept Prep: Graphs of log functions

Example 1. Make a sketch of the graph of the function given by $y = \log x$

First we'll make a table of values. Note that $\log(-10)$ is undefined, because the base, 10, is a positive number and there is no exponent that will give us -10 : $10^{??} = -10$.

Similarly, the log of **any negative number** is undefined. If we try to evaluate $\log(0)$, we have the same problem.

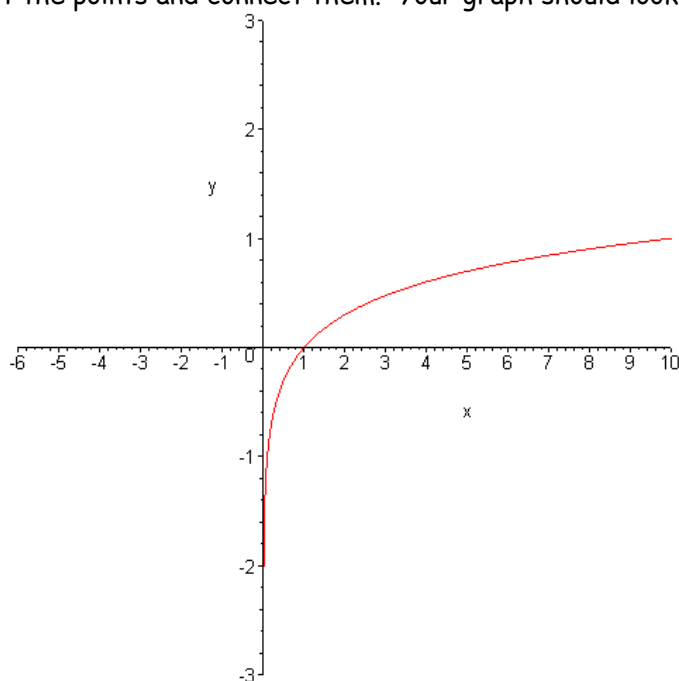
We say the domain of the function given by $y = \log x$ is $(0, \infty)$.

In the table below, we've chosen numbers easy to work with ($x = 0.01, 0.1, 1, \text{ and } 10$). In the other cases, we can use our calculators to approximate the values.

Section 3.4

x	y
0.01	-2
0.1	-1
1	0
2	.30
5	.70
7	.85
8	.90
10	1
100	2

Next, we plot the points and connect them. Your graph should look something like this:



Notice that since the log of 0 is undefined, the graph hugs the vertical axis. This happens because the inputs can get really, really close to 0, but not equal 0. We write:

$$x \rightarrow 0^+$$

which we read "as x approaches 0 from the right." Notice that as these input values get very small, like $\frac{1}{1000}$, the output values get large in the negative direction. Complete the following table to convince yourself of this fact.

Section 3.4

x	$\log x$
$\frac{1}{10}$	
$\frac{1}{100}$	
$\frac{1}{1000}$	
$\frac{1}{10,000}$	
...	
10^{-10}	
...	...

We say that the vertical axis is a [vertical asymptote](#) because as $x \rightarrow 0^+$, the function values, $f(x) \rightarrow -\infty$.

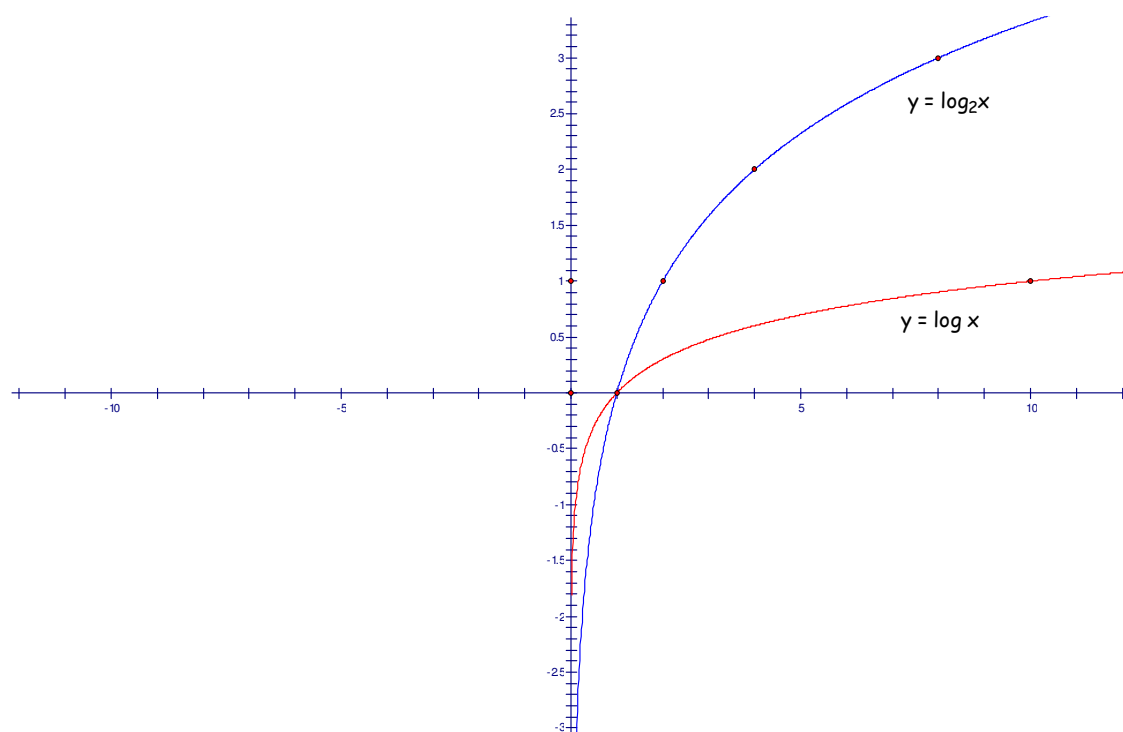
Example 2. Make a graph of each of the functions given by $y = \log_2 x$ and $y = \log_4 x$

Our table of values for $y = \log_2 x$ is the following. Note that as with the log function base 10, only positive numbers have output values. Note also that using powers of 2 makes calculations easy.

x	$y = \log_2 x$
-2	Undefined
-1	Undefined
0	Undefined
$1/4$	-2
$1/2$	-1
1	0
2	1
4	2
8	3

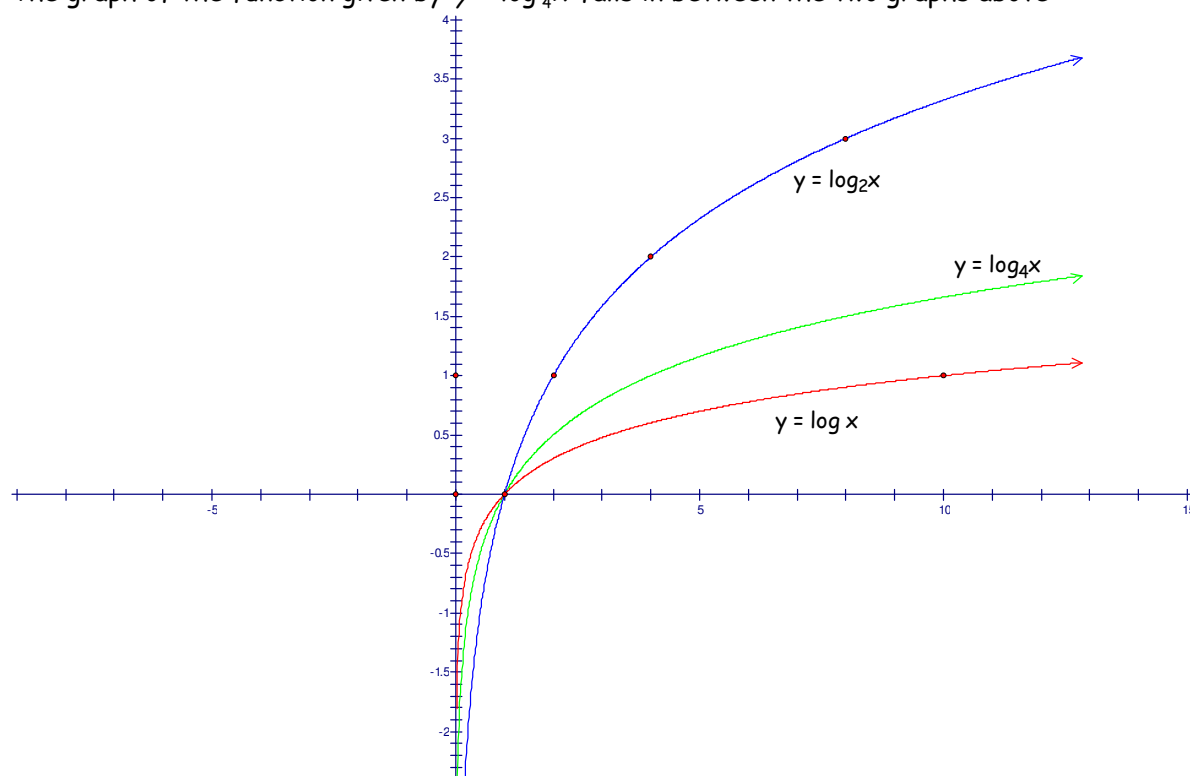
Note that the graph of this function is wider (farther from the x -axis) than the graph of $y = \log x$ when the base was 10. Discuss with your colleagues why this is so.

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Notice again that the vertical axis (y -axis) is a vertical asymptote of the graph of the function because as $x \rightarrow 0^+$, the function values, $f(x) \rightarrow -\infty$.

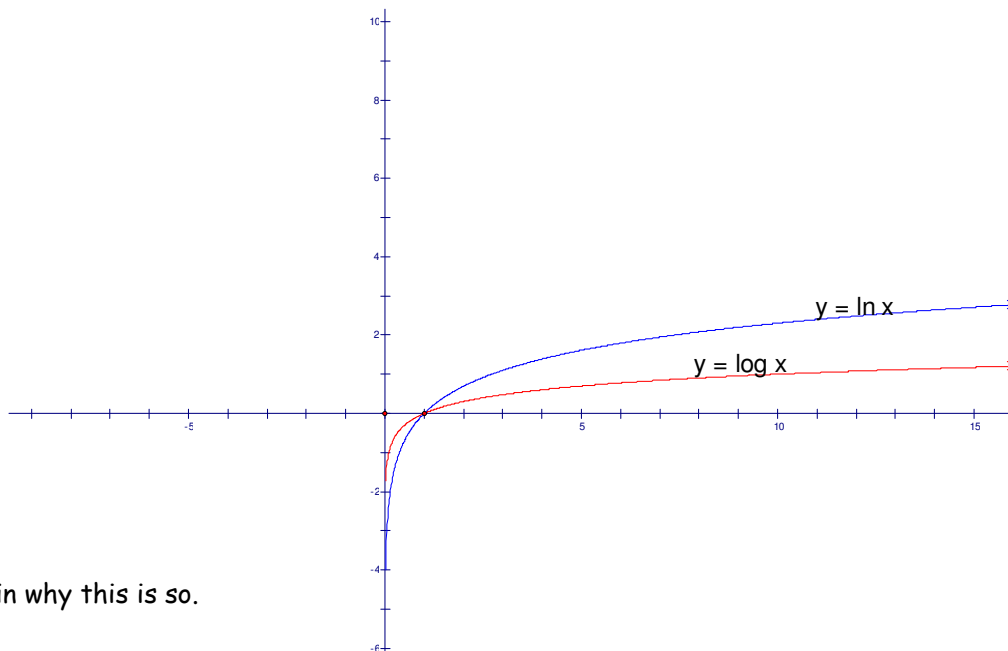
The graph of the function given by $y = \log_4 x$ falls in between the two graphs above:



Explain why this is so.

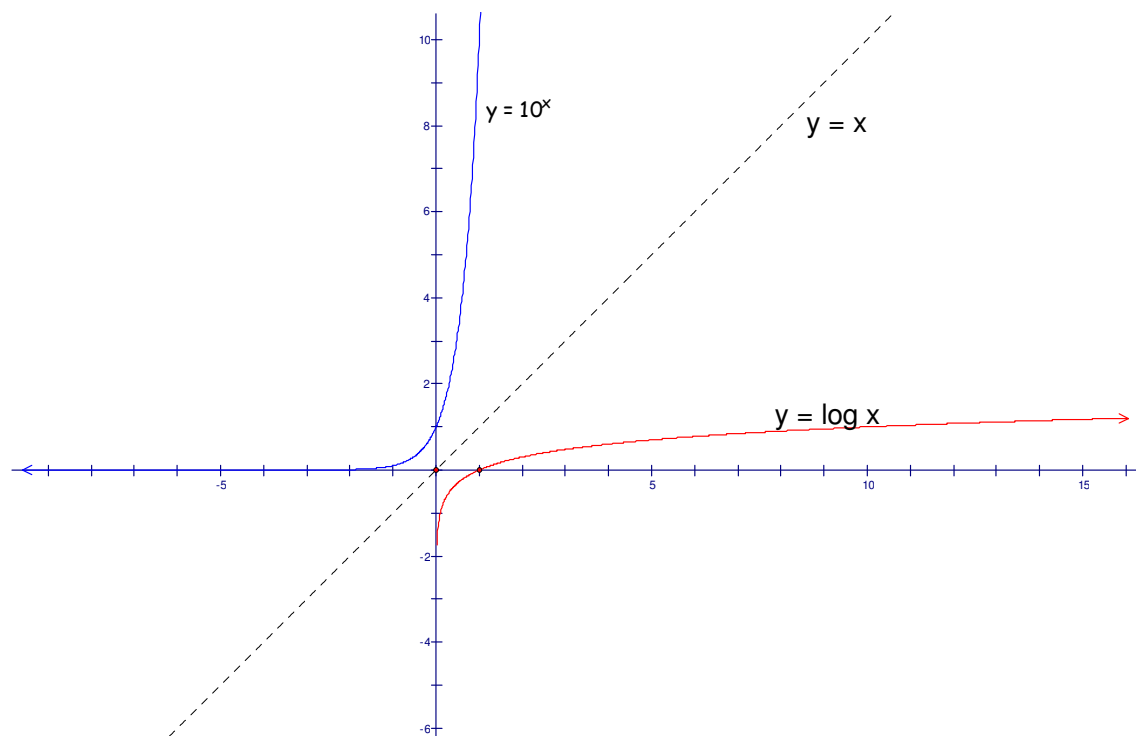
✓ Checkpoint: Graphs of Logarithmic Functions

Example 3. Note how the graph of $y = \ln x$ compares to the graph of $y = \log x$.



Explain why this is so.

Example 4. Compare the graphs of $y = 10^x$ and $y = \log x$ on the same graph. How are they related?



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The graphs are mirror images of each other across the line $y = x$.

If we look at the table of values for each of the above functions, we notice that if we use the outputs of $y = 10^x$ as inputs for $y = \log x$, we obtain the inputs of $y = 10^x$.

x	$y = 10^x$	x	$y = \log x$
-2	1/100	1/100	-2
-1	1/10	1/10	-1
0	1	1	0
1	10	10	1
2	100	100	2
3	1000	1000	3

Use these outputs as inputs here.

✓ Checkpoint: Graphs of Logarithmic Functions 2

The Domain of the Log Function

As we noticed above, the log of a negative number is undefined. Recall that no matter what the value of a positive base a is, $a^{\text{any number}}$ is positive. Convince yourself of this fact by trying to evaluate the following logarithms.

Example 5.

- a) $\log(-100)$ b) $\log_2(-4)$
 c) $\log_3(0)$ d) $\ln(-5)$

SOLUTION

- a) Using the definition of logarithm, $\log(-100)$ is the exponent we need to raise 10 to in order to obtain -100 :

$$10^{??} = -100$$

DOES NOT EXIST!

Section 3.4

- b) Using the definition of logarithm, $\log_2(-4)$ is the exponent we need to raise 2 to in order to obtain -4 :

$$2^{??} = -4$$

DOES NOT EXIST!

- c) Using the definition of logarithm, $\log_3(-9)$ is the exponent we need to raise 3 to in order to obtain -9 :

$$3^{??} = -9$$

DOES NOT EXIST!

- d) Using the definition of logarithm, $\ln(-5)$ is the exponent we need to raise e to in order to obtain -5 :

$$e^{??} = -5$$

DOES NOT EXIST!

The domain of the function $y = \log_a x$, for any base, $a > 0$, where $a \in \mathbb{R}$ is $(0, \infty)$

Example 6. Find the domain of each of the following functions.

a) $y = \ln x$

b) $y = \log(3x)$

c) $y = \log(x - 5)$

d) $y = \ln(3x + 48)$

e) $y = \log(x^2 - 1)$

f) $y = \ln(x^2 - 9)$

SOLUTION

a) $y = \ln x$ Since the log of a negative number does not exist, the domain is $(0, \infty)$.

b) $y = \log(3x)$

The expression after "log" must be positive, so we set it > 0 and solve:

$$3x > 0$$

$$x > 0$$

The domain therefore, is $(0, \infty)$

c) $y = \log(x - 5)$

The expression after "log" must be positive,
so we set it > 0 and solve:

$$x - 5 > 0$$

$$x > 5$$

The domain therefore, is $(5, \infty)$

d) $y = \ln(3x + 48)$

The expression after "ln" must be positive,
so we set it > 0 and solve:

$$3x + 48 > 0$$

$$3x > -48$$

$$x > -16$$

The domain therefore, is $(-16, \infty)$.

Notice that some negative numbers fall in this domain. For example, -10 is in the domain since the expression after "ln" would then be $3(-10) + 48 = 18$, which is positive.

e) $y = \log(x^2 - 1)$

Finding the domain for this function takes more work since the expression is non-linear. You may need to refresh the use of [sign charts](#), which we review briefly here. For more detail, see [section 2.4](#) of the Fundamental Mathematics V e-book.

Find the critical numbers for the function by
setting the expression after "log" = 0:

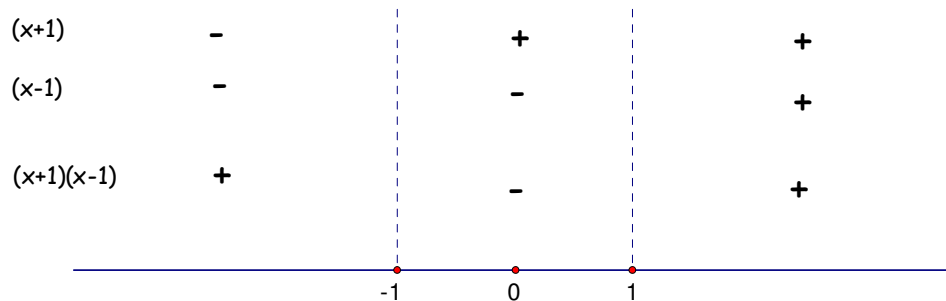
$$x^2 - 1 = 0$$

$$(x + 1)(x - 1) = 0$$

$$x = -1, x = 1$$

Use these critical numbers as interval endpoints on a sign chart and then use test numbers to determine the sign of the factors in these intervals:

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We choose the intervals where the product $(x+1)(x-1)$ is positive.

The domain, therefore, is $(-\infty, -1) \cup (1, \infty)$

f) $y = \ln(x^2 - 9)$

Again we use a sign chart here:

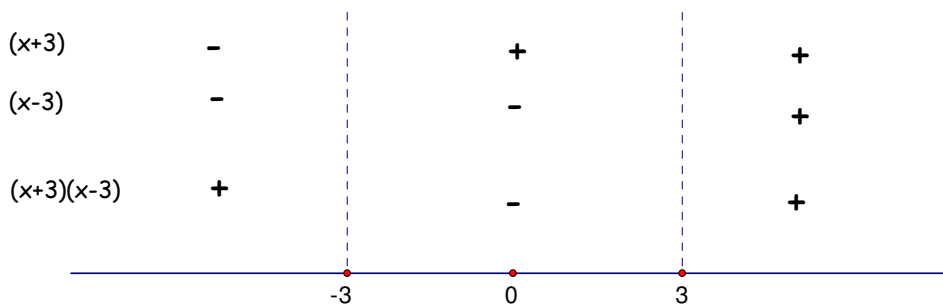
Find the critical numbers for the function by setting the expression after "log" = 0:

$$x^2 - 9 = 0$$

$$(x+3)(x-3) = 0$$

$$x = -3, x = 3$$

Use these critical numbers as interval endpoints on a sign chart and then use test numbers to determine the sign of the factors in these intervals:



We choose the intervals where the product $(x+3)(x-3)$ is positive.

The domain, therefore, is $(-\infty, -3) \cup (3, \infty)$

✓ Checkpoint: Domains of Logarithmic Functions

Section 3.4

3.4 Graphs and Domains of Logarithmic Functions: Homework Exercises

Characterize the graph of each of the following (#1-24), naming the domain, vertical asymptote, range, x -intercept, and one extra point. Then sketch the graph *by hand* using this information.

1. $y = \log_3 x$

2. $y = \log_6 x$

3. $y = \log_5 x$

4. $y = \log_9 x$

5. $y = \log_{\frac{1}{2}} x$

6. $y = \log_{\frac{1}{3}} x$

7. $y = \ln(2x)$

8. $y = 2\log(x)$

9. $y = \log(x - 2)$

10. $y = \log(x + 5)$

11. $y = \log(x - 3)$

12. $y = \log(x + 4)$

13. $y = \log(10 - x)$

14. $y = \ln(2 - 5x)$

15. $y = \ln(7 - 3x)$

16. $y = \log\left(\frac{3x - 5}{6}\right)$

17. $y = \log x + 5$

18. $y = \log x + 1$

19. $y = \ln x - 4$

20. $y = \ln x - 3$

21. Find the domains of the following functions

a) $y = \log(x^2 - 9)$

b) $y = \log(x^2 - 4)$

c) $y = \log(x^2 - 2)$

d) $y = \log_3(2x^2 - x - 1)$

e) $y = \ln(x^2 - 7x + 12)$

f) $y = \log(x^2 + 8x + 15)$

g) $y = \ln(x^2)$

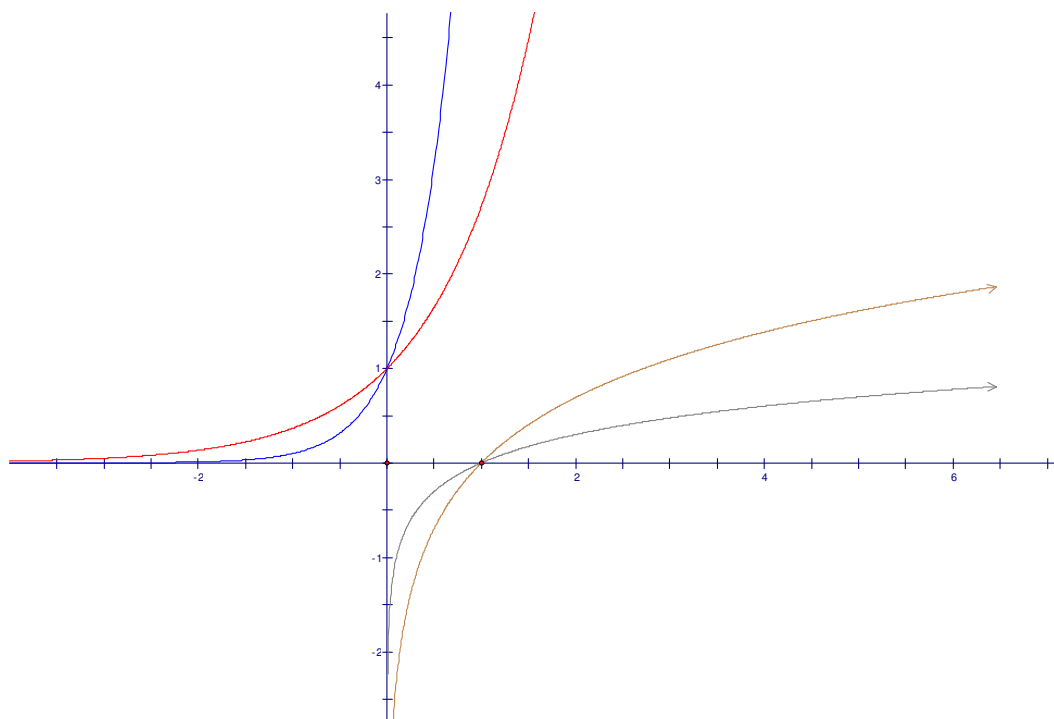
h) $y = (\ln x)^2$

22. For what values of x is $\log_3 x > \ln x$? (Hint: Compare the graphs.)

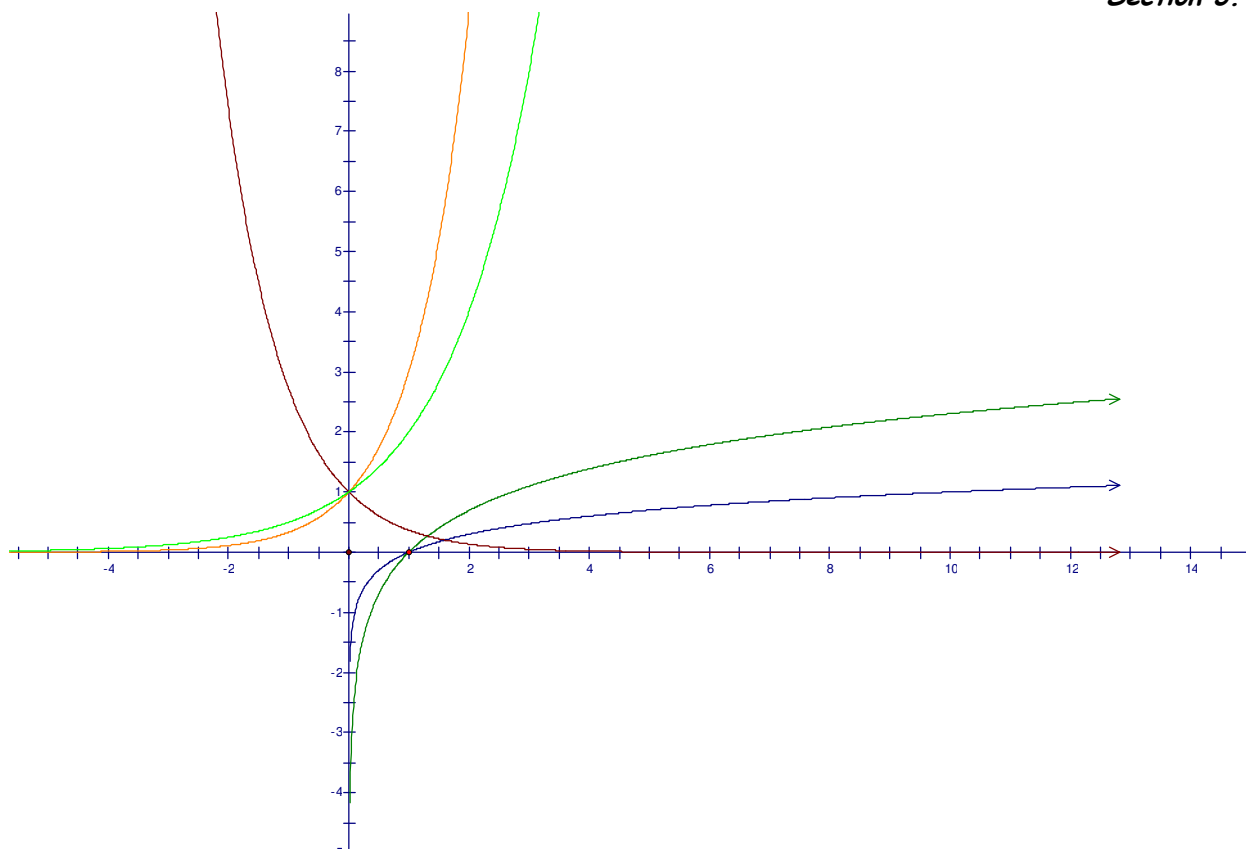
23. For what values of x is $\log_3 x > \log_5 x$? (Hint: Compare the graphs.)

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24. Without using a calculator, match the functions $y = 10^x$, $y = e^x$, $y = \log x$, $y = \ln x$ with their graphs below.



25. Without using a calculator, match the functions $y = \log x$, $y = 3^x$, $y = \ln x$, $y = e^{-x}$, $y = 2^x$ with their graphs below.

Section 3.4**3.5 Solving Logarithmic Equations****Section 3.5**

Goals: To solve logarithmic equations of the form: $\log_a x = b$ where $a \in \mathbb{R}$ and $a > 0$.

Terms to know:

- Base (of exponent)
- Exponent
- Exponential form
- Logarithm
- Logarithmic equation

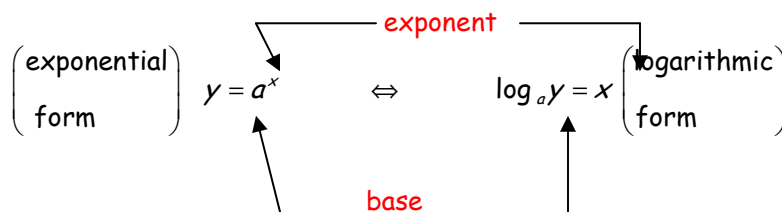
Skill Prep: Exponential form vs. logarithmic form

The simplest type of logarithmic equation is one like

$$\log x = 1000.$$

To solve logarithmic equations you need to understand the basic relationship between exponential and logarithmic equations. You've been working with them for a while, now. Let's review this basic relationship.

Section 3.5



Example 1. Change the following to logarithmic form:

a) $5^2 = 25$ b) $3^4 = 81$

c) $10^3 = 1000$ d) $2^{-3} = \frac{1}{8}$

SOLUTION

a) $5^2 = 25$

The base is 5, the exponent is 2. Since the logarithm is the exponent, we can write it like this:

$$\log_5 25 = 2$$

b) $3^4 = 81$

The base is 3, the exponent is 4. Since the logarithmic is the exponent, we can write it like this:

$$\log_3 81 = 4$$

c) $10^3 = 1000$

The base is 10, the exponent is 3. Since the logarithmic is the exponent, we can write it like this:

$$\log 1000 = 3$$

d) $2^{-3} = \frac{1}{8}$

The base is 2, the exponent is -3. Since the logarithm is the exponent, we can write it like this:

$$\log_2 \frac{1}{8} = -3$$

Example 2. Change the following to exponential form:

a) $\log_4 16 = 2$ b) $\log_3 \frac{1}{3} = -1$

c) $\log 100 = 2$ d) $\log_5 \sqrt{5} = \frac{1}{2}$

SOLUTION

a) The base is 4, the exponent (logarithm) is 2, so

$$4^2 = 16$$

Section 3.5

- b) The base is 3, the exponent (logarithm) is -1, so $3^{-1} = \frac{1}{3}$
- c) The base is 10, the exponent (logarithm) is 2, so $10^2 = 100$
- d) The base is 5, the exponent (logarithm) is $\frac{1}{2}$, so $5^{1/2} = \sqrt{5}$

Recall that the log of a negative number is undefined. For example, $\log(-100)$. Also the base needs to be a positive number, but not = 1.

For $\log_a x$ to be defined,

$$x > 0; \text{ and}$$

$$a > 0, a \neq 1$$

Example 3. Solve the following equations

a) $\log_5 x = 2$ b) $\log_{1/2} 8 = x$

c) $\log_x(7x - 10) = 2$

a) $\log_5 x = 2$

We use the definition of logarithm here as we convert to exponential form.

$$5^2 = x$$

$$25 = x$$

Check:

$$\log_5 25 = 2 \text{ since } 5^2 = 25$$



Graphical check

b) $\log_{1/2} 8 = x$

We use the definition of logarithm and convert to exponential form:

$$\left(\frac{1}{2}\right)^x = 8$$

Section 3.5

This now is an exponential equation: write both sides with the same base:

$$\left(\frac{1}{2}\right)^x = 8$$

$$(2^{-1})^x = 2^3$$

$$2^{-x} = 2^3$$

$$-x = 3$$

$$x = -3$$

Check:

$$\log_{1/2} 8 = -3 \quad \text{since} \quad \left(\frac{1}{2}\right)^{-3} = (2^{-1})^{-3} = 2^3 = 8$$



Graphical check

c) $\log_x(7x - 10) = 2$

We use the definition of logarithm and convert to exponential form:

$$x^2 = 7x - 10$$

Solve the quadratic as usual:

$$x^2 = 7x - 10$$

$$x^2 - 7x + 10 = 0$$

$$(x - 5)(x - 2) = 0$$

$$x = 5 \text{ or } x = 2$$

Check: If $x = 5$, then $\log_x(7x - 10) = \log_5(7 \times 5 - 10) = \log_5(25) = 2$



If $x = 2$, then $\log_x(7x - 10) = \log_2(7 \times 2 - 10) = \log_2(4) = 2$



Graphical check

 Checkpoint Log Equations
More worked examples

3.5 Solving Logarithmic Equations: Homework Exercises

Solve each of the following equations.

1. $\log_4 x = \frac{1}{2}$

2. $\log_{27} x = \frac{1}{3}$

3. $\log(x+3) = 2$

4. $\log(2x-10) = 3$

5. $\log_{\frac{1}{2}} x = 4$

6. $\log_{\frac{1}{4}} x = 2$

7. $\log_2(x-5) = 4$

8. $\log_3(x-1) = 4$

9. $e^{\ln x} = 7$

10. $e^{\ln x} = 9$

11. $\log_{\frac{x}{3}} x = 2$

12. $\log_{\frac{x}{4}} x = 3$

13. $\log(x^2 - 15x) = 2$

14. $\log_4(x^2 - 6x) = 2$

15. $\log x^2 = -2$

16. $\log x^3 = -3$

17. $|\log 2x| = -1$

18. $|\log(x+2)| = 0$

19. $\ln(x^2 - 2) = \ln(-x)$

20. $\ln(x^2 - 27) = \ln(6x)$

21. $\log(8x-7) = \log e$

22. $\log_5(2x^2 - 45) = \log_5 x$

23. $\log_{\frac{1}{2}}|6x-9| = \log_{\frac{1}{2}}|-6|$

24. $\log \frac{x+1}{x+3} - \log x = 0$

CHAPTER 4

Introduction to Rational Functions

4.1 Graphs of Simple Rational Functions

Goals:

- Understand the notion of asymptote
- Be able to accurately graph by hand the graph of the common reciprocal functions
- Graph a reflection of a reciprocal function through the x-axis
- Graph a vertical stretch or shrink of a reciprocal function

Terms to know:

- asymptote
- decreasing function
- domain
- end behavior
- increasing function
- range

Concept Prep: Graphs of Simple Rational Functions

The Graph of $f(x) = \frac{1}{x}$

In the [concept prep](#) exercise, you created various tables of values for different functions. Here we consider the table for the function, f , given by $f(x) = \frac{1}{x}$.

x	$f(x)$
1	1
2	$\frac{1}{2}$
3	$\frac{1}{3}$
5	$\frac{1}{5}$
10	$\frac{1}{10}$
20	$\frac{1}{20}$
30	$\frac{1}{30}$
50	$\frac{1}{50}$
100	$\frac{1}{100}$
1000	$\frac{1}{1000}$

We see that as x gets very large, the outputs get close to 0, since the reciprocal of a very large number is a very small number. (If you cut a pizza into more and more pieces, those pieces get smaller and smaller.) In symbols, we write: as $x \rightarrow \infty$, $f(x) \rightarrow 0$ and we say "as x approaches infinity, $f(x)$ approaches 0."

For positive real numbers, $\frac{1}{\text{BIG NUMBER}} = \text{small number}$

Plot these points on a set of coordinate axes. You should notice that the points are getting very close to the x-axis ($y = 0$) as $x \rightarrow \infty$.

Section 4.1

x	$f(x)$
-1	-1
-2	$-\frac{1}{2}$
-3	$-\frac{1}{3}$
-5	$-\frac{1}{5}$
-10	$-\frac{1}{10}$
-20	$-\frac{1}{20}$
-30	$-\frac{1}{30}$
-50	$-\frac{1}{50}$
-100	$-\frac{1}{100}$
-1000	$-\frac{1}{1000}$

Similarly, if the x values above were all negative, the output values would also be negative, but also getting close to 0. We write, as $x \rightarrow -\infty$, then $f(x) \rightarrow 0$ and we say "as x approaches negative infinity, then $f(x)$ approaches 0."

Plot these points on a set of coordinate axes. You should notice that these points are also getting very close to the x -axis ($y = 0$) as $x \rightarrow -\infty$.

Since $y \rightarrow 0$ as $x \rightarrow \infty$ and as $x \rightarrow -\infty$, we say that the line $y = 0$ is the horizontal **asymptote** of the graph. The **end behavior** (the behavior on the far left and far right side of the graph) determines this horizontal asymptote of a graph.

Horizontal Asymptote

The line $y = 0$ is a **horizontal asymptote** for a function f , if, as the input, x , increases or decreases without bound, the output, $f(x)$, approaches 0. Symbolically, we write:

$$\text{as } x \rightarrow \infty, f(x) \rightarrow 0 \quad \text{or} \quad \text{as } x \rightarrow -\infty, f(x) \rightarrow 0$$

Symbol**Meaning**

$x \rightarrow -\infty$	x decreases without bound, "x approaches negative infinity"
$x \rightarrow \infty$	x increases without bound, "x approaches infinity"
$f(x) \rightarrow h$	The function value tends toward h , " $f(x)$ approaches h "

You might notice that the same phenomenon occurs for the function g given by $g(x) = \frac{1}{x^2}$.

As $x \rightarrow \infty$, $g(x) \rightarrow 0$ (as x approaches infinity $g(x)$ approaches 0) and as $x \rightarrow -\infty$, $g(x) \rightarrow 0$. Thus, the line $y = 0$ is the horizontal asymptote of the graph. The line $y = 0$ is the horizontal asymptote of the graph of a simple rational function of the form $f(x) = \frac{1}{x^n}$, for any n . You might convince yourself of this fact by choosing $n = 3$, $n = 4$, $n = 5$, etc.

Now, we consider the output values when x gets close to zero. Since f is undefined at 0, we consider values of x very close, but not equal, to 0. Values of x can approach 0 from either the right side (where the numbers are positive) or from the left side (where the numbers are negative).

Section 4.1

The notation $x \rightarrow 0^+$ means x is approaching 0 from the **right** or positive side and the notation $x \rightarrow 0^-$ means x is approaching 0 from the **left** or negative side. First we consider values of $f(x)$ when $x \rightarrow 0^+$.

x	$f(x)$
1	1
$\frac{1}{2}$	2
$\frac{1}{3}$	3
$\frac{1}{4}$	4
$\frac{1}{5}$	5
$\frac{1}{10}$	10
$\frac{1}{20}$	20
$\frac{1}{30}$	30
$\frac{1}{50}$	50
$\frac{1}{100}$	100
$\frac{1}{1000}$	1000

Note that the values $f(x)$ are getting very large in the positive direction. Symbolically, we write $f(x) \rightarrow \infty$. So, as $x \rightarrow 0^+$, $f(x) \rightarrow \infty$

For positive real numbers, $\frac{1}{\text{small number}} = \text{BIG NUMBER}$

Try graphing the points in this table. As the x -coordinates gets very close to 0, the y -coordinates increase without bound -- they get bigger and bigger. The line $x = 0$ is what mathematicians call a **vertical asymptote**. The word "asymptote" comes from Greek roots meaning "not falling together." The graph of a function and its vertical asymptote(s) do not meet.

Similarly, if x gets closer and closer to 0 from the negative side, the output values would tend toward negative infinity. Thus in symbols we write, as $x \rightarrow 0^-$ $f(x) \rightarrow -\infty$.

Symbol	Meaning
$x \rightarrow h^+$	x approaches h from the right side.
$x \rightarrow h^-$	x approaches h from the left side.

Vertical Asymptote

The line $x = 0$ is a vertical asymptote for a function, f , if as x approaches 0, $f(x)$ increases or decreases without bound:

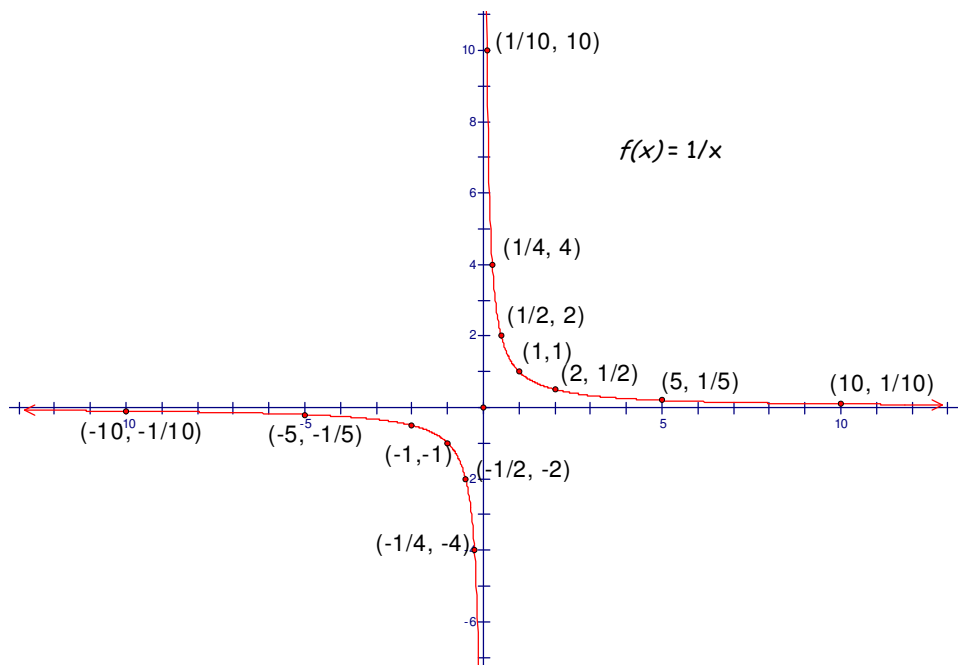
$$\text{as } x \rightarrow 0^+, f(x) \rightarrow \pm\infty \quad \text{or} \quad \text{as } x \rightarrow 0^-, f(x) \rightarrow \pm\infty$$

Note that the vertical asymptote occurs where the function is undefined. The **domain** of f , given by $f(x) = \frac{1}{x}$, is the set of all real numbers except 0. We write $\{x \mid x \neq 0\}$ or

Section 4.1

$(-\infty, 0) \cup (0, \infty)$ to indicate this domain.

Graphing the above points on the coordinate plane and connecting them results in the following graph:



The graph of $f(x) = \frac{1}{x}$

The range of f is $\{y \mid y \neq 0\}$, that is, all real numbers except 0. The function is decreasing over its entire domain, that is, the values $f(x)$ are decreasing over $(-\infty, 0) \cup (0, \infty)$.

Notice that the graph has no x -intercepts or y -intercepts. Suppose we try to find these:

To find the x -intercept, set $y=0$:

$$y = f(x) = \frac{1}{x} = 0$$

or $\frac{1}{x} = 0$, which is the same as

$$\frac{1}{x} = \frac{0}{1}. \quad \text{Cross multiply to obtain}$$

$$1 \cdot 1 = x \cdot 0.$$

So $1 = 0$, which is not true. Thus, the graph has no x -intercepts

Section 4.1

To find the y-intercept, set $x = 0$:

$f(0) = \frac{1}{0}$, which is undefined. Thus the graph has no y-intercept.

Checkpoint 4.1 A

The Graph of $g(x) = \frac{1}{x^2}$

Complete the following tables of values for the function g given by $g(x) = \frac{1}{x^2}$.

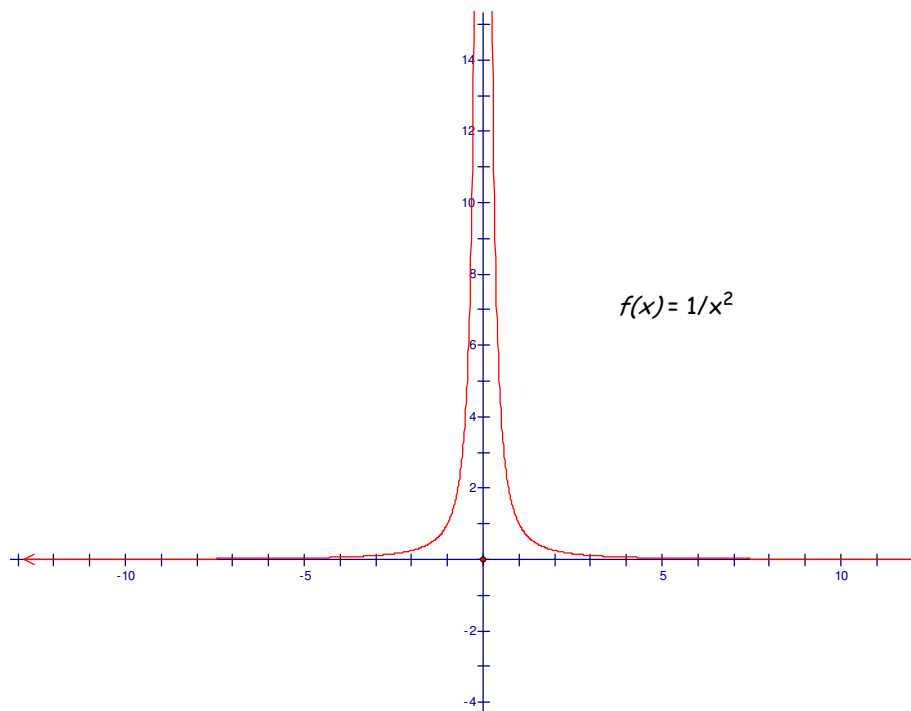
x	$g(x)$	x	$g(x)$	x	$g(x)$
1		-1		1	
2		-2		$\frac{1}{2}$	
5		-5		$\frac{1}{5}$	
10		-10		$\frac{1}{10}$	
50		-50		$\frac{1}{50}$	
100		-100		$\frac{1}{100}$	
1000		-1000		$\frac{1}{1000}$	

Complete the following:

- as $x \rightarrow \infty$, $g(x) \rightarrow ??$
- as $x \rightarrow -\infty$, $g(x) \rightarrow ??$ What is the horizontal asymptote?
- as $x \rightarrow ??$, $g(x) \rightarrow \infty$
- as $x \rightarrow ??$, $g(x) \rightarrow \infty$ What is the vertical asymptote?

When sketching the above points and connecting them, your graph should look something like this:

Section 4.1



How is this graph of the function g similar to the graph of $f(x) = \frac{1}{x}$? How is it different?

The domain of g is $\{x \mid x \neq 0\}$.

The range is $\{y \mid y > 0\}$ or $(0, \infty)$.

The vertical asymptote is $x = 0$.

The horizontal asymptote is $y = 0$.

The function is increasing on $(-\infty, 0)$ and decreasing on $(0, \infty)$.

There are no x -intercepts and no y -intercepts.

Checkpoint 4.1 B

The Effect of Multiplying by a Constant, c

Recall in section 2.3 that multiplying an exponential function by a factor of c resulted in a vertical stretch of the graph. Let's take a look at what happens when we multiply a simple rational function

by a constant, c . We first compare tables of function values. Let $f(x) = \frac{1}{x}$ and

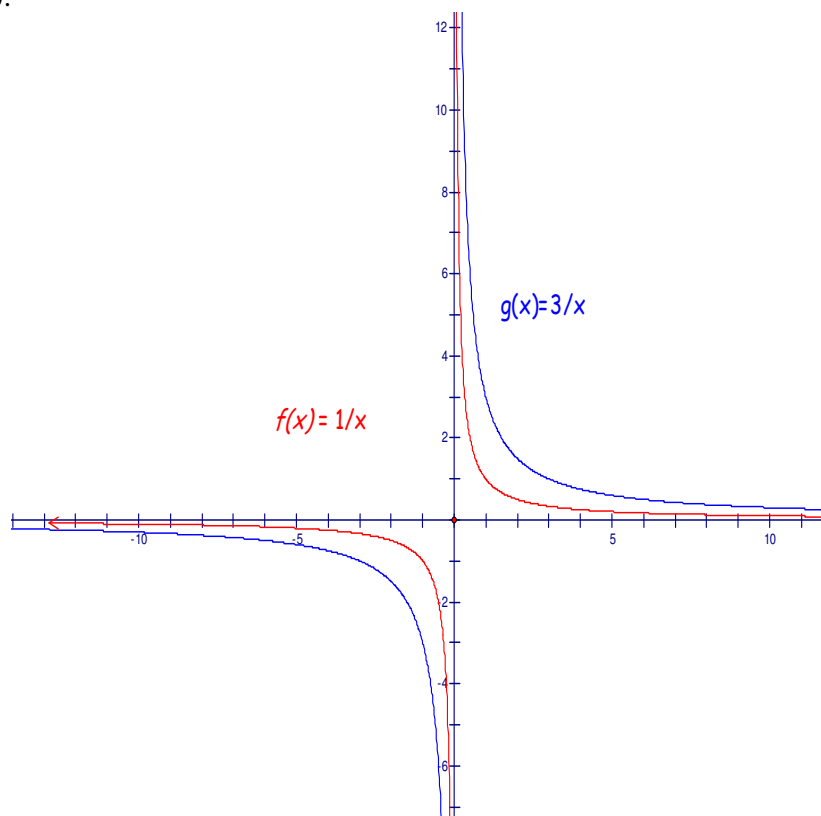
let $g(x) = 3 \cdot \frac{1}{x} = \frac{3}{x}$.

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x	$f(x) = \frac{1}{x}$	$g(x) = \frac{3}{x}$
1	1	3
2	$\frac{1}{2}$	$\frac{3}{2}$
3	$\frac{1}{3}$	1
10	$\frac{1}{10}$	$\frac{3}{10}$
20	$\frac{1}{20}$	$\frac{3}{20}$
50	$\frac{1}{50}$	$\frac{3}{50}$
100	$\frac{1}{100}$	$\frac{3}{100}$

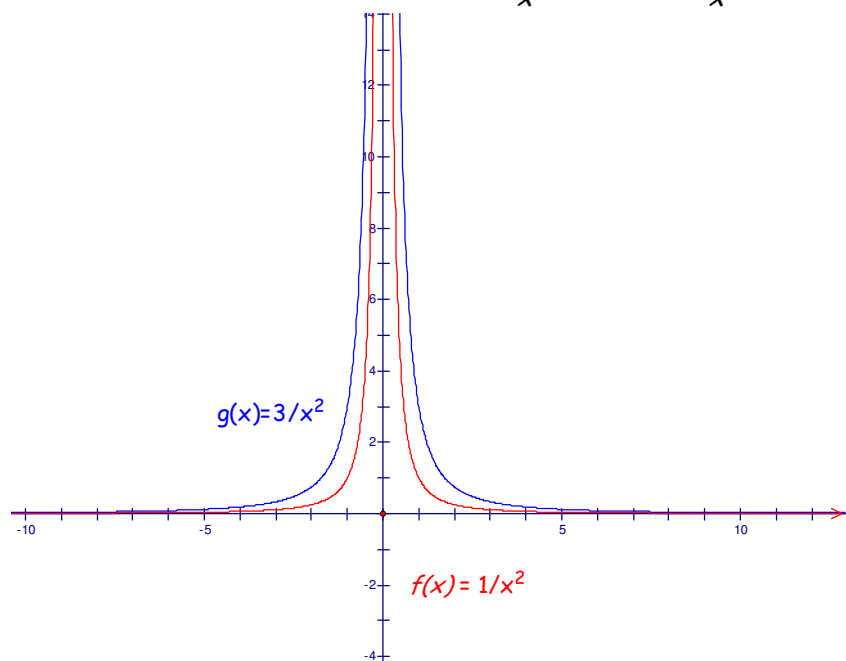
x	$f(x) = \frac{1}{x}$	$g(x) = \frac{3}{x}$
1	1	3
$\frac{1}{2}$	2	6
$\frac{1}{3}$	3	9
$\frac{1}{10}$	10	30
$\frac{1}{20}$	20	60
$\frac{1}{50}$	50	150
$\frac{1}{100}$	100	300

Since each output of the function g is three times the output of f , the graph of g is the graph of f stretched by a factor of 3. Imagine holding the graph of f at the upper and lower ends and pulling it (stretching it). The values of g decrease more slowly as $x \rightarrow \infty$ and rise faster as $x \rightarrow 0^+$. The vertical and horizontal asymptotes are the same for the graphs of both functions. Carefully compare the plotted point for f and g so that you understand why they graphs appear as they do in the sketch below.

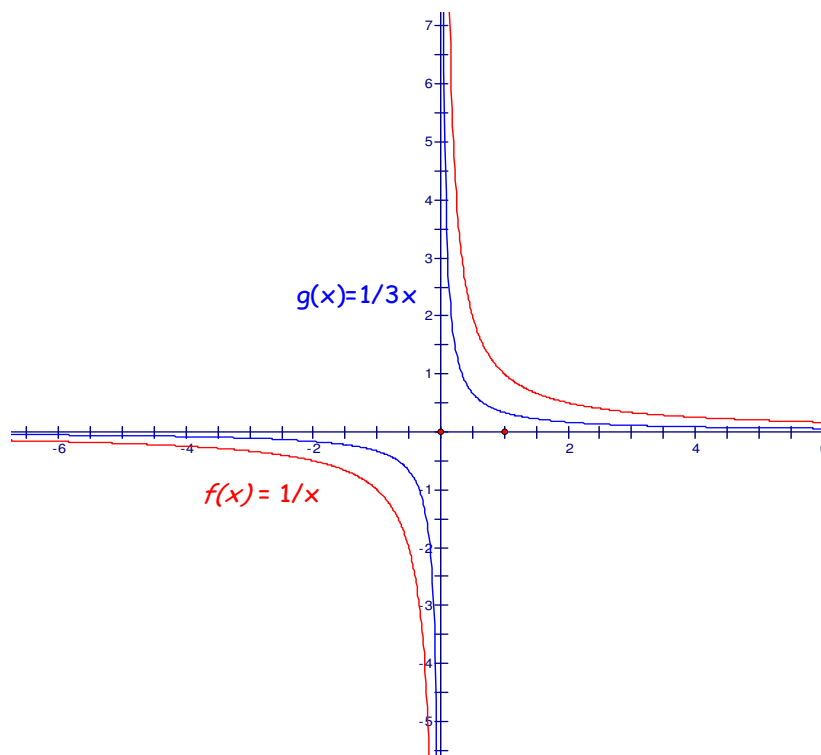


Section 4.1

Note a similar comparison between the graphs of $g(x) = \frac{1}{x^2}$ and $h(x) = \frac{3}{x^2}$.



Using the same reasoning, let's suppose $0 < c < 1$. Consider for example the functions f and g given by $f(x) = \frac{1}{x}$ and $g(x) = \frac{1}{3} \cdot \frac{1}{x} = \frac{1}{3x}$. Noting that the outputs of g are *one-third* the outputs of f , we can compare the two graphs:



Vertical Stretch or Compression

Given a simple rational function, f , and a new function g such that $g(x) = c \cdot f(x)$, then:

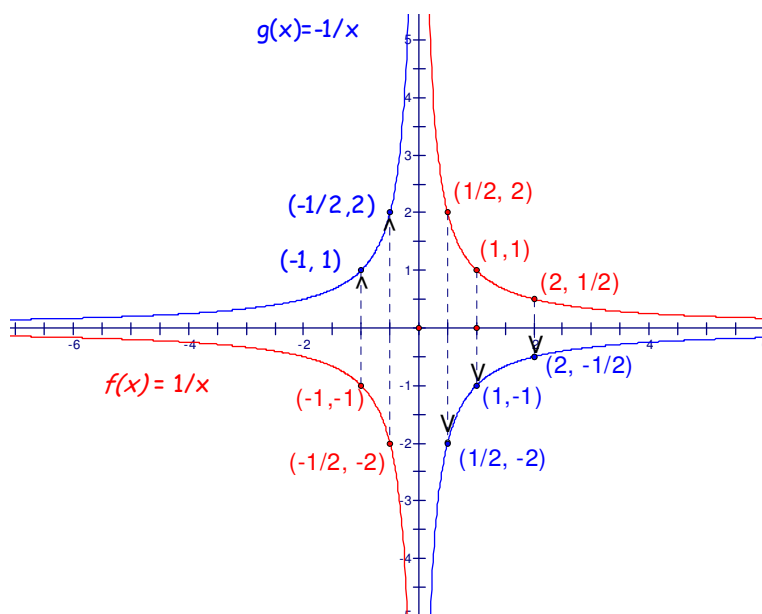
- If $c > 1$, then the graph of g is a **vertical stretch** of the graph of f by a factor of c .
- If $0 < c < 1$, then the graph of g is a **vertical compression** of the graph of f by a factor of c .

What if $c < 0$?

How would a negative value of c affect a graph? Consider the function f given by $f(x) = \frac{1}{x}$ and the function g given by $g(x) = -1 \cdot f(x) = -\frac{1}{x}$. Completing the table below clearly gives outputs for g which are opposite the outputs for f . Before reading on, make your own sketch of the graph of $g(x) = -f(x) = -\frac{1}{x}$.

x	$f(x) = \frac{1}{x}$	$g(x) = -\frac{1}{x}$
1	1	-1
2	$\frac{1}{2}$	$-\frac{1}{2}$
3	$\frac{1}{3}$	$-\frac{1}{3}$
5	$\frac{1}{5}$	$-\frac{1}{5}$
10	$\frac{1}{10}$	$-\frac{1}{10}$
20	$\frac{1}{20}$	$-\frac{1}{20}$
30	$\frac{1}{30}$	$-\frac{1}{30}$
50	$\frac{1}{50}$	$-\frac{1}{50}$
100	$\frac{1}{100}$	$-\frac{1}{100}$
1000	$\frac{1}{1000}$	$-\frac{1}{1000}$

Since each output of the function g is opposite times the output of f , the graph of g is the graph of f reflected through the x -axis or "flipped upside down."

**Reflecting a graph**

Given a function, f , and another function g given by $g(x) = -f(x)$. The graph of g is a **reflection through the x -axis** of the graph of f .

[More Worked Examples](#)

4.1 Graphs of Simple Rational Functions - Homework Exercises

Characterize the graph of each of the following by naming

- the domain
- the end behavior: As $x \rightarrow \infty, y \rightarrow ?$ and as $x \rightarrow -\infty, y \rightarrow ?$
- the asymptotes
- the intercepts

Use this information to sketch the graph *by hand*, finding at least one additional point on either side of the vertical asymptote. Then name

- the range and
- the interval(s) over which the function is increasing and/or decreasing.

Use your graphing calculator as a check only.

$$1. f(x) = \frac{1}{x^3}$$

$$2. f(x) = \frac{1}{x^5}$$

$$3. f(x) = \frac{1}{x^4}$$

$$4. f(x) = \frac{1}{x^6}$$

$$5. f(x) = \frac{5}{x}$$

$$6. f(x) = \frac{8}{x}$$

$$7. f(x) = \frac{2}{x}$$

$$8. f(x) = \frac{4}{x}$$

$$9. f(x) = \frac{3}{x^2}$$

$$10. f(x) = \frac{10}{x^2}$$

$$11. f(x) = \frac{5}{x^2}$$

$$12. f(x) = \frac{4}{x^2}$$

$$13. f(x) = \frac{4}{x^3}$$

$$14. f(x) = \frac{5}{x^3}$$

$$15. f(x) = \frac{1}{4x}$$

$$16. f(x) = \frac{1}{5x}$$

$$17. f(x) = \frac{1}{2x}$$

$$18. f(x) = \frac{1}{6x}$$

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19. $f(x) = \frac{1}{2x^2}$

20. $f(x) = \frac{1}{3x^2}$

21. $f(x) = \frac{1}{4x^2}$

22. $f(x) = \frac{1}{5x^2}$

23. $f(x) = -\frac{1}{x}$

24. $f(x) = -\frac{3}{x}$

25. $f(x) = -\frac{2}{x}$

26. $f(x) = -\frac{5}{x}$

27. $f(x) = -\frac{1}{x^2}$

28. $f(x) = -\frac{3}{x^2}$

29. For what values of x is $\frac{1}{x} > \frac{1}{x^2}$?

30. For what values of x is $\frac{1}{x} < \frac{1}{x^2}$?

31. For what values of x is $\frac{1}{x} > \frac{1}{x^3}$?

32. For what values of x is $\frac{1}{x} < \frac{1}{x^3}$?

33. For what values of x is $\frac{3}{x} > \frac{1}{x}$?

34. For what values of x is $\frac{3}{x} < \frac{1}{x}$?

35. For what values of x is $-\frac{1}{x} > \frac{1}{x}$?

36. For what values of x is $-\frac{1}{x} < \frac{1}{x}$?

ANSWERS TO ODD NUMBERED PROBLEMS**Section 4.2****4.2 More About Asymptotes**

Goals:

- Given a formula for a rational function, the student will be able to name the vertical asymptote(s) of its graph
- Graph a transformation of a simple rational function using vertical or horizontal shifts
- Understands the effects of shifting on the asymptotes of the graph

Terms to know:

- asymptote
- domain of a function
- function
- rational function

Prep assignment

The Effect of Adding a Constant, k

You may recall from [section 2.3](#) that adding a constant to an exponential function resulted in a vertical shift of the original graph. For example, the graph of $y = 2^x + 3$ was a vertical shift up 3 of the graph of $y = 2^x$. The graph of $y = 5^x - 2$ was a shift down 2 of the graph of $y = 5^x$. Let's explore the effect of adding a constant to a simple rational function. We begin by comparing a table of values for the reciprocal functions, $f(x) = \frac{1}{x}$ and $g(x) = \frac{1}{x} + 3$. First consider $x \rightarrow \infty$, that is, x is approaching positive infinity.

x	$f(x) = \frac{1}{x}$	$g(x) = \frac{1}{x} + 3$
1	1	$1+3=4$
2	$\frac{1}{2}$	$\frac{1}{2} + 3 = 3\frac{1}{2}$
5	$\frac{1}{5}$	$3\frac{1}{5}$
10	$\frac{1}{10}$	$3\frac{1}{10}$
20	$\frac{1}{20}$	$3\frac{1}{20}$
50	$\frac{1}{50}$	$3\frac{1}{50}$
100	$\frac{1}{100}$	$3\frac{1}{100}$
1000	$\frac{1}{1000}$	$3\frac{1}{1000}$

We notice that as $x \rightarrow \infty$, $g(x)$ seems to be approaching a value of 3, since the fractional part of $g(x)$ is getting smaller and smaller. Since we are adding 3 to $\frac{1}{x}$ for each output value, and since $\frac{1}{x} \rightarrow 0$ as $x \rightarrow \infty$, we have $g(x) \rightarrow 3$. In a similar fashion, as $x \rightarrow -\infty$, $g(x) \rightarrow 3$, that is, as x approaches negative infinity, $g(x)$ approaches 3. (You might complete a table of values for negative values of x if you are not convinced.) Since this end behavior determines the horizontal asymptote, the horizontal asymptote of the graph of $g(x) = \frac{1}{x} + 3$ is $y = 3$.

Now let's examine what happens to the values $g(x)$ as $x \rightarrow 0^+$, that is, as x approaches 0 from the right (positive) side.

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x	$f(x) = \frac{1}{x}$	$g(x) = \frac{1}{x} + 3$
1	1	4
$\frac{1}{2}$	2	5
$\frac{1}{5}$	5	8
$\frac{1}{10}$	10	13
$\frac{1}{20}$	20	23
$\frac{1}{50}$	50	53
$\frac{1}{100}$	100	103
$\frac{1}{1000}$	1000	1003

Since $f(x) \rightarrow \infty$ as $x \rightarrow 0^+$, clearly adding 3 to each output will result in $g(x) \rightarrow \infty$. The chart above confirms this. Similarly, as $x \rightarrow 0^-$, that is, as x approaches 0 from the left (negative) side, we note that $g(x) \rightarrow -\infty$. Thus the vertical asymptote of the graph of $g(x) = \frac{1}{x} + 3$ is $x = 0$, the same as it is for the graph of $f(x) = \frac{1}{x}$.

In conclusion, the horizontal asymptote of the graph of $g(x) = \frac{1}{x} + 3$ is $y = 3$ and the vertical asymptote is $x = 0$.

[Checkpoint 4.2 A](#)

Asymptotes for $f(x) = \frac{1}{x} + k$

The **horizontal asymptote** of a rational function of the form $f(x) = \frac{1}{x} + k$ is $y = k$ since as $x \rightarrow \infty$, $y \rightarrow k$ and as $x \rightarrow -\infty$, $y \rightarrow k$.

The **vertical asymptote** is $x = 0$ since as $x \rightarrow 0^+$ or $x \rightarrow 0^-$, $y \rightarrow \infty$ or $y \rightarrow -\infty$.

Example 1. Name the asymptotes of the graph of $g(x) = \frac{1}{x} + 8$.

Note that 8 is added on to the output

The horizontal asymptote is $y = 8$.

of $f(x) = \frac{1}{x}$,

which has $y = 0$ as a horizontal asymptote.

Section 4.2

Note that as $x \rightarrow 0^+$ or as $x \rightarrow 0^-$, $y \rightarrow \infty$ or $y \rightarrow -\infty$. At $x = 0$, g is undefined.

The vertical asymptote is $x = 0$.

Example 2. Name the asymptotes of the graph of $h(x) = \frac{3}{x} - 4$

Note that 4 is subtracted from the output of $f(x) = \frac{3}{x}$, which has $y = 0$ as a horizontal asymptote.

The horizontal asymptote is $y = -4$.

Note that as $x \rightarrow 0^+$ or as $x \rightarrow 0^-$, $y \rightarrow \infty$ or $y \rightarrow -\infty$. At $x = 0$, h is undefined.

The vertical asymptote is $x = 0$.

Example 3. Name the asymptotes of the graph of $r(x) = \frac{5}{x^2} + 1$

Note that 1 is added to the output of $f(x) = \frac{5}{x^2}$, which has $y = 0$ as a horizontal asymptote.

The horizontal asymptote is $y = 1$.

Note that f as $x \rightarrow 0^+$ or as $x \rightarrow 0^-$, $y \rightarrow \infty$ or $y \rightarrow -\infty$. At $x = 0$, r is undefined.

The vertical asymptote is $x = 0$.

Example 4. Name the asymptotes of the graph of $s(x) = -\frac{3}{x^2} - 7$

Note that 7 is subtracted from the output of $f(x) = -\frac{3}{x^2}$, which has $y = 0$ as a horizontal asymptote.

The horizontal asymptote is $y = -7$.

Note that as $x \rightarrow 0^+$ or as $x \rightarrow 0^-$, $y \rightarrow \infty$ or $y \rightarrow -\infty$. At $x = 0$, s is undefined.

The vertical asymptote is $x = 0$.

Example 5. Name the asymptotes of the graph of $m(x) = \frac{5}{x^3} + 10$

Note that 10 is added to the output of $f(x) = \frac{5}{x^3}$, which has $y = 0$ as an asymptote.

The horizontal asymptote is $y = 10$.

Section 4.2

Note that as $x \rightarrow 0^+$ or as $x \rightarrow 0^-$, $y \rightarrow \infty$
or $y \rightarrow -\infty$. At $x = 0$, s is undefined.

The vertical asymptote is $x = 0$.

Checkpoint 4.2 B*Finding the Domain of a Rational Function*

To find the domain of a rational function, consider the values of the input, x , which make the function undefined. The domain is all real numbers EXCEPT these values.

Example 6. Find the domain of the function f given by $f(x) = \frac{1}{x-3}$

Set the denominator equal to 0 and solve:

$$\begin{aligned}x - 3 &= 0 \\x &= 3\end{aligned}$$

The domain is all real numbers EXCEPT this value.

So, the domain is all real numbers except 3.

Write the answer in set notation or as an interval

$$\{x \mid x \neq 3\} \quad \text{OR} \quad (-\infty, 3) \cup (3, \infty)$$

Example 7. Find the domain of the function g given by $g(x) = \frac{5}{x+1} + 8$

Set the denominator equal to 0 and solve:

$$\begin{aligned}x + 1 &= 0 \\x &= -1\end{aligned}$$

The domain is all real numbers EXCEPT this value.

So, the domain is all real numbers except -1.

Write the answer in set notation or as an interval

$$\{x \mid x \neq -1\} \quad \text{OR} \quad (-\infty, -1) \cup (-1, \infty)$$

Example 8. Find the domain of the function h given by $h(x) = \frac{3}{x^2 - 2x - 15} - 7$

Set the denominator equal to 0 and solve:

$$\begin{aligned}x^2 - 2x - 15 &= 0 \\(x - 5)(x + 3) &= 0 \\ \text{so } x &= 5 \text{ OR } x = -3\end{aligned}$$

The domain is all real numbers EXCEPT this value.

So, the domain is all real numbers except 5 and -3.

Write the answer in set notation or as an interval

$$\begin{aligned}\{x \mid x \neq 5 \text{ and } x \neq -3\} \quad \text{OR} \\(-\infty, -3) \cup (-3, 5) \cup (5, \infty)\end{aligned}$$

Vertical Asymptotes Revisited

Recall from [section 4.1](#) that the vertical asymptote of a reciprocal function, f , given by $f(x) = \frac{c}{x}$ or $f(x) = \frac{c}{x^2}$ was the line $x = 0$, the y -axis. In this section we explore vertical asymptotes of rational functions of the form $f(x) = \frac{c}{x-h}$ and $f(x) = \frac{c}{x-h} + k$. We first consider the function f given by $f(x) = \frac{1}{x-3}$.

We noted above that this function is undefined at $x = 3$. In the chart below, we consider the values of the function as x approaches 3 from the right. We start with 4 as the input, then gradually move closer and closer to 3. You might convince yourself that the output values are indeed correct.

x	$f(x)$
4	1
$3\frac{1}{2}$	2
$3\frac{1}{4}$	4
$3\frac{1}{5}$	5
$3\frac{1}{10}$	10
$3\frac{1}{20}$	20
$3\frac{1}{50}$	50
$3\frac{1}{100}$	100
$3\frac{1}{1000}$	1000

We see that the outputs are increasing without bound, that is, $y \rightarrow \infty$. If we make a similar chart with the inputs approaching 3 from the left, with x values of $2, 2\frac{1}{2}, 2\frac{3}{4}, 2\frac{9}{10}, 2\frac{99}{100}$ for example, we notice that the outputs are decreasing without bound, that is, $y \rightarrow -\infty$. You might make such a chart and convince yourself that this is true.

So now we have $y \rightarrow \infty$ as $x \rightarrow 3^+$ and $y \rightarrow -\infty$ as $x \rightarrow 3^-$. Thus, the line $x = 3$ is the vertical asymptote of the graph of f . We generalize the definition of vertical asymptote from section 4.1:

Vertical Asymptote

Given a constant, h , the line $x = h$ is a vertical asymptote for a function, f , if as x approaches h , $f(x)$ increases or decreases without bound:

$$\text{as } x \rightarrow h^+, f(x) \rightarrow \pm\infty \quad \text{or} \quad \text{as } x \rightarrow h^-, f(x) \rightarrow \pm\infty$$

If we make similar tables for functions f of the form $f(x) = \frac{1}{x-h}$, we note that $f(x) \rightarrow \infty$ as $x \rightarrow h^+$ and $f(x) \rightarrow -\infty$ as $x \rightarrow h^-$. Click [here](#) for more such examples.

Vertical Asymptote of a Rational Function

For a rational function of the form $f(x) = \frac{1}{x-h}$, the line $x = h$ is a vertical asymptote.

Thus, the vertical asymptote of the graph of g in example 7 above is $x = -1$. The function h in example 8 above has 2 vertical asymptotes: $x = 5$ and $x = -3$

[Checkpoint 4.2 C](#)*Putting it all together: Graphing a Rational Function*

Example 9. Name the domain and asymptotes of the function h given by $h(x) = \frac{8}{x-4}$. Find the intercepts, then graph the function and name the range. Also identify the values of x for which the function is increasing and the values of x for which it is decreasing.

Set the denominator equal to 0 to find the domain:

$$\begin{aligned} x - 4 &= 0 \\ x &= 4 \end{aligned}$$

So the domain is $\{x \mid x \neq 4\}$ or $(-\infty, 4) \cup (4, \infty)$

The domain provides us with information about the vertical asymptote:

$x = 4$ is the vertical asymptote

The end behavior provides information about the horizontal asymptote:

As $x \rightarrow \infty$, $y \rightarrow 0$ AND as $x \rightarrow -\infty$, $y \rightarrow 0$, the line $y = 0$ is the horizontal asymptote.

Find the y-intercept by setting $x = 0$

$h(0) = \frac{8}{0-4} = -\frac{8}{4} = -2$, so the point $(0, -2)$ is on the graph

Section 4.2

Find the x -intercept by setting
 $y = h(x) = 0$

$$0 = \frac{8}{x-4}. \quad \text{Cross multiply to obtain:}$$

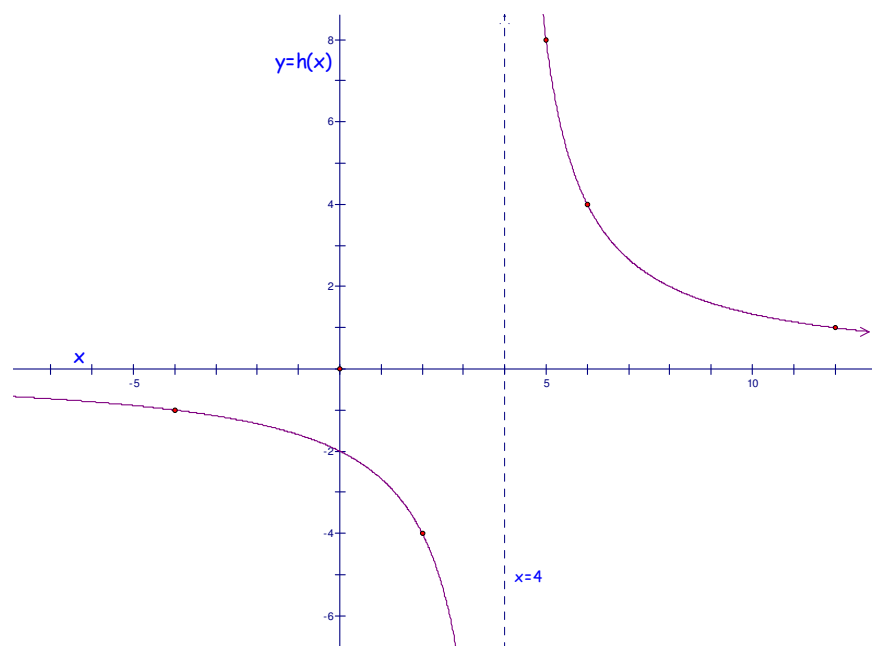
$$0 = 8, \quad \text{which is false.}$$

Thus there are NO x -intercepts

Finding a few extra points will make our graph more accurate. Picking a few points on either side of the vertical asymptote(s) is a good idea.

x	-4	2	5	6	12
y	-1	-4	8	4	1

Plotting the above, we obtain:



$$\text{The graph of } h(x) = \frac{8}{x-4}$$

Note that the range (the set of all outputs) is the set of all real numbers except 0. The function is decreasing over its entire domain, that is it is decreasing over $(-\infty, 4) \cup (4, \infty)$.

Example 10. Name the domain and asymptotes of the function h given by $g(x) = \frac{1}{x+3} - 2$. Find the intercepts, then graph the function and name the range. Also identify the values of x for which the function is increasing and the values of x for which it is decreasing.

Set the denominator equal to 0 to find the domain:

$$x + 3 = 0$$

$$x = -3$$

So the domain is $\{x \mid x \neq -3\}$ or $(-\infty, -3) \cup (-3, \infty)$

Section 4.2

The domain provides us with information about the vertical asymptote:

$x = -3$ is the vertical asymptote

The end behavior provides information about the horizontal asymptote:

As $x \rightarrow \infty$, $y \rightarrow -2$ AND as $x \rightarrow -\infty$, $y \rightarrow -2$, the line $y = -2$ is the horizontal asymptote.

Find the y-intercept by setting $x = 0$

$g(0) = \frac{1}{0+3} - 2 = \frac{1}{3} - 2 = -\frac{5}{3}$, so the point $\left(0, -\frac{5}{3}\right)$ is on the graph.

Find the x-intercept by setting $y = g(x) = 0$

$$0 = \frac{1}{x+3} - 2. \text{ Add 2 to both sides:}$$

$$2 = \frac{1}{x+3} \quad \text{Cross multiply to obtain:}$$

$$2(x+3) = 1 \quad \text{Solve:}$$

$$2x = -5$$

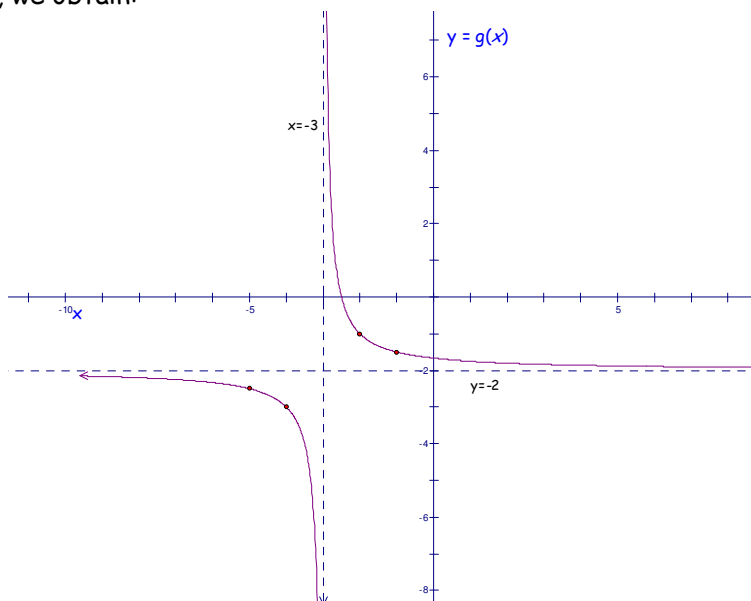
$$x = -\frac{5}{2}$$

So the point $\left(-\frac{5}{2}, 0\right)$ is on the graph.

Finding a few extra points will make our graph more accurate. Picking a few points on either side of the vertical asymptote is a good idea.

x	-1	-2	-4	-5
y	-1.5	-1	-3	$-\frac{1}{2}$

Plotting the above, we obtain:



The graph of $g(x) = \frac{1}{x+3} - 2$

Section 4.2

Note that the range (the set of all outputs) is the set of all real numbers except -2 , that is $\{y \mid y \neq -2\}$. The function is decreasing over its entire domain, that is it is decreasing over $(-\infty, -3) \cup (-3, \infty)$.

Example 11. Name the domain and asymptotes of the function h given by $f(x) = \frac{1}{(x-2)^2} - 3$.

Find the intercepts, then graph the function and name the range. Also identify the values of x for which the function is increasing and the values of x for which it is decreasing.

Set the denominator equal to 0 to find the domain:

$$x - 2 = 0$$

$$x = 2$$

So the domain is $\{x \mid x \neq 2\}$ or $(-\infty, 2) \cup (2, \infty)$

The domain provides us with information about the vertical asymptote:

$x = 2$ is the vertical asymptote

The end behavior provides information about the horizontal asymptote:

As $x \rightarrow \infty$, $y \rightarrow -3$ AND as $x \rightarrow -\infty$, $y \rightarrow -3$, the line $y = -3$ is the horizontal asymptote.

Find the y-intercept by setting $x = 0$

$$f(0) = \frac{1}{(0-2)^2} - 3 = \frac{1}{4} - 3 = -\frac{11}{4} = -2\frac{3}{4}, \text{ so the point}$$

$\left(0, -\frac{11}{4}\right)$ is on the graph.

Find the x-intercept(s) by setting $y = f(x) = 0$

$$0 = \frac{1}{(x-2)^2} - 3. \text{ Add 3 to both sides:}$$

$$3 = \frac{1}{(x-2)^2} \quad \text{Cross multiply to obtain:}$$

$$3(x-2)^2 = 1 \quad \text{Solve:}$$

$$(x-2)^2 = \frac{1}{3}$$

$$x-2 = \pm\sqrt{\frac{1}{3}}$$

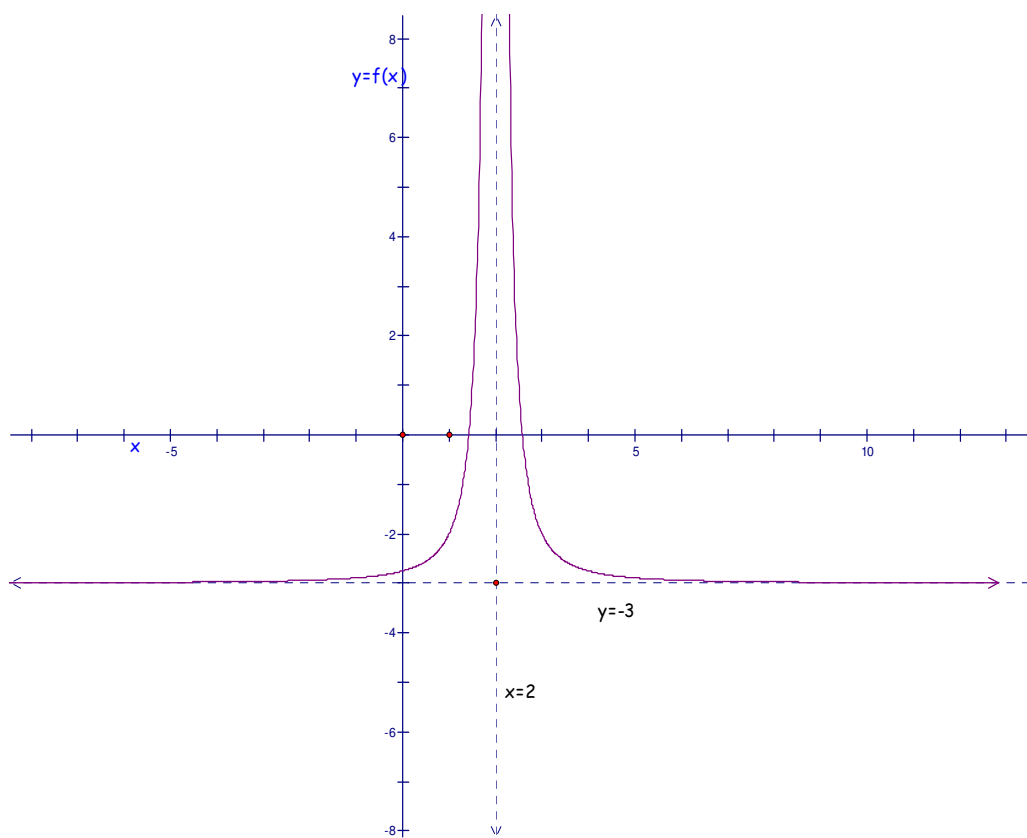
$$x = 2 \pm \sqrt{\frac{1}{3}} = 2 \pm \frac{\sqrt{3}}{3},$$

so $x \approx 2.6$ or $x \approx 1.4$ are the x-intercepts

Picking a few points on either side of the vertical asymptote is a good idea

x	1	-2	3	4
y	-2	≈ -2.9	-2	-2.75

Section 4.2



The graph of $f(x) = \frac{1}{(x-2)^2} - 3$

The range is $\{y \mid y > 0\}$. The function is increasing over $(-\infty, 2)$ and decreasing over $(2, \infty)$.

[More worked examples](#)

4.2 More About Asymptotes - Homework Exercises

Determine the horizontal and vertical asymptotes of the graph of the functions represented in #1-10

1. $f(x) = \frac{1}{x+1} - 2$

2. $y = \frac{1}{x^2} + 5$

3. $g(x) = \frac{1}{(x-2)^2} - 5$

4. $f(x) = 1 + \frac{1}{(x+3)^2}$

5. $y = \frac{1}{(x-1)^3} + 3$

6. $f(x) = -2 + \frac{1}{(x+2)^3}$

7. $f(x) = 4 + \frac{1}{(x+2)^2}$

8. $y = \frac{1}{(x-5)^2} - 3$

9. $g(x) = -3 + \frac{1}{(x+6)^6}$

10. $f(x) = \frac{1}{(x+7)^5} + 7$

For # 11-36, characterize the graph of the given function, naming

g) the domain

h) the end behavior: As $x \rightarrow \infty, y \rightarrow ?$ and as $x \rightarrow -\infty, y \rightarrow ?$

i) the asymptotes and

j) intercepts.

Use this information to sketch the graph *by hand*, finding at least one additional point on either side of the vertical asymptote. Be sure to show all asymptotes with a dashed line. Then name

k) the range and

l) the interval(s) over which the function is increasing and/or decreasing.

11. $f(x) = \frac{1}{x-4}$

12. $f(x) = \frac{1}{x+3}$

13. $f(x) = \frac{1}{x+5}$

14. $y = \frac{1}{x-1}$

Section 4.2

15. $f(x) = \frac{1}{\left(x - \frac{5}{2}\right)^2}$

16. $g(x) = \frac{1}{(x+1)^2}$

17. $y = \frac{1}{(\pi + x)^2}$

18. $f(x) = \frac{1}{\left(x - \frac{7}{2}\right)^2}$

19. $f(x) = \frac{1}{x} - 5$

20. $h(x) = \frac{1}{x} + 3$

21. $y = \frac{1}{x^2} + \frac{9}{4}$

22. $f(x) = \frac{1}{x^2} - 2.4$

23. $f(x) = -8 + \frac{1}{x^2}$

24. $y = 6 + \frac{1}{x^2}$

25. $g(x) = \frac{1}{x^3} - \sqrt{10}$

26. $y = \sqrt{5} + \frac{1}{x^3}$

27. $f(x) = \frac{1}{x-2} + 3$

28. $y = 4 + \frac{1}{x+2}$

29. $f(x) = \frac{1}{x+4} - 1$

30. $y = -3 + \frac{1}{x-5}$

31. $g(x) = \frac{1}{(x+4)^2} - 5$

32. $y = 6 + \frac{1}{(3+x)^2}$

33. $h(x) = 4 + \frac{1}{(x-3)^3}$

34. $y = 3 + \frac{1}{(\sqrt{2} + x)^3}$

35. $f(x) = \frac{1}{(x-2)^3} - \frac{10}{3}$

36. $g(x) = -4.5 + \frac{1}{(x+8)^3}$

ANSWERS TO ODD NUMBERED PROBLEMS

4.3 Different Forms of Rational Functions

Goals: Students will be able to:

- Combine terms and express a function in rational form as a ratio of polynomials
- Name the horizontal asymptote of the graph of a rational function given as the ratio of 2 polynomials with like degree

Terms to know:

- asymptote
- domain of a function
- function
- leading coefficient
- least common denominator
- rational function

Prep assignment: Combining Fractions

Prep assignment: Finding the LCD

Changing the Form of a Rational Function

In this section we review the technique of combining terms of a rational function.

Example 1. Combine terms in the function g and express it as a ratio of polynomials:

$$g(x) = \frac{1}{x+1} - 2$$

Rewrite the constant term with a denominator of 1, then find the common denominator.

$$g(x) = \frac{1}{x+1} - \frac{2}{1}$$

The common denominator is $(x+1)$

Multiply the constant term by 1, that is multiply the constant term by the least common denominator
least common denominator

$$g(x) = \frac{1}{x+1} - \frac{2(x+1)}{1(x+1)}$$

Expand using the distributive property, write using single denominator, and combine like terms.

$$\begin{aligned} \frac{1}{x+1} - \frac{2(x+1)}{1(x+1)} &= \frac{1 + -2(x+1)}{x+1} \\ &= \frac{1 - 2x - 2}{x+1} \\ &= \frac{-2x - 1}{x+1} \end{aligned}$$

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Example 2. Combine terms in the function h and express it as a ratio of polynomials:

$$h(x) = \frac{1}{x^2} + 5$$

Rewrite the constant term with a denominator of 1, then find the common denominator.

$$h(x) = \frac{1}{x^2} + \frac{5}{1}$$

The common denominator is x^2

Multiply the constant term by 1, that is, multiply the constant term by the least common denominator
least common denominator

$$h(x) = \frac{1}{x^2} + \frac{5x^2}{1x^2}$$

Write using a single denominator

$$h(x) = \frac{1 + 5x^2}{x^2}$$

Example 3. Combine terms in the function f and express it as a ratio of polynomials:

$$f(x) = \frac{1}{(x+4)^2} - 6$$

Rewrite the constant term with a denominator of 1, then find the common denominator.

$$f(x) = \frac{1}{(x+4)^2} - \frac{6}{1}$$

The common denominator is $(x+4)^2$

Multiply the constant term by 1, that is, multiply the constant term by the least common denominator
least common denominator

$$f(x) = \frac{1}{(x+4)^2} - \frac{6(x+4)^2}{1(x+4)^2}$$

Expand the binomial, write using a single denominator, and combine like terms.

$$\begin{aligned} \frac{1}{(x+4)^2} - \frac{6(x+4)^2}{1(x+4)^2} &= \frac{1 - 6(x^2 + 8x + 16)}{(x+4)^2} \\ &= \frac{1 - 6x^2 - 48x - 96}{(x+4)^2} \end{aligned}$$

Factoring out the negative is optional.

$$\begin{aligned} &= \frac{-6x^2 - 48x - 95}{(x+4)^2} \\ &= \frac{-(6x^2 + 48x + 95)}{(x+4)^2} \end{aligned}$$

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Example 4. Combine terms in the function h and express it as a ratio of polynomials:

$$h(x) = -4 + \frac{1}{(x+2)^3}$$

Rewrite the constant term with a denominator of 1, then find the common denominator.

$$h(x) = \frac{-4}{1} + \frac{1}{(x+2)^3}$$

The common denominator is $(x+2)^3$

Multiply the constant term by 1, that is, multiply the constant term by the least common denominator
least common denominator

$$h(x) = \frac{-4(x+2)^3}{1(x+2)^3} + \frac{1}{(x+2)^3}$$

Expand the binomial and write using a single denominator

$$h(x) = \frac{-4(x+2)^3}{1(x+2)^3} + \frac{1}{(x+2)^3}$$

$$= \frac{-4(x^3 + 6x^2 + 12x + 8) + 1}{(x+2)^3}$$

Distribute the -4

Combine like terms.

$$= \frac{(-4x^3 - 24x^2 - 48x - 32) + 1}{(x+2)^3}$$

Factoring out the negative is optional.

$$= \frac{-4x^3 - 24x^2 - 48x - 31}{(x+2)^3}$$

$$= \frac{-(4x^3 + 24x^2 + 48x + 31)}{(x+2)^3}$$

Checkpoint 4.3A

Horizontal Asymptotes Revisited

Consider the horizontal asymptotes of the functions in the examples above.

<i>Original function</i>	<i>Horizontal asymptote</i>
1. $g(x) = \frac{1}{x+1} - 2$	$y = -2$
2. $h(x) = \frac{1}{x^2} + 5$	$y = 5$
3. $f(x) = \frac{1}{(x+4)^2} - 6$	$y = -6$
4. $f(x) = -4 + \frac{1}{(x+2)^3}$	$y = -4$

Now consider the combined form of these same functions:

<i>Original function</i>	<i>Combined form</i>	<i>Horizontal asymptote</i>
1. $g(x) = \frac{1}{x+1} - 2$	$g(x) = \frac{-2x-1}{x+1}$	$y = -2$
2. $h(x) = \frac{1}{x^2} + 5$	$h(x) = \frac{1+5x^2}{x^2}$	$y = 5$
3. $f(x) = \frac{1}{(x+4)^2} - 6$	$f(x) = \frac{-6x^2 - 48x - 95}{(x+4)^2}$	$y = -6$
4. $f(x) = -4 + \frac{1}{(x+2)^3}$	$f(x) = \frac{-4x^3 - 24x^2 - 48x - 31}{(x+2)^3}$	$y = -4$

Take a close look at the combined form and the horizontal asymptotes. Do you see a pattern? Would you be able to determine the horizontal asymptote if given only the combined form of a rational function?

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Try it! Find the horizontal asymptote of the following. Then check on your graphing calculator.

A. $h(x) = \frac{3x^2 - 2x + 1}{x^2 - 4}$

B. $f(x) = \frac{8x^3 + 4x - 3}{4x^3 + 1}$

C. $r(x) = \frac{x + 1}{2x + 5}$

D. $g(x) = \frac{6x^3 - 4x + 1}{2x^3 + 3}$

The horizontal asymptote of these functions is the leading coefficient of the numerator divided by the leading coefficient of the denominator. Looking back at the steps for combining two terms of a rational function will help us understand why this is the case. Consider the procedure in Example 1:

Multiply the constant term by 1, that is multiply

the constant term by the

least common denominator
least common denominator

$$g(x) = \frac{1}{x+1} - \frac{2(x+1)}{1(x+1)}$$

We multiplied the least common denominator - which happens to be the denominator of the first term - by the constant term, which happens to be the value of the asymptote. The resulting numerator has the SAME DEGREE as the denominator and the SAME LEADING COEFFICIENT as the asymptote.

The horizontal asymptote of the graph of a rational function in combined form WITH POLYNOMIALS OF EQUAL DEGREE IN THE NUMERATOR AND DENOMINATOR is the **leading coefficient of the numerator divided by the leading coefficient of the denominator.**

If the degree of the polynomials are **not** the same, this rule does **not** apply.

Example 5. Name the horizontal asymptote of the graph of the function r given by

$$r(x) = \frac{8x - 3}{4x - 1}. \text{ Then find the } x\text{-intercept(s) and the } y\text{-intercept.}$$

Since the degree of the numerator = the degree of the denominator, the horizontal asymptote is the leading coefficient over the leading coefficient.

The leading coefficient of the numerator is 8 and the leading coefficient of the denominator is 4, so the horizontal asymptote is $y = \frac{8}{4} = 2$ or $y = 2$

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To find the y-intercept, set $x = 0$

$$r(0) = \frac{8(0) - 3}{4(0) - 1} = \frac{-3}{-1} = 3.$$

The point $(0, 3)$ is on the graph.

To find the x-intercept, set $y = 0$

$$0 = \frac{8x - 3}{4x - 1}$$

$$0 = 8x - 3$$

$$3 = 8x$$

$$\frac{3}{8} = x$$

So the point $\left(\frac{3}{8}, 0\right)$ is on the graph

We could cross multiply or just set the numerator = 0 and solve.

Example 6. Name the horizontal asymptote of the graph of the function s given by

$s(x) = \frac{2x^2 - x - 10}{x^2 - 2x - 3}$. Then find the x-intercept(s) and the y-intercept.

The horizontal asymptote is the leading coefficient of the numerator over the leading coefficient of the denominator

The leading coefficient of the numerator is 2 and the leading coefficient of the denominator is 1, so the horizontal asymptote is $y = \frac{2}{1}$ or $y = 2$

To find the y-intercept, set $x = 0$

$$s(0) = \frac{2(0)^2 - 0 - 10}{(0)^2 - 2(0) - 3} = \frac{-10}{-3} = \frac{10}{3}$$

The point $\left(0, \frac{10}{3}\right)$ is on the graph

To find the x-intercept(s), set $y = 0$

$$0 = \frac{2x^2 - x - 10}{x^2 - 2x - 3}$$

$$0 = 2x^2 - x - 10$$

$$0 = (2x - 5)(x + 2)$$

so

$$2x - 5 = 0 \quad \text{or} \quad x + 2 = 0$$

$$2x = 5$$

$$x = \frac{5}{2} \quad \text{or} \quad x = -2$$

We could cross multiply or just set the numerator = 0 and solve.

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So the points $\left(\frac{5}{2}, 0\right)$ and $(-2, 0)$ are on the Graph.

Checkpoint 4.3B*Separating a Rational Function into Two Terms*

Now suppose we needed to go in the reverse direction. That is, suppose we start with rational function in combined form and want to write it as a function with two terms.

Example 7. Separate the function h given by $h(x) = \frac{4x-5}{x-2}$ into two terms such that the second term is a constant.

First we need to identify the horizontal asymptote.

The horizontal asymptote is $y = 4$

The separated form of h will look like

this: $h(x) = \frac{\text{Something}}{x-2} + 4$, since the

denominator will be the same and the horizontal asymptote indicates a vertical shift up 4.

Set the two forms equal to each other.

$$\frac{4x-5}{x-2} = \frac{\text{Something}}{x-2} + 4$$

Solve for the "Something." Abbreviate "Something" with an S .

$$\frac{4x-5}{x-2} = \frac{S}{x-2} + 4$$

Multiply both sides by the LCD.

$$(x-2) \frac{4x-5}{x-2} = \left(\frac{S}{x-2} + 4 \right) (x-2)$$

Solve for S

$$4x-5 = S + 4(x-2)$$

$$4x-5 = S + 4x-8$$

$$3 = S$$

Write the separated form.

$$h(x) = \frac{3}{x-2} + 4$$

More worked examples

Section 4.3

4.3 Different Forms of Rational Functions - Homework Exercises

For each of the functions represented below,

- Determine the horizontal and vertical asymptotes of the graph;
- Combine terms and express the function as a ratio of polynomials. Be sure to expand all binomials in the numerator and combine like terms.

1. $f(x) = \frac{1}{x+1} - 2$

2. $f(x) = -3 + \frac{1}{x-4}$

3. $f(x) = \frac{1}{x+7} + 7$

4. $f(x) = \frac{1}{x+7} + 7$

5. $y = \frac{1}{x^2} + 5$

6. $y = \frac{1}{x^2} - 7$

7. $y = \frac{1}{x^2} - 4$

8. $y = \frac{1}{x^2} + 8$

9. $g(x) = \frac{1}{(x-2)^2} - 5$

10. $f(x) = 1 + \frac{1}{(x+3)^2}$

11. $g(x) = 4 + \frac{1}{(x+2)^2}$

12. $f(x) = 4 + \frac{1}{(x+2)^2}$

13. $y = \frac{1}{(x-1)^3} + 3$

14. $f(x) = -2 + \frac{1}{(x+2)^3}$

15. $f(x) = -2 + \frac{1}{(x+2)^3}$

16. $f(x) = -1 + \frac{1}{(x+3)^3}$

17. $f(x) = 4 + \frac{1}{(x+3)^2}$

18. $y = \frac{1}{(x-5)^2} - 3$

19. $g(x) = -2 + \frac{1}{(x+6)^3}$

20. $f(x) = \frac{1}{(x+7)^3} + 7$

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Name the horizontal asymptote of each of the following rational functions. Then find the x -intercept and the y -intercept.

21. $f(x) = \frac{3x-1}{x+5}$

22. $f(x) = \frac{4-2x}{x+3}$

23. $h(x) = \frac{5x+3}{x-7}$

24. $h(x) = \frac{7x+1}{x+5}$

25. $g(x) = \frac{1-4x}{x+9}$

26. $g(x) = \frac{3-5x}{x-7}$

27. $f(x) = \frac{4-3x}{x+6}$

28. $f(x) = \frac{8-10x}{x-3}$

29. $r(x) = \frac{6-5x}{10x+7}$

30. $r(x) = \frac{7-8x}{4x+1}$

31. $s(x) = \frac{3x^2+4x+1}{x^2-4}$

32. $s(x) = \frac{2x^2-7x-15}{x^2-9}$

33. $m(x) = \frac{5x^2-3x-2}{9-x^2}$

34. $m(x) = \frac{5x^2-25}{4-x^2}$

35. $h(x) = \frac{x^2-5x+1}{3x^2-2x-8}$

36. $g(x) = \frac{3x^2+x-10}{x^2+2x-6}$

37. $f(x) = \frac{x^2-3x+1}{4x^2-2x-3}$

38. $h(x) = \frac{x^2-7x+2}{x^2-x-12}$

39. $f(x) = \frac{x^2+5x+7}{3x^2+3x-6}$

40. $f(x) = \frac{x^2-2x+6}{x^2-6x-1}$

For each of the functions represented below, separate the function into two terms such that the second term is a constant. Then name the vertical and horizontal asymptotes.

41. $f(x) = \frac{3x-1}{x+5}$

42. $f(x) = \frac{2x-3}{x+4}$

43. $s(x) = \frac{4-2x}{x+3}$

44. $s(x) = \frac{3-4x}{x+1}$

45. $g(x) = \frac{4x+9}{8x-2}$

46. $f(x) = \frac{7x-21}{2x+5}$

47. $h(x) = \frac{2x^2}{1+x^2}$

48. $g(x) = \frac{3x^2}{2+x^2}$

49. $f(x) = \frac{x^2-4}{3x^2}$

50. $r(x) = \frac{x^2-9}{4x^2}$

ANSWERS TO ODD NUMBERED PROBLEMS

