MATH 10033
Fundamentals of Mathematics III

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Contents

Chapter 1

Intro to FM III, Graphs and Functions

Welcome to Fundamentals of Mathematics III! We will begin this course by learning how to graph equations and functions and comparing the different shapes obtained. Then we will continue with some rules of algebra for systems of equations, radicals, exponentials and logarithms and using these concepts in problem solving. We will emphasize simplifying and solving problems by hand, but it will be important to have a graphing calculator handy for a few exercises and as a way to check answers. A free computer-based graphing calculator can be found for example at www.graphcalc.com!

We will try to keep the language informal. In particular we will be using the symbols $\rightarrow$ for "implies" and $\leftrightarrow$ for "is equivalent to" quite a bit. For example $x = 3 \rightarrow x^2 = 9$ because when $x = 3$, $x^2 = 9$ must be true. However, $x^2 = 9$ does not imply $x = 3$ because when $x^2 = 9$, x could be 3 or -3. When two statements are equivalent (i.e. imply each other) and that information is important, we will join the statements with $\leftrightarrow$. For example $2x = 6 \leftrightarrow x = 3$ which is true because we could multiply both sides of $2x = 6$ by $\frac{1}{2}$ to get $x = 3$, or multiply both sides of $x = 3$ by 2 to get $2x = 6$. The $\leftrightarrow$ is used a lot in solving equations and indicates the original equation has the same solution set as the final equation.

Exact answers, especially in fraction form, will be emphasized. So you will see most answers given with an "=" for exact. But when rounding is necessary or more meaningful, we will emphasize a non-exact or approximate value with "$\approx$."

If you are reading this book on a computer, you can navigate quickly from one part of the book to another using the bookmarks on the left. Click on the "+" to expand the bookmarks out until you reach the part of the book to which you want to navigate to. Then, click on the desired chapter, section or subsection. In addition, near each "Try This" problem there is a $\boxed{\text{Ⓢ}}$ marker for navigation to its complete solution and the $\boxed{a}$ markers can be used to navigate to the final answers to homework problems. These solutions and final answers can also be found at the end of the book.

Well, it's time to get started with some concepts about graphs of equations and functions. With a strong effort now, you should feel the rewards throughout the course. Good luck!

1.1 Basic Graphing Concepts

Points Satisfying an Equation

To say a point satisfies an equation in two variables (normally x and y) means that the point balances the equation when its x- and y-coordinates are substituted into the equation. For example the point $(2, -3)$ satisfies the equation $2x + 4y = -8$ since substituting 2 in for x and -3 in for y does check out true: $2 \cdot (2) + 4 \cdot (-3) \stackrel{?}{=} -8$ is true. Normally there are many points that satisfy a given equation. For example the point $(-4, 0)$ also satisfies the equation $2x + 4y = -8$ since $2 \cdot (-4) + 4 \cdot (0) \stackrel{?}{=} -8$ is again true. However, the point $(0, -4)$ does not since $2 \cdot (0) + 4 \cdot (-4) \neq -8$.

Example 1.1 *Which points satisfy the equation* $y = \frac{1}{2}x - 4$: $A = (2, -3)$, $B = (\frac{5}{2}, \frac{1}{4})$, $C = (4, -2)$*?*

Solution: For each point, substitute the x- and y-coordinates into the equation $y = \frac{1}{2}x - 4$ and see if the two sides of the equation balance out. For point A, $-3 \stackrel{?}{=} \frac{1}{2} \cdot (2) - 4$ is true since $\frac{1}{2} \cdot (2) - 4 = 1 - 4 = -3$.

But for point B, $\frac{1}{4} \stackrel{?}{=} \frac{1}{2} \cdot \left(\frac{5}{2}\right) - 4$ is false since $\frac{1}{2} \cdot \left(\frac{5}{2}\right) - 4 = \frac{5}{4} - 4 = \frac{-11}{4} \neq \frac{1}{4}$.

Finally for point C, $-2 \stackrel{?}{=} \frac{1}{2} \cdot (4) - 4$ is true since $\frac{1}{2} \cdot (4) - 4 = 2 - 4 = -2$.

So we know points A and C satisfy the equation but point B does not. ∎

Example 1.2 *Determine the value of* b *so that the point* $(-3, b)$ *satisfies the equation* $\frac{x}{2} - 2y = 1$.

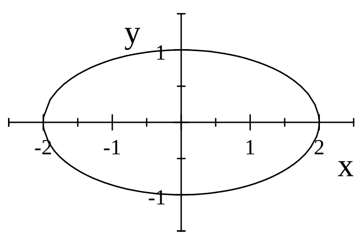
Solution: Since the point $(-3, b)$ must satisfy the equation $\frac{x}{2} - 2y = 1$ just plug in -3 for x and b for y. Then solve for b. Here are the steps:

sub in -3 for x and b for y,	$\frac{-3}{2} - 2b = 1$
multiply both sides by the LCD,	$2 \cdot (\frac{-3}{2} - 2b) = 2 \cdot (1)$
distribute left hand side,	$-3 - 4b = 2$
add 3 to both sides,	$-4b = 5$
divide both sides by -4,	$b = \frac{-5}{4}$. ∎

After reading some examples, hopefully you are ready to exercise your mind! The next problem will help you review some arithmetic and graphing facts from Fundamentals of Mathematics I. *And, it has a really nice graph too.*

Try This 1.1 *The graph of the equation $x^2 + 4y^2 = 4$ is the ellipse below. For each point $A - E$ below, determine if the point satisfies $x^2 + 4y^2 = 4$ and plot the point to see whether the point lies on the graph of the ellipse. Which points satisfy the equation? Which points lie on the ellipse?*

$A = (-2, 0)$,
$B = (0, 1)$,
$C = (-1, \frac{\sqrt{3}}{2})$,
$D = (\sqrt{3}, \frac{1}{2})$,
$E = (2, -1)$.



Ellipse $x^2 + 4y^2 = 4$

Ⓢ

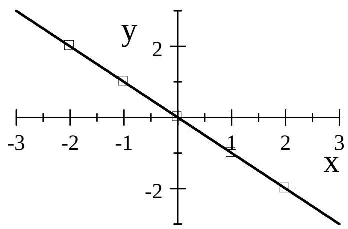
The Graph of an Equation

By definition, the graph of an equation is the collection of all points that satisfy the equation. In the next two examples, you will notice that very different shapes can result from different kinds of equations.

Example 1.3 *Determine and plot five points that satisfy the equation $x + y = 0$. What shape does the graph appear to have?*

Solution: It's easy to come up with points that satisfy $x + y = 0$. Just pick x-coordinate and y-coordinates so that one number is opposite in sign of the other. Below is a table and graph with five such points:

x		y
2	$\rightarrow$	-2
1	$\rightarrow$	-1
0	$\rightarrow$	0
-1	$\rightarrow$	1
-2	$\rightarrow$	2



Graph of $x + y = 0$

The points should appear to form a line. In Chapter 2 we will see why! ∎

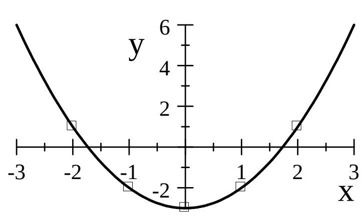
Example 1.4 *Determine and plot at least two points to the left and to the right of the origin that satisfy the equation $x^2 - y = 3$. What shape does the graph appear to have?*

Solution: Since this equation is more complicated, it would be best to have one of the variables isolated. Normally, we isolate whichever variable is easiest. In this problem we can isolate y in two simple steps:

$$x^2 - y = 3 \leftrightarrow x^2 = y + 3 \leftrightarrow x^2 - 3 = y.$$

So we will graph the equation $y = x^2 - 3$. Once we select an x-value, we can compute the y-value. Below is a table of points and a sketch of the graph of $y = x^2 - 3$.

x		y
-2	$\rightarrow$	1
-1	$\rightarrow$	-2
0	$\rightarrow$	-3
1	$\rightarrow$	-2
2	$\rightarrow$	1



Graph of $x^2 - y = 3$

Does the graph look familiar? It's a U-shaped graph (a "parabola" that opens upward). ∎

Evaluating and Graphing Functions

The function concept is one of the most important topics in all of mathematics. Mathematics can be done without functions, but it tends to be simplified and easier to express with functions.

Basically, a function (without getting into technicalities) is a rule that starts with an input value (normally x) and determines a unique output value (normally y). Often the function rule is written as an equation like $y = x^2 - 3$. If we start with an x-value, we can determine a corresponding y-value. For example,

$$\begin{array}{lc|c|cl} \text{if } x = 3 & \mapsto & \text{Equation} & \mapsto & \text{then } y = (3)^2 - 3 = 9 - 3 = 6 \\ \text{if } x = -1 & \mapsto & y = x^2 - 3 & \mapsto & \text{then } y = (-1)^2 - 3 = 1 - 3 = -2 \end{array}$$

To emphasize that the input value determines the output value, function notation was introduced. The function notation for the rule $y = x^2 - 3$ might be $f(x) = x^2 - 3$.

The $f(x)$, pronounced "f of x", is notation that means the x-value determines a y-value by the function f. The letter f is often used to name functions, but any other name is fine. To evaluate a function means to apply the function rule with a specified input value to determine an output value.

For example using the function $f(x) = x^2 - 3$, the notation $f(3)$ represents the output or y-value when the function f is evaluated with $x = 3$; and in this case the function evaluation gives $f(3) = (3)^2 - 3 = 9 - 3 = 6$. The function notation also allows us to state an answer without words, e.g. "$f(3) = 6$" is the same as "if $x = 3$ then $y = 6$."

Similarly with $f(x) = x^2 - 3$, $f(-1) = (-1)^2 - 3 = 1 - 3 = -2$ and that means when the input value for the function f is $x = -1$, the output or y-value is -2.

Another example of a function rule would be the formula for the perimeter of a square, $P = 4l$. If we know that the side length of a square is l units, then the perimeter P is just 4 times l units. With function notation the rule $P = 4l$ would be written as $P(l) = 4l$. And, for example, "$P(5)$" would mean "perimeter of a square if the side length is 5 units." If we actually did the function evaluation we would get $P(5) = 4 \cdot (5) = 20$ units. Do you agree?

Example 1.5 *A function is defined by* $g(x) = 3x^2 - 4x + 2$. *Evaluate* $g(x)$ *with* $x = -2$, $x = -1$, $x = 0$, $x = 1$ *and* $x = 2$.

Solution: Remember function evaluation is just inputting in a specified value and calculating using the function rule to determine the output value. You might as well know now, using parentheses around the input value is an extremely good habit. So, to evaluate the function $g(x)$ with $x = -2$ we will keep -2 in parentheses:

$$g(-2) = 3 \cdot (-2)^2 - 4 \cdot (-2) + 2 = 3 \cdot (4) + 8 + 2 = 12 + 8 + 2 = 22.$$

Without parentheses around -2, we could get the wrong answer because:

$$3 \cdot -2^2 - 4 \cdot -2 + 2 = 3 \cdot -4 + 8 + 2 = -12 + 8 + 2 = -2.$$

Here are the rest of the correct answers using parentheses:

$$g(-1) = 3 \cdot (-1)^2 - 4 \cdot (-1) + 2 = 3 \cdot (1) + 4 + 2 = 9;$$
$$g(0) = 3 \cdot (0)^2 - 4 \cdot (0) + 2 = 3 \cdot (0) - 0 + 2 = 2;$$
$$g(1) = 3 \cdot (1)^2 - 4 \cdot (1) + 2 = 3 \cdot (1) - 4 + 2 = 1; \text{ and}$$
$$g(2) = 3 \cdot (2)^2 - 4 \cdot (2) + 2 = 3 \cdot (4) - 8 + 2 = 6. \blacksquare$$

Example 1.6 *Cost C in dollars for driving a rental car x miles is determined by the function $C(x) = .25x + 50$. (a) Compute $C(250)$. (b) Explain what $C(250)$ means. (c) Compute $C(0)$. What does $C(0)$ mean? Why would $C(0)$ tend to not equal 0?*

Solution: For part (a) we just calculate: $C(250) = .25 \cdot (250) + 50 = 112.5$ dollars. For part (b), $C(x)$ would be the cost for driving the rental car x miles. Therefore $C(250)$ is the cost for driving the rental car 250 miles which happens to be \$112.50.

Finally, $C(0) = .25 \cdot (0) + 50 = 50$ (dollars) which is the cost for driving 0 miles. $C(0)$ is usually not 0 because there are always costs for setting up a service even if the service is not used. $\blacksquare$

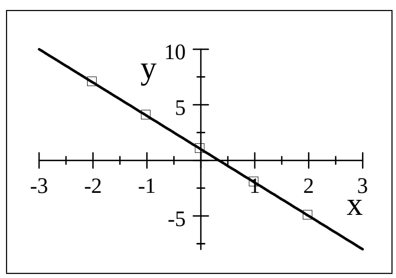
Now that we have seen how functions work, we will discuss a simple but important topic concerning functions–their domains. The domain of a function is the set of all meaningful input values. Generally, we need to rely on context or arithmetic rules to determine the domain of the function. For example for the cost function from the last example $C(x) = .25x + 50$ the domain is the set of all numbers greater than or equal to 0 since the input value x–which represents number of miles–can not be negative! For a function like $W(x) = \frac{3}{x-1}$, if x were allowed to be 1 then there would be a division by 0 (an invalid arithmetic operation–do you remember why?). So the domain of W would be the set of all numbers not equal to 1. However, for a function like $g(x) = 3x^2 - 4x + 2$ which does not have any particular meaning and which doesn't have any arithmetic operation restrictions has domain the set of all real numbers.

Lastly, we will discuss what is meant by the graph of a function. The graph of a function f is the collection all points of the form $(x, f(x))$. In other words, the graph is the set of all points where the y-coordinates come from evaluating the function with all possible x-coordinate values. To graph a function we will normally pick some x-values, calculate the y-values using the function rule, organize the information into a table and then plot the points.

Example 1.7 *Graph the function* $L(x) = -3x + 1$.

Solution: The domain of this function is the set of all real numbers. So pick negative and positive values for x, use $L(x)$ to calculate the y-values and then the plot corresponding points:

x		y
2	$\rightarrow$	$L(2) = -5$
1	$\rightarrow$	$L(1) = -2$
0	$\rightarrow$	$L(0) = 1$
-1	$\rightarrow$	$L(-1) = 4$
-2	$\rightarrow$	$L(-2) = 7$



Graph of $L(x) = -3x + 1$

Perhaps you have noticed that graphing a function is just like graphing an equation in which y is isolated. If not, this connection should become more clear with more practice as the course ensues! ∎

Homework Problems

1. Plug $(-3, -2)$ into each equation below. Which equation or equations does the point $(-3, -2)$ satisfy?

(a) $2x + 3y = 0$;

(b) $x = -\frac{3}{2}y$;

(c) $y = x^2 + 7$;

(d) $x + \frac{1}{2}y^2 = -1$;

(e) $|x| + 1 = 2|y|$;

2. Redo Problem 1, but with the point $(-3, 2)$. Which equation(s) does the point $(-3, 2)$ satisfy?

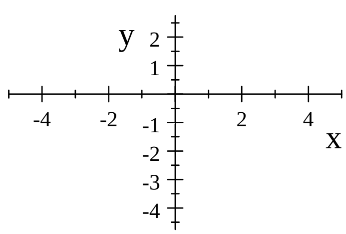
3. Determine the value for a so that point $(a, -5)$ satisfies the equation $3x - 5 = 2y$.

4. Determine the value for b so that point $(-2, b)$ satisfies the equation $3 - x^2 = 2y$.

5. Determine the value for a so that point $(a, \sqrt{2})$ satisfies the equation $4x = y^2$.

6. For each row in the table, determine the missing y-values so the points satisfy the equation $y = \frac{1}{2}x - 1$.Then, plot the points.

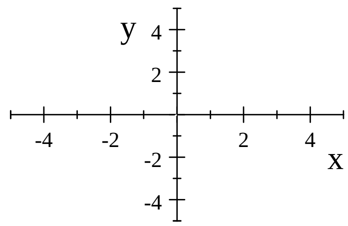
x		y
5	$\rightarrow$	
3	$\rightarrow$	
0	$\rightarrow$	
-3	$\rightarrow$	
-5	$\rightarrow$	



7. What shape does the Problem 6 graph have? (a) Linear, (b) U-shaped, (c) V-shaped, (d) S-shaped, (e) Sideways V-Shaped, (f) Sideways U-shaped or (g) Sideways half U-Shaped.

8. For each row in the table, determine the missing x-values so the points satisfy the equation $x = |y - 1|$.Then, plot the points.

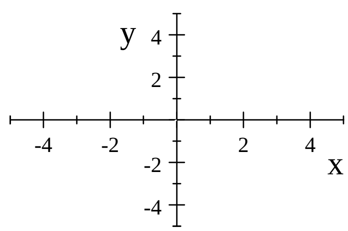
x		y
	$\leftarrow$	-2
	$\leftarrow$	-1
	$\leftarrow$	0
	$\leftarrow$	1
	$\leftarrow$	2
	$\leftarrow$	3



9. What shape does the Problem 8 graph have? (a) Linear (b) U-shaped (c) V-shaped (d) S-shaped (e) Sideways half V-Shaped?

10. For each row in the table, determine the missing y-values so the points satisfy the equation $y = (x + 2)^2$.

x		y
-4	$\rightarrow$	
-3	$\rightarrow$	
-2	$\rightarrow$	
-1	$\rightarrow$	
0	$\rightarrow$	



11. What shape does the Problem 9 graph have: (a) Linear, (b) U-shaped, (c) V-shaped, (d) S-shaped, (e) Sideways V-Shaped, (f) Sideways U-shaped or (g) Sideways half U-Shaped?

12. Evaluate $f(x) = 2x^2 - 7x$ with $x = -2$; $x = -1$; $x = 1$; and $x = 2$.

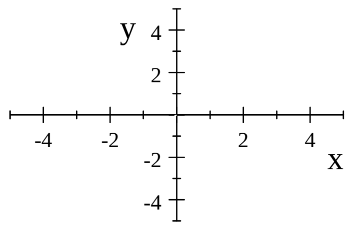
13. Evaluate $f(x) = -x^2 + 1$ with $x = -2$; $x = -1$; $x = 1$; and $x = \sqrt{2}$.

14. Evaluate $f(x) = 2x^3 - .1x^2$ with $x = -1$; $x = \frac{-1}{2}$; $x = \frac{1}{2}$; and $x = 1$. Express your answers as integers or fractions.

15. Evaluate $f(x) = x^3 - x^2$ with $x = -1$; $x = \frac{-1}{2}$; $x = \frac{1}{2}$; and $x = 1$. Express your answers as integers or fractions.

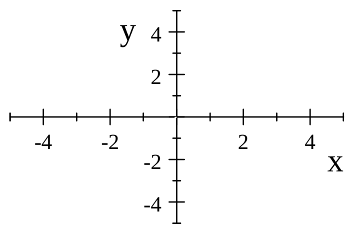
16. By plotting several points, graph the function $f(x) = -x + 1$.

x		y
	$\longrightarrow$	
	$\longrightarrow$	
	$\longrightarrow$	
	$\longrightarrow$	
	$\longrightarrow$	



17. By plotting several points, graph the function $g(x) = 5 - x^2$.

x		y
	$\longrightarrow$	
	$\longrightarrow$	
	$\longrightarrow$	
	$\longrightarrow$	
	$\longrightarrow$	



18. A function is defined by $h(x) = 6$. Evaluate $h(x)$ with $x = -1$, $x = 0$ and $x = 1$. In a sentence, explain why this type of function might be called a "constant function."

19. For the function $i(x) = 2x^3 + 3x^2 - 8x + 3$, verify that $i(1)$, $i(-3)$ and $i(\frac{1}{2})$ each equal 0. Do you think $i(x)$ always produces a value of 0?

20. Can any number be square-rooted?; Can any number be squared?; Can any number be divided by 3?

21. For which of the following functions is the domain the set of all real numbers: (a) $f(x) = \sqrt{x}$; (b) $g(x) = x^2$; (c) $h(x) = \frac{x}{3}$?

22. The function C for calculating the cost of making x chairs is $C(x) = 50x + 30$ assuming at least 10 chairs are made. What would be the domain for the function C?

23. Give an example of a function $j(x)$ and numbers a and b that counterexamples the false statement $j(a + b) = j(a) + j(b)$. $\boxed{a}$

Chapter 2

Lines and Their Equations

Across history mathematics has been used to solve many kinds of problems. One basic tool is "mathematical modeling" where for example inequalities, equations or formulas are used to describe how quantities are related to each other. Linear equation forms are amongst the most common mathematical models. Perhaps you remember linear equation forms from the past? The three most common ones are:

General Form	$Ax + By = C$
Slope-Intercept Form	$y = mx + b$
Point-Slope Form	$y - y_1 = m(x - x_1)$

Later we will discuss where these forms come from. For now just notice that in each form the implied power of x and y is 1.

2.1 Graphing Linear Equations

Recall that to make the graph of any equation in two variables, we must determine points (x, y) that satisfy the equation. The easiest situation is when x or y is isolated. If neither x nor y is isolated, we will isolate whichever is easier. Since graphing calculators and function notation both require y to be isolated, we will tend to prefer to isolate y.

Example 2.1 *Graph the equation* $x + \frac{5}{2}y = 5$.

Solution: Do you recognize how this equation fits the $Ax + By = C$ form? So, we should be expecting the graph to be a line. Now in terms of the actual work, it's easiest to rearrange the equation to isolate x:

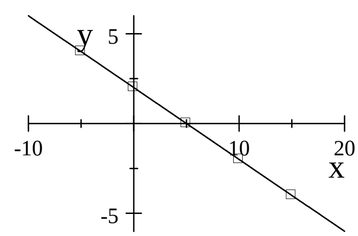
$$x + \frac{5}{2}y = 5 \leftrightarrow x = -\frac{5}{2}y + 5.$$

Since x is isolated, we will pick values of y then calculate x. To avoid fractions we will pick even integer values of y. For example:

$$y = -4 \to x = \frac{-5}{2} \cdot (-4) + 5 = -5 \cdot (-2) + 5 = 10 + 5 = 15.$$
$$y = -2 \to x = \frac{-5}{2} \cdot (-2) + 5 = -5 \cdot (-1) + 5 = 5 + 5 = 10.$$

Etc. Here are additional points and the graph:

$x = -\frac{5}{2}y + 5$		y
15	$\leftarrow$	-4
10	$\leftarrow$	-2
5	$\leftarrow$	0
0	$\leftarrow$	2
-5	$\leftarrow$	4



Graph of $x = -\frac{5}{2}y + 5$

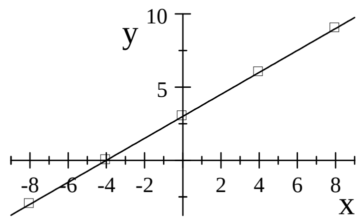
Example 2.2 *Graph the equation* $3x - 4y = -12$.

Solution: Neither variable is very easy to isolate, so out of general preference we will isolate y:

$$\begin{array}{ll} & 3x - 4y = -12 \\ \text{isolate the } y \text{ term,} & \leftrightarrow -4y = -3x - 12; \\ \text{isolate } y, & \leftrightarrow \frac{-1}{4}(-4y) = \frac{-1}{4}(-3x - 12); \\ \text{apply laws of arithmetic,} & \leftrightarrow \left(\frac{-1}{4} \cdot \frac{-4}{1}\right) y = \frac{-1}{4} \cdot \frac{-3}{1}x + \frac{-1}{4} \cdot \frac{-12}{1}; \\ \text{simplify,} & \leftrightarrow y = \frac{3}{4}x + 3. \\ \text{Therefore:} & 3x - 4y = -12 \leftrightarrow y = \frac{3}{4}x + 3. \end{array}$$

Working with $y = \frac{3}{4}x + 3$ and multiples of 4 for x to avoid fractions produces the following table of points and graph:

x		$y = \frac{3}{4}x + 3$
-8	$\rightarrow$	-3
-4	$\rightarrow$	0
0	$\rightarrow$	3
4	$\rightarrow$	6
8	$\rightarrow$	9



Graph of $3x - 4y = -12$

Finding Intercepts

In the last example with the equation $3x - 4y = -12$ (or equivalently $y = \frac{3}{4}x + 3$), two of the points are actually "intercepts." An intercept is a point that lies on the x- or y-axis. More specifically, an x-intercept is a point on a graph that lies on the x-axis; a y-intercept is a point on a graph that lies on the y-axis. In the last example, the point $(-4, 0)$ was an x-intercept; and the point $(0, 3)$ was a y-intercept.

Finding intercepts without making a graph is simple:

How to Find Intercepts
X-intercepts are points on the x-axis where the y-coordinate is 0. So, set $y = 0$. Then solve for x. Y-intercepts are points on the y-axis where the x-coordinate is 0. So, set $x = 0$. Then solve for y.

Example 2.3 *Without making a graph, determine the intercepts of the line with equation* $3x - 4y = -12$.

Solution: Pretend like you don't already know the answers from the last example! To find the x-intercepts, set $y = 0$ and solve for x:

$$3x - 4 \cdot (0) = -12 \leftrightarrow 3x = -12 \leftrightarrow \frac{1}{3} \cdot (3x) = \frac{1}{3} \cdot (-12) \leftrightarrow x = -4.$$

So, $(-4, 0)$ is the only x-intercept. To find the y-intercepts set $x = 0$ and solve for y:

$$3 \cdot (0) - 4y = -12 \leftrightarrow -4y = -12 \leftrightarrow \frac{-1}{4} \cdot (-4y) = \frac{-1}{4} \cdot (-12) \leftrightarrow y = 3.$$

So, $(0, 3)$ is the only y-intercept. ∎

Example 2.4 *Find the intercepts of the graph of the linear equation $3x = 5y + 8$, then graph the equation.*

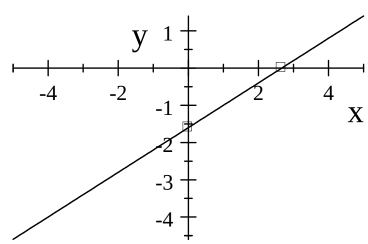
Solution: To find the x-intercept(s), set $y = 0$ and solve for x:

$$3x = 5 \cdot (0) + 8 \leftrightarrow 3x = 8 \leftrightarrow \tfrac{1}{3} \cdot (3x) = \tfrac{1}{3} \cdot (8) \leftrightarrow x = \tfrac{8}{3}.$$

So the only x-intercept is the point $\left(\frac{8}{3}, 0\right)$. To find the y-intercept(s), set $x = 0$ and solve for y:

$$3 \cdot (0) = 5y + 8 \leftrightarrow 0 = 5y + 8 \leftrightarrow$$
$$-8 = 5y \leftrightarrow \tfrac{1}{5} \cdot (-8) = \tfrac{1}{5} \cdot (5y) \leftrightarrow \tfrac{-8}{5} = y.$$

So, $(0, \frac{-8}{5})$ is the only y-intercept. Here's a graph of the line (right) using the x- and y- intercepts. Since $\frac{8}{3} = 2\frac{2}{3}$ and $\frac{-8}{5} = -1\frac{3}{5}$, the x- and y-intercepts are in the correct locations! ∎



Graph of $3x = 5y + 8$

Did you notice in the last two examples with lines that there were a total of two intercepts? Do you think that's always true for lines? Please give this problem a try.

Try This 2.1 *Draw several lines in the coordinate plane. Include at least one horizontal line and one vertical line. How many x-intercepts could a line have? How many y-intercepts could a line have? How many total intercepts could a line have? (Count the origin as both an x- and a y-intercept.)* Ⓢ

Non-linear graphs add new possibilities for the total number of intercepts. As the next example shows, three intercepts are possible with parabolas.

Example 2.5 *Without graphing, determine the x- and y-intercepts of the graph of $x = 9 - y^2$.*

Solution: Note that $x = 9 - y^2$ is not a linear equation because of the y^2. To find the x-intercept, set $y = 0$ and solve for x. This gives:

$$x = 9 - (0)^2 \leftrightarrow x = 9.$$

So the x-intercept is $(9, 0)$. To find the y-intercept, set $x = 0$ and solve for y. That gives:

$$0 = 9 - y^2 \leftrightarrow 9 = y^2.$$

By mental math, $y = \pm 3$. So, there are two y-intercepts: $(0, 3)$ and $(0, -3)$ and an x-intercept at $(9, 0)$; therefore, a total of three intercepts. ∎

Generally, intercepts can be important because they show where a graph crosses the axes. In applications, intercepts often have special meaning.

Example 2.6 *The market value in dollars of a certain computer follows the formula $V = -550t + 1925$ where t represents the age of the computer in years. Determine the intercepts for the graph of $V = -550t + 1925$. What do the intercepts mean in the context of this problem?*

Solution: For the t-intercept, set $V = 0$ and solve for t. This gives:

$$0 = -550t + 1925 \leftrightarrow 550t = 1925 \leftrightarrow t = \tfrac{1925}{550} = \tfrac{7}{2} = 3.5$$

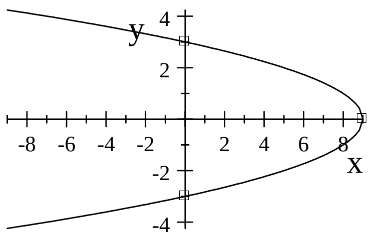
So $(3.5, 0)$ is the t-intercept, and it means the market value of the computer reaches \$0 at age 3.5 years.

For the V-intercept, set $t = 0$ and solve for V. This gives:

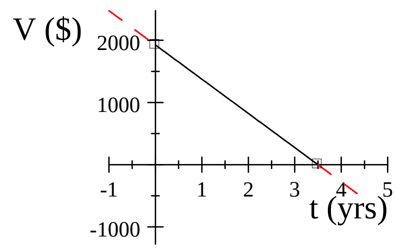
$$V = -550 \cdot (0) + 1925 = 1925.$$

So $(0, 1925)$ is the V-intercept which means the (initial) purchase price of the computer is \$1925. ∎

In closing, we have generated graphs for the last two examples to emphasize the important geometric nature of intercepts. Do you see how the answers we obtained using algebra agree with what the graphs tell us?



Graph of $x = 9 - y^2$

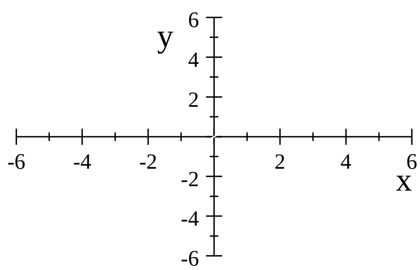


Graph of $V = -550t + 1925$

Homework Problems

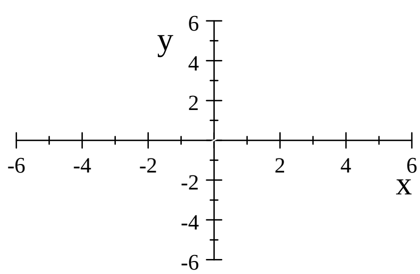
1. Graph the linear equation $y = \frac{4}{3}x - 5$ by finding at least three points that satisfy the equation.

x	y



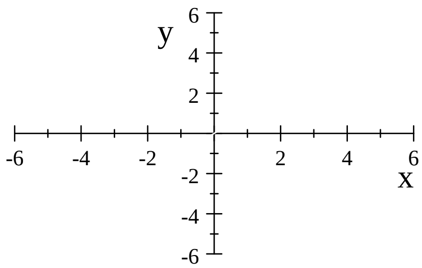
2. Graph the linear equation $x = 3y - 2$ by finding at least three points that satisfy the equation.

x	y



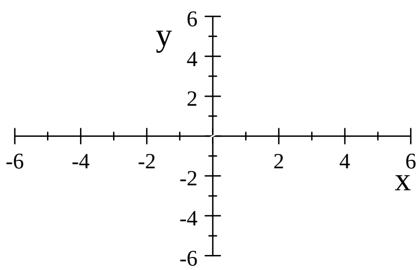
3. Solve the equation $2x + 3y = 3$ for y. Then, graph the equation by finding at least three points that satisfy the equation.

x	y



4. Solve the equation $2x - 10y = 5$ for y. Then, graph the equation by finding at least three points that satisfy the equation.

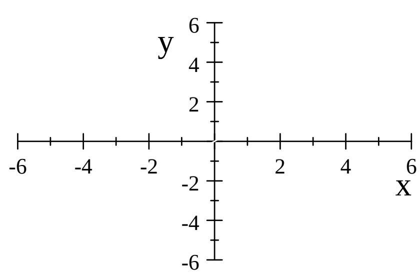
x	y



5. Why must all of the graphs in Problems 1-4 be lines?

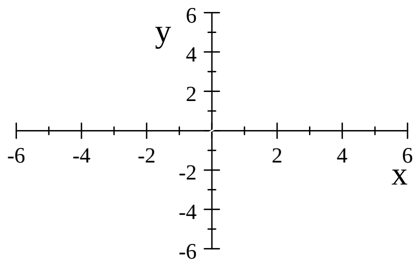
6. Determine the x- and y-intercepts of the graph of $y = 3x - 6$ and one additional point on the graph. Then, graph the equation using the intercepts and an additional point.

x	y



7. Determine the x- and y-intercepts of the graph of $-y = \frac{2}{3}x + 1$ and one additional point on the graph. Then, graph the equation using the intercepts and an additional point.

x	y



8. If a person says that the x-intercept of a graph is $x = 3$, what does the person really mean? Hint: remember, intercepts are points.

9. Suppose the number of people y who have heard a rumor x days after the rumor is started is given the equation $y = 120x + 12$.

   (a) After one week how many people have heard the rumor?

   (b) Determine the number of days it takes before 5200 people have heard the rumor. Round your final answer to the nearest day.

   (c) Determine the y-intercept for this graph? What is the meaning of the y-intercept in this problem?

   (d) Determine the x-intercept for this graph? Does the x-intercept make sense in this problem? Why or why not?

10. Suppose the number of Z-Mart stores z has been increasing according to $z = 150t + 1875$ while the number of M-Mart stores has been decreasing according to the $m = -25t + 3500$, where t represents the number of years since 1990. For each of the following years, determine if the number of M-Mart stores is less than, equal to or greater than the number of Z-Mart stores: 1993, 1995, 1997, 1999, 2001 and 2003.

11. For the equation $m = -25t + 3500$ in Problem 10, determine the m- and t-intercepts of the graph. Explain what each intercept means.

12. Considering all possible circles in the coordinate plane:

   (a) How many x-intercepts could a circle have? How about y-intercepts?

   (b) How many intercepts could a circle have? (Count the origin as both an x- and y-intercept.)

   (c) Draw circles that show the fewest and most possible intercepts. $\boxed{a}$

2.2 The Critical Concept of Slope

Slope is the most important topic concerning lines. Equations of lines depend on their slope, as do most uses of lines. The first major idea about slope is that all lines, even horizontal and vertical lines, have an assigned slope:

The Slope Formula
If (x_1, y_1) is a "first" point and (x_2, y_2) is a different "second" point on a line, then the slope formula $m = \frac{y_2 - y_1}{x_2 - x_1}$ determines the slope the line. Possible slopes are any positive or negative, 0 or undefined.

The value $y_2 - y_1$ in the slope formula is called the "rise" or "change in y" because it measures how much y changes moving from a first to a second point. Similarly, $x_2 - x_1$is called the "run" or "change in x." So, slope m is the ratio of rise to run moving from a first to a second point on the line.

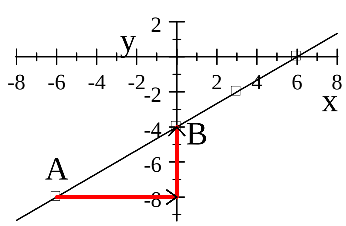
There are also two other key facts about slope which we will illustrate using the line L (graphed below) which has equation $y = \frac{2}{3}x - 4$ and several points on line L including $(-6, -8)$, $(0, -4)$, $(3, -2)$ and $(6, 0)$.

Fact 1: Slope is the same no matter which point is first or second. For example using $A = (-6, -8)$ and $B = (0, -4)$ on line L:

With A as a first point and B as a second point then:

$y_2 - y_1 = (-4) - (-8) = 4$ and
$x_2 - x_1 = (0) - (-6) = 6$.
So, $m = \frac{4}{6} = \frac{2}{3}$.

In the graph below, the $m = \frac{4}{6} \frac{[\text{rise}]}{[\text{run}]}$ is shown moving from A to B.

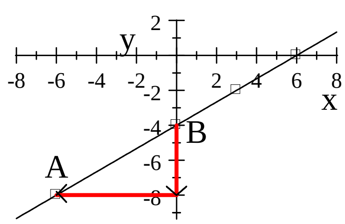


Line L w/ $m = \frac{4}{6} = \frac{2}{3}$

With B as a first point and A as a second point then:

$y_2 - y_1 = (-8) - (-4) = -4$ and
$x_2 - x_1 = (-6) - (0) = -6$.
So, $m = \frac{-4}{-6} = \frac{2}{3}$.

In the graph below, the $m = \frac{-4}{-6} \frac{[\text{rise}]}{[\text{run}]}$ is shown moving from B to A.



Same line w/ $m = \frac{-4}{-6} = \frac{2}{3}$

Fact 2: Slope is the same no matter which two points are used. Again, using line L and different pairs of points on L, the slope for each pair works out to be $\frac{2}{3}$:

$(3,-2)$ and $(6,0) \rightarrow m = \frac{(0)-(-2)}{(6)-(3)} = \frac{2}{3}$.
$(-6,-8)$ and $(6,0) \rightarrow m = \frac{(0)-(-8)}{(6)-(6)} = \frac{8}{12} = \frac{2}{3}$.
$(-6,-8)$ and $(3,-2) \rightarrow m = \frac{(-2)-(-8)}{(3)-(-6)} = \frac{6}{9} = \frac{2}{3}$.
Are convinced that slope doesn't change?

Did you happen to notice that the slope $\frac{2}{3}$ also appears in the equation $y = \frac{2}{3}x - 4$ as the coefficient of x? Keep that in mind for Section 2.4! But before moving on, it's important for you to practice with the slope formula.

Try This 2.2 *For the line L with equation $y = 4x - 5$:*

(a) Determine the x-intercept, y-intercept and one additional point on the line L. Use those three points to calculate the slope with three different pairs of points. What value did you get each time?
(b) How does the slope value correspond to the coefficients you see in the equation $y = 4x - 5$?
(c) How does the y-intercept correspond to the coefficients you see in the equation $y = 4x - 5$? $\boxed{\text{S}}$

Finally, let's consider an example of how the slope concept occurs in applications where a quantity is steadily increasing or decreasing over time.

Example 2.7 *Suppose from the time of purchase to six years later a* \$2300 *computer completely depreciates in value. Determine the average rate of decrease for the value of the computer.*

Solution: If we organize the information into points of the form (t, v), where t stands for time passed (in years) and v stands for value (in dollars), we can use the slope formula to determine the average rate of decrease in value over time. The two pieces of information we have correspond to the points $(0, 2300)$ and $(6, 0)$. So the average rate of *change* in value is:

$$m = \frac{(0)-(2300)}{(6)-(0)} = \frac{-2300}{6} = \frac{-1150\ [\text{dollars}]}{3\ [\text{year}]} = -383\tfrac{1}{3}\ \frac{[\text{dollars}]}{[\text{year}]}$$

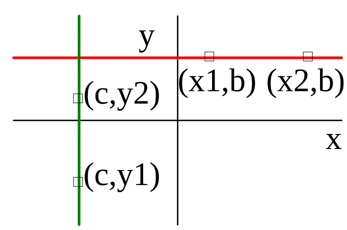
So taking into account the minus sign, the average rate of *decrease* in value is \$1150 per 3 years, or about \$383.33 per year. ∎

Horizontal and Vertical Lines

Take a moment and visualize a horizontal line, a vertical line and any other line that slopes upward or downward across the plane. The fact that vertical lines do not travel left/right and horizontal lines do not travel up/down makes them very special lines. Here are two key facts about horizontal and vertical lines that we will discuss in this section:

Horizontal and Vertical Line Equation and Slope Facts
For any real numbers b and c, Any horizontal line has an equation of the form $y = b$ and slope equal to 0. Any vertical line has an equation of the form $x = c$ and an undefined slope.

These facts imply that the equations of horizontal and vertical lines have either the x or the y missing from the equation. This is completely possible because in the General Form $Ax + By = C$, either A or B could be zero eliminating either the x or y. We will use the graph (right) to help us "derive" the horizontal and vertical line facts:



Horizontal/Vertical Lines

Fact 1: Any horizontal line has an equation of the form y = b and slope equal to 0. First, look closely at the horizontal line. What you should notice is that as you move from point to point the x-coordinate changes but the y-coordinate does not. In other words y must always be the same value b no matter what x is. That makes the equation of any horizontal line $y = b$. There is no x in the equation because the x-value does not affect the y-value (which must always be b).

As for the slope of any horizontal line, the rise or change in y is $y_2 - y_1 = b - b = 0$. So $m = \frac{0}{x_2 - x_1} = 0$. It's better to say horizontal lines have "slope 0" rather than "no slope" which is unclear in meaning.

Fact 2: Any vertical line has an equation of the form x = c and an undefined slope. Go back up to the graph and look carefully at the vertical line. This time notice that as you move from point to point the

y-coordinate changes but the x-coordinate does not. In other words x must always be the same value c no matter what y is. That makes the equation of the vertical line just $x = c$. There is no y in the equation because the y-value does not affect the x-value (which must always be c).

As for the slope of any vertical line, the run or change in x is $x_2 - x_1 = c - c = 0$. So $m = \frac{y_2 - y_1}{0}$ is undefined. Again, it's better to say vertical lines have "undefined slope" rather than "no slope."

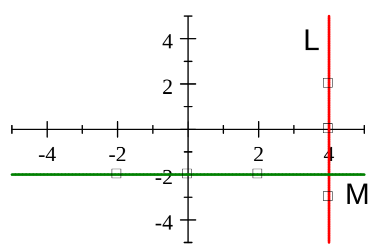
Example 2.8 *Graph the linear equations $x = 4$ and $y = -2$.*

Solution: First note statements like $x = 4$ and $y = -2$ can be confused for statement about numbers (like $x = 4$ is the solution to this problem) rather than equations of lines. We need to rely on context to know we are talking about lines.

Now, the graph of the linear equation $x = 4$ is the line where each point has an x-coordinate that must be 4. So, its graph must be a vertical line. Next, the graph of the linear equation $y = -2$ is the line where each point has a y-coordinate of -2. So, its graph must be a horizontal line. To accurately graph the lines we will as usual plot two or more points on each line:

line L	
x	y
4	2
4	-3
4	0

line M	
x	y
-2	-2
2	-2
0	-2



Lines $x = 4$ and $y = -2$

Example 2.9 *Determine the equation of the vertical line that passes through the point $(-1, -2)$.*

Solution: Vertical lines have equation $x = c$ (y does not appear in the equation). Since the line must go through the point $(-1, -2)$, the equation must be $x = -1$ so that the x-coordinate -1 fits properly into the equation. Therefore, the final answer is the linear equation $x = -1$. ∎

Example 2.10 *Determine the equation of the line that passes through* $(4, 2)$ *and is perpendicular to the line with equation* $x - 5 = 0$.

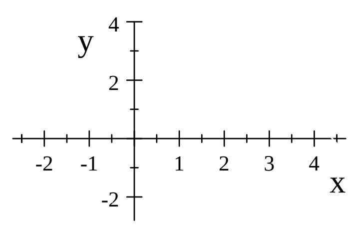
Solution: In this problem, the given equation is $x - 5 = 0 \leftrightarrow x = 5$, which corresponds to a vertical line. Hence the solution will need to correspond to a horizontal line. Since all horizontal lines have the form $y = b$ and the line must go through $(4, 2)$, the linear equation will be $y = 2$. Final answer! ∎

In conclusion, horizontal and vertical lines are best handled as simple special cases of lines in general. Watch for horizontal or vertical line clues, like a missing x or y, a slope of 0 or an undefined slope.

Homework Problems

1. For the line L which passes through the points $A = (2, -3)$ and $B = (-3, 1)$:

   (a) Determine the slope of line L using Slope Formula.
   (b) On the axes (right), plot points A and B.
   (c) Illustrate what the slope means using line segments and arrows.



2. A line passes through the points $(\frac{2}{3}, 0)$ and $(-1, 2)$. Determine the slope of the line.

3. A line passes through the points $(-1, 0)$ and $(-1, 5)$. Determine the slope of the line. What special type of line is this?

4. A line passes through the points $(-2, 2 \cdot \sqrt{2},)$ and $(2, 4 \cdot \sqrt{2})$. Determine the slope of the line.

5. A line crosses the x-axis at -3 and the y-axis at -4. Determine the slope of the line.

6. Determine the slope of the line with equation $-x = 2 - 3y$ by determining two points on the line and calculating the slope.

7. Determine the slope of the line with equation $-x = 2 - 3y$ by rearranging the equation so y is isolated.

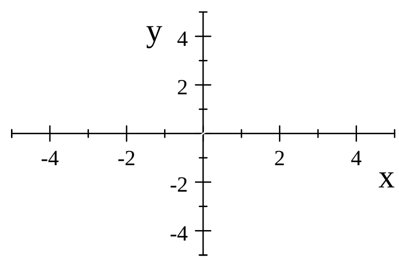
8. If you know the equation of a line, which method of finding its slope is easier: the method in Problem 6 or Problem 7?

9. Determine the equation of a line which has slope 0 and passes through the point $(3, -2)$. What special type of line is this?

10. Line L passes through the point $(-2, 1)$ and is parallel to the line M which has equation $x = 4$. Determine the equation of line L.

11. Graph lines L and M from Problem 10 on the axes below. Fill in the tables with two points each for lines L and M.

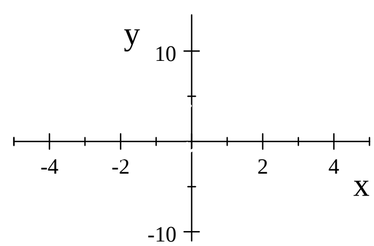
line L		line M	
x	y	x	y



12. Determine the equation of a line L that passes through the point $(4, -4)$ and is perpendicular to the y-axis. What special type of line is line L?

13. Line L passes through the points $(0, -1)$ and $(2, 3)$ and line M passes through the points $(-1, 2)$ and $(0, 4)$.

(a) Determine the slope of line L.
(b) Determine the slope of line M.
(c) Graph lines L and M (right).
(d) How do lines L and M appear to be related?



14. Which statement or statements below are false:

(a) The line through the points $(-1, 6)$ and $(3, 7)$ has positive slope.

(b) The line $x + 4 = 1$ has slope 0.

(c) The line $x + y = -3$ has x-intercept at $(0, -3)$.

(d) The line $y = 3$ is parallel to the y-axis.

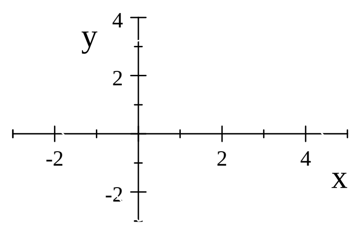
(e) Any vertical line and horizontal line will intersect each other.

15. In 1995 Jean's salary was \$36000; then \$37500 in year 2000. In 1996 Sally's salary was \$33700; then \$34400 in 1998. Whose average rate of increase in salary is greater?

16. A company produces 10 million computers in 1995 then 13 million computers in 1999. If the rate of production continues to grow at a steady rate, how many computers will have to be produced in 2005?

17. A quadrilateral has vertices $A = (-3, 2)$, $B = (0, -3)$, $C = (2, 4)$ and $D = (5, -1)$.

(a) Graph the quadrilateral (right).
(b) Use slope to prove that the opposite sides in $ABDC$ are parallel.
(c) What kind of quadrilateral is $ABDC$?



18. Line L passes through $(-2, 3)$ and $(3, 6)$ and line M passes through $(-9, -5)$ and $(1, w)$. What should w be so that the lines L and M have the same slope?

19. Suppose the cost of producing a radio commercial is 10% greater each year over the previous year.

(a) If the producing cost was \$1000 in 1990, what would be the cost in 1991?; in 1992?; and in year 1993? Record your answers as points, where the x-coordinate is the number of years past 1990 and the y-coordinate is the production cost.

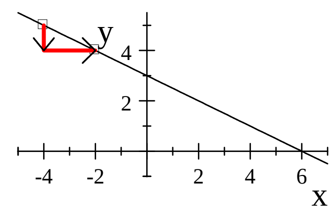
(b) Is the production cost rising with a steady slope? $\boxed{a}$

2.3 The Point-Slope Form of a Line

Perhaps you have heard the statement "any two points determine a line." Indeed, if you know just two points on a line there is one and only one line connecting the two points. This a special feature of lines. For example, two points would not be enough to determine a U-shaped graph (for more about this see Homework Problem 11 at the end of this section).

Another way to uniquely describe a line is by a point on the line and the slope of a line. For example, there is a unique line L which goes through the point $(-4, 5)$ and has slope $-\frac{1}{2}$:

To draw the line L start at point $(-4, 5)$ then travel to another point on the line using the slope. For example thinking of the slope as $\frac{-1 \quad \text{[change in y]}}{2 \quad \text{[change in x]}}$, the slope leads to a second point $(-2, 4)$. Then, join the two points with a line.



In this section our goal will be to determine the equation of a line given a point on the line and the slope of the line using the Point-Slope Form. We begin by defining the Point-Slope Form and then solving some examples.

The Point-Slope Form
If a line has slope m and passes through the point (x_1, y_1), then the Point-Slope Form of the line is $y - y_1 = m(x - x_1)$. (Our normal procedure will be to isolate y so the final answer is unique.)

Example 2.11 *Determine the equation of the line L that has slope 2 and passes through the point $(3, 0)$. (y should be isolated in the final answer).*

Solution: The Point-Slope Form tell us the equation of the line L is $y - (0) = 2(x - (3))$ which simplifies to $y = 2x - 6$. ∎

Example 2.12 *Determine the equation of the line L above which passes through the point $(-2, 4)$ and has slope $\frac{-1}{2}$.*

Solution: Substituting the given point and slope into the Point-Slope Form yields $y - (4) = \frac{-1}{2}(x - (-2))$. Then just continue to simplify and isolate y for the final answer *(remember this is preferred form because of calculators and function notation)*:

$$\begin{aligned} & y - 4 = \tfrac{-1}{2}(x+2) \\ \leftrightarrow\ & y - 4 = \tfrac{-1}{2}x - 1 \\ \leftrightarrow\ & y = \tfrac{-1}{2}x + 3. \end{aligned}$$

Hence, the equation of line L is $y = \frac{-1}{2}x + 3$. ∎

Example 2.13 *Determine the equation of the line L that passes through the point $(4, 6)$ and $(-3, 5)$. (y should be isolated in the final answer).*

Solution: First, the Slope Formula gives:

$$m = \tfrac{(5)-(6)}{(-3)-(4)} = \tfrac{-1}{-3-4} = \tfrac{-1}{-7} = \tfrac{1}{7}$$

Then we apply the Point-Slope Form with either point and simplify:

$$\begin{array}{l|l} y-(6) = \frac{1}{7}(x-(4)) & y-(5) = \frac{1}{7}(x-(-3)) \\ \leftrightarrow y = \frac{1}{7}x - \frac{4}{7} + 6 & \leftrightarrow y = \frac{1}{7}x + \frac{3}{7} + 5 \\ \leftrightarrow y = \frac{1}{7}x + \frac{38}{7}. & \leftrightarrow y = \frac{1}{7}x + \frac{38}{7}. \end{array}$$

Either way, the equation of line L is $y = \frac{1}{7}x + \frac{38}{7}$. ∎

See how easy it is to determine the equation of a line? Just be extra careful with vertical lines. Because vertical lines have undefined slope, it's impossible to use the Point-Slope Form with them. It's best and easiest just to remember that vertical lines have the form $x = c$. It's also practical to remember horizontal lines have the form $y = b$, or you could use the Point-Slope Form with $m = 0$.

Deriving the Point-Slope Form

Finally, let's take a moment to see where the Point-Slope Form comes from. The derivation is actually rather short but elegant. If we know a line passes though the point (x_1, y_1) and has slope m, and if we use (x, y) to represent another arbitrary second point on the line, the slope formula tell us $m = \frac{(y)-(y_1)}{(x)-(x_1)} = \frac{y-y_1}{x-x_1}$. Then, multiply left and right hand sides of this equation by $(x - x_1)$ and simplify:

$$m \cdot (x - x_1) = \frac{y-y_1}{x-x_1} \cdot (x - x_1)$$
$$\leftrightarrow m \cdot (x - x_1) = y - y_1$$
$$\leftrightarrow y - y_1 = m \cdot (x - x_1), \text{ the Point-Slope Form. } \blacksquare$$

OK, here's a problem for you to try. Despite the word "prove," it is actually an easier problem. So be brave!

Try This 2.3 *If a line L has y-intercept at $(0, b)$ and slope m, use the Point-Slope Form to prove the equation of line L is actually $y = mx + b$.* $\boxed{Ⓢ}$

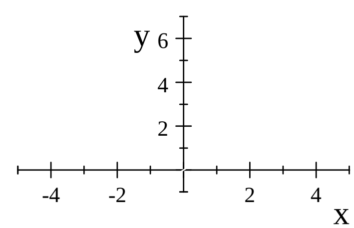
Homework Problems

1. Determine the equation of the line that has slope -2 and passes through the point $(-3, -1)$. Express the final answer with y isolated.

2. Determine the equation of the line that has slope $\frac{-2}{3}$ and passes through the x-axis at 4. Express the final answer with y isolated.

3. A line L has slope $\frac{-2}{3}$ and passes through the point $(-1, 4)$.

   a. Use the slope and point to graph the line L (to right).
   b. Determine the equation of the line L. Express the final answer with y isolated.
   c. Use the equation to determine the value of y when x is 3. Does your answer agree with the graph?
   d. Use the equation to determine the value of x when y is 6. Does your answer agree with the graph?



4. A line M has slope 2 and passes through the point $(-1, -4)$.

   (a) Determine an equation of the line M. Isolate y in the final answer.

   (b) Use the equation to determine the value of y when x is $\frac{5}{4}$. Express your final answer as a fraction.

   (c) Use the equation to determine the value of x when y is -2.3. Express your final answer as a decimal.

5. Line N passes through the points $(-4,-2)$ and $(2,1)$. Determine an equation of the line N. Isolate y in the final answer.

6. Line O passes through the point $(2,3)$ has the same slope as line N (previous problem).

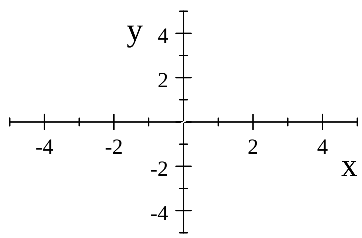
   (a) Determine an equation of the line O. Express your final answer with y isolated.

   (b) Graph lines N and O on the axes below. Make sure you determine and plot at least two points for line O.

   (c) What do you notice about the graphs of lines N and O?

line N	
x	y
-4	-2
2	1

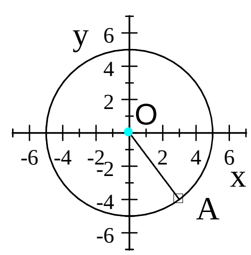
line O	
x	y



7. Line P crosses the x-axis at $\frac{1}{2}$ and the y-axis at $\frac{-3}{2}$. Determine the equation of line P.

8. The circle below has its center at the origin O, has radius 5 units long and passes through the point $A=(3,-4)$.

   (a) Graph the line Q that passes through $A=(3,-4)$ and has slope $\frac{3}{4}$. Besides $(3,-4)$, what other point did you use to graph line Q?
   (b) What do you notice about the line Q and the circle?
   (c) Determine the slope of the line segment from the origin O to A.
   (d) How does the slope of line segment OA relate to the slope $\frac{3}{4}$ of line Q?



9. It is known that 0 degrees Celsius corresponds to 32 degrees Fahrenheit and Fahrenheit degrees F and Celsius degrees C rise/fall at a steady rate of 9 degrees Fahrenheit for every 5 degrees Celsius. Thinking of F as y and C as x, use the Point-Slope Form to derive the formula $F=\frac{9}{5}C+32$.

10. *Imagine* the amount of medicine y (in ounces) a child takes increases at a steady rate according the weight of the child x (in pounds). If a 20 pound child should get a 3 ounce dose and a 50 pound child should get a 5 ounce dose, use the Point-Slope Form to determine the equation for y and x. Then, use the equation to determine the proper dose for a 40 pound child.

11. On piece of paper, plot any two points. Draw two different parabolas that pass through those two points. $\boxed{a}$

2.4 The General and Slope-Intercept Forms

The General Form of a linear equation is $Ax + By = C$ where A, B and C are understood to be real numbers. A and B can not both be 0 to avoid zeroing out the left hand side of the equation. In common examples, A and B are integers and have an LCD of 1. For example $A = 2$, $B = -1$ and $C = 0$ yield the linear equation $2x - y = 0$. Or, $A = 0$, $B = -1$ and $C = 9$ yield $-y = 9 \leftrightarrow y = -9$ which corresponds to a horizontal line.

The General Form $Ax + By = C$
The General Form for a linear equation is $Ax + By = C$, where A, B and C are real numbers and A and B can not both be 0.

Lines are described in General Form for various reasons. One reason is to have the x's and y's line up nicely in a system of equations like $\left\{ \begin{smallmatrix} 3x - y = 7 \\ x - y = 5 \end{smallmatrix} \right\}$ (studied in Chapter 4). Another reason for the General Form is to avoid using fractions. For example, the linear equation $y = -\frac{3}{4}x + \frac{1}{2}$ could be rewritten into General Form with integers only by multiplying both sides of the equation by its LCD of 4 and bringing the x-term to the left:

$$
\begin{aligned}
&y = -\tfrac{3}{4}x + \tfrac{1}{2} \\
&\quad \leftrightarrow 4 \cdot (y) = 4 \cdot (-\tfrac{3}{4}x + \tfrac{1}{2}) \\
&\quad \leftrightarrow 4y = -3x + 2 \\
&\quad \leftrightarrow 3x + 4y = 2.
\end{aligned}
$$

Now the General Form $3x + 4y = 2$ may look nicer, but the equivalent equation $y = -\frac{3}{4}x + \frac{1}{2}$, in Slope-Intercept Form, is actually more useful. Let's review some previous discoveries about the Slope-Intercept Form:

The Slope-Intercept Form $y = mx + b$

In the Slope-Intercept form $y = mx + b$, m represents the slope of the line and b corresponds to the y-intercept of the line at $(0, b)$. The Slope-Intercept Form does not apply for vertical lines because m is undefined.

Example 2.14 *Given a line with equation $4x + 2y = 1$, determine the slope and y-intercept of the line. Use the slope to find another point on the line.*

Solution: Isolate the y in $4x + 2y = 1$ to obtain the Slope-Intercept Form,

$$\begin{aligned}&4x + 2y = 1\\ &\ \leftrightarrow 2y = -4x + 1\\ &\ \leftrightarrow \tfrac{1}{2} \cdot (2y) = \tfrac{1}{2} \cdot (-4x + 1)\\ &\ \leftrightarrow y = -2x + \tfrac{1}{2}.\end{aligned}$$

So the slope of the line is -2 and the y-intercept is $(0, \frac{1}{2})$. Thinking of the slope -2 as $\frac{-2 \text{ [change in y]}}{1 \text{ [change in x]}}$, another point on the line would be $(1, \frac{-3}{2})$. ∎

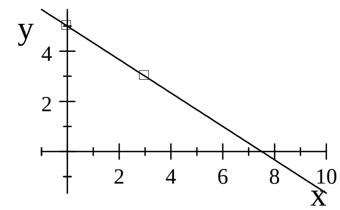
2.4.1 Linear Functions

Another reason why the Slope-Intercept Form stands out as very important is that linear functions are written in this form. A linear function has the form $f(x) = mx + b$. Linear functions, which are essentially the same as linear equations in Slope-Intercept Form, are very easy to graph because the slope and y-intercept are already part of the function rule.

Example 2.15 *Graph the linear function $f(x) = \frac{-2}{3}x + 5$ using the slope and y-intercept. In addition, determine the x-intercept of the graph.*

Solution: Recall that when graphing a function, x still acts as the x-coordinate while $f(x)$ serves as the y-coordinate.

Because $f(x) = \frac{-2}{3}x + 5$ is in Slope-Intercept Form, the graph will be a line with slope $\frac{-2}{3} \frac{\text{[change in y]}}{\text{[change in x]}}$ and y-intercept $(0, 5)$. According to the slope, starting at the point $(0, 5)$, a second point on the line would be $(3, 3)$ as shown to the right:



Graph of $f(x) = \frac{-2}{3}x + 5$

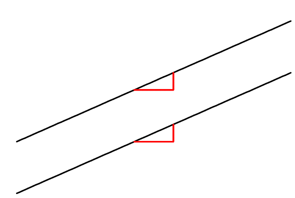
To determine the x-intercept, set $y = 0$ and solve for x. Since y is $f(x)$, that means $0 = \frac{-2}{3}x + 5$. Then, solve for x step by step:

$$\begin{aligned} &0 = \tfrac{-2}{3}x + 5 \\ &\quad \leftrightarrow 3 \cdot (0) = 3 \cdot \left(\tfrac{-2}{3}x + 5\right) \\ &\quad \leftrightarrow 0 = -2x + 15 \\ &\quad \leftrightarrow 2x = 15 \\ &\quad \leftrightarrow x = \tfrac{15}{2}. \end{aligned}$$

Therefore, $(\frac{15}{2}, 0)$ is the x-intercept. ∎

Parallel Lines

Recall that two lines are parallel (in a plane) if they never intersect. But in a more practical way we could say two lines are parallel if they have the same slope but are not the same lines. A typical pair of parallel lines might look like the ones to the right.


Parallel Lines

Example 2.16 *Suppose line L has equation $y = 2x + 4$ and line M has equation $-2x + y = 0$. Are lines L and M equal, parallel or neither.*

Solution: Line L is already Slope-Intercept Form. Line M in Slope-Intercept Form is $y = 2x$. Based on their equations, we can see the lines have the same slope but are not equal (have different y-intercepts). So the lines are parallel. ∎

Example 2.17 *Suppose line L has equation $2x - 7y = 14$ and line M has equation $-3x = \frac{-21}{2}y - 21$. Are lines L and M equal, parallel or neither.*

Solution: Get each line into Slope-Intercept Form:

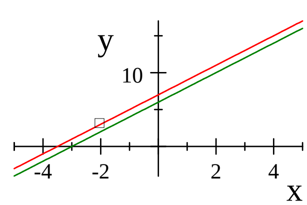
Line L	Line M
$2x - 7y = 14$	$-3x = \frac{-21}{2}y - 21$
$\leftrightarrow -7y = -2x + 14$	$\leftrightarrow -3x + 21 = \frac{-21}{2}y$
$\leftrightarrow y = \frac{-1}{7}(-2x + 14)$	$\leftrightarrow \frac{2}{-21}(-3x + 21) = y$
$\leftrightarrow y = \frac{2}{7}x - 2.$	$\leftrightarrow \frac{2}{7}x - 2 = y$
	$\leftrightarrow y = \frac{2}{7}x - 2.$

So lines L and M both have equation $y = \frac{2}{7}x - 2$. Hence the lines are equal, not parallel. ∎

Example 2.18 *Determine the equation of the line F that passes through $(-2, 3)$ and is parallel to the line G with equation $2x - y = -6$.*

Solution: Lines F and G must have the same slope in order to be parallel. Since $2x - y = -6 \leftrightarrow y = 2x + 6$, we know line G must have slope 2. Hence, line F has slope of 2. Line F passes through $(-2, 3)$, so its equation from the Point-Slope Form is $y - 3 = 2(x - (-2))$, which simplifies to $y = 2x + 7$. ∎

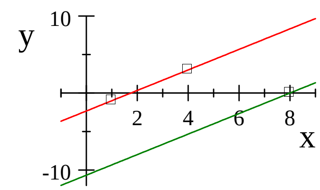
By the way, here's a graph of lines F and G from the last example. Can you tell which line is F and which one is G? An easy way is to use the y-intercepts. Line F with equation $y = 2x + 7$ is the top line with y-intercept at $(0, 7)$. ∎



Lines F and G

Example 2.19 *Suppose line J passes through the points $(4, 3)$ and $(1, -1)$. Determine the equation of the line K that is parallel to line L and has x-intercept at $(8, 0)$.*

Solution: J has slope $m = \frac{(-1)-(3)}{(1)-(4)} = \frac{-4}{-3} = \frac{4}{3}$. Being parallel to J, line K must have the same slope. Hence, the equation for line K would be $y - 0 = \frac{4}{3}(x - 8)$, which is the same as $y = \frac{4}{3}x - \frac{32}{3}$. ∎



Lines J and K

Example 2.20 *Determine the equation of the line that is parallel to the y-axis and passes through the point $(3, 4)$.*

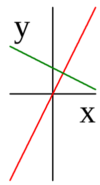
Solution: This line must be vertical so m is undefined which rules out using the Point-Slope Form. Recall that vertical lines have equations of the form $x = c$. Since $(3, 4)$ is on the line, the equation must be $x = 3$. ∎

Hopefully, those parallel line examples were clear. If not, now's a good time to ask your instructor some questions! Let's summarize the key fact about parallel lines:

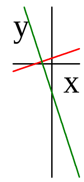
Summary of Parallel Lines Facts
If two lines (in a plane) are given to be parallel then automatically their slopes equal. However, to conclude two lines (in a plane) are parallel, the lines must have the same slope but not be the same line.

Perpendicular Lines

According to definition, two lines are perpendicular if they meet at a 90 degree angle. For example, any horizontal line is perpendicular to any vertical line. Now for non-vertical/non-horizontal perpendicular lines, we might visualize examples like these:



Perpendicular lines



Perpendicular Lines

Reading the graphs from left to right, do you notice in the first graph how one line goes up quickly and the other line goes down slowly, whereas in the second graph one line goes down quickly and the other goes up slowly? The slopes in both examples are opposites in that one is positive and one is negative and one is large in magnitude and the other small. The exact relation turns out to be exactly a "negative reciprocal" relation. Examples of negative reciprocals are: 2 and $\frac{-1}{2}$; $\frac{-3}{2}$ and $\frac{2}{3}$; and -1 and 1. The general rule for perpendicular lines is:

How to Determine if Two Lines are Perpendicular
Two lines are perpendicular if their slopes are negative reciprocals or if one line is vertical and one is horizontal.

Example 2.21 *Determine the equation of the line L that passes through the point $(1, -2)$ and is perpendicular to the line M with equation $x + 2y = 0$. Graph both lines.*

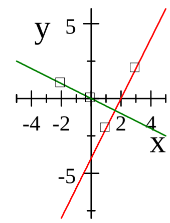
Solution: Isolating y in the equation $x+2y = 0$ yields the Slope-Intercept Form $y = \frac{-1}{2}x$. Therefore line M has slope $\frac{-1}{2}$.

So line L must have slope 2. Therefore, the Point-Slope Form equation of line L is $y - (-2) = 2(x - (1))$ which simplifies to $y + 2 = 2x - 2 \leftrightarrow y = 2x - 4$.

Here's a graph of the two lines.

$y = 2x - 4$	
x	y
1	-2
3	2

$y = \frac{-1}{2}x$	
x	y
0	0
-2	1



Lines L and M

Notice how we used (at least) two points on each line to make an accurate graph? By the way, if you use a graphing calculator to graph perpendicular lines, make sure you use the "zoom square" option. Otherwise the lines may not look perpendicular. ∎

Example 2.22 *Are the lines with equations $3x + 2y = 9$ and $6x - 4y = 0$ equal lines, parallel lines, perpendicular lines or none of these.*

Solution: Get both lines into Slope-Intercept Form:

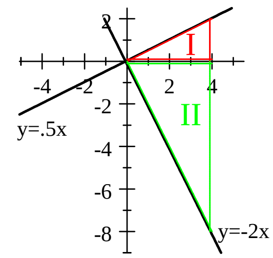
$$\begin{array}{l|l} 3x + 2y = 9 & 6x - 4y = 0 \\ \quad \leftrightarrow 2y = -3x + 9 & \quad \leftrightarrow -4y = -6x \\ \quad \leftrightarrow y = \frac{1}{2}(-3x + 9) & \quad \leftrightarrow y = \frac{-1}{4}(-6x) \\ \quad \leftrightarrow y = \frac{-3}{2}x + \frac{9}{2}. & \quad \leftrightarrow y = \frac{3}{2}x \\ \quad \rightarrow m = \frac{-3}{2} & \quad \rightarrow m = \frac{3}{2}. \end{array}$$

The two slopes $\frac{-3}{2}$ and $\frac{3}{2}$ are not equal and are not negative reciprocals. Hence the lines are not equal, not parallel and not perpendicular. ∎

OK here's a problem that proves with geometry that the lines $y = \frac{1}{2}x$ and $y = -2x$ are perpendicular. Of course, we know those lines should be perpendicular because their slopes, $\frac{1}{2}$ and -2, are negative reciprocals. You will need to use the full version of the Pythagorean Theorem which states the three sides of a triangle a, b and c (longest side) satisfying $a^2 + b^2 = c^2$ is equivalent to the triangle being a right triangle.

Try This 2.4 *Below is a graph of the linear equations $y = \frac{1}{2}x$ and $y = -2x$ and two right triangles, Triangles I and II. Let Triangle III be the triangle obtained by combining Triangle I and Triangle II into a single triangle.*

(a) How long are the height and base of Triangle I? By the Pythagorean Theorem, how long is the hypotenuse of Triangle I?
(b) How long are the height and base of Triangle II? By the Pythagorean Theorem, how long is the hypotenuse of Triangle II?
(c) How long are the three sides of Triangle III?
(d) Verify that the sides of the Triangle III satisfy the Pythagorean Theorem.
(e) What conclusion can you draw about Triangle III and the two lines? $\boxed{Ⓢ}$



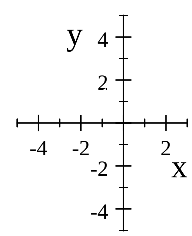
Homework Problems

1. Find the slope and y-intercept of the line given by $x - 2y = -4$.

2. Find the slope and y-intercept of the line given by $-x = \frac{1}{2}y + 1$.

3. Referring back to Problems 1-2:

   Graph the lines from Problems 1-2 using their slopes and y-intercepts on the axes to the right. Are the lines related in any special way?



4. Line C has equation $y = x - 5$. If a line D is to be parallel to line C what should be the slope of line D? If a line E is to be perpendicular to line C what should be the slope of line E?

5. *Imagine* the value of a stock fund A is governed by the formula $V = 120x + 5000$, whereas the value of stock fund B is governed by the formula $V = 110x + 5500$. Which stock fund has greater value in the short run? Which one has greater value in the long run? Explain.

6. A soccer ball is traveling down the line $2y + x = 10$. If a defensive player runs to intercept the ball at a $90°$ angle, what should be the slope of the runner's path?

7. Line L passes through points $(3,4)$ and $(-1,-8)$. Line M has equation $y = 3x - 5$. Are L and M identical, parallel, perpendicular or none of these?

8. Line L has equation $2x+3y = 6$. Line M has equation $.5x = -.75y+9$. Are these lines identical, parallel, perpendicular or neither?

9. Line L has equation $2x + 3y = 7$. Line M has equation $-3x + 2y = 0$. Are these lines identical, parallel, perpendicular or none of these?

10. Using slope, prove quadrilateral $ABDC$ with vertices at $A = (-1,4)$, $B = (6,2)$, $C = (-3,-3)$ and $D = (4,-5)$ is a rectangle.

11. The fee F charged by a plumber for x hours of work is given by the function $F(x) = 75x + 35$.

    (a) Compute $F(3.5)$ and $F(9.5)$ and interpret what these values mean.

    (b) What do the 75 and 35 represent mathematically? What do they mean in the context of this problem?

12. In terms of slope and y-intercept, in what way would the graphs of $f(x) = 2x - 1$, $g(x) = 4x + 1$ and $h(x) = 6x - 10$ be similar? In what way would they differ? Which graph would be highest when $x = 0$? Which graph would be highest when $x = 1000$?

13. *Imagine* line L has General Form $2x - 6y = -3$ while line M has General Form $4x - 3y = 12$.

    (a) Determine the x- and y-intercepts of the lines L and M.

    (b) Look at your answers to part (a) carefully. Describe an easy way to determine the x- and y-intercepts of a line when the line is in General Form $Ax + By = C$. $\boxed{a}$

Chapter 3

Conversion Factors

3.1 A Basic Way to Use Conversion Factors

You have seen conversion factors many times before this course. For example, two of the most common conversion factors are $\frac{12 \text{ inches}}{1 \text{ foot}}$ and its reciprocal $\frac{1 \text{ foot}}{12 \text{ inches}}$. Conversion factors are actually ratios that relate a measurement of one type to its equivalent measure of a second type (e.g. inches and feet). Since the numerator and denominator of a conversion factor are considered equivalent, conversion factors are actually forms of the number 1.

Conversion factors are used to convert all measurements of one type to their equivalent measures of the second type. A typical conversion, no matter how it is done, will ultimately require multiplication and/or division, but it can be hard to guess (based only on intuition) which one and when. The following method for doing conversions is simple and takes away the need for guess work!

> The Multiplication Method for Conversion Factors
>
> Step 1: If necessary, look up the conversion factor information. (Memorize simple everyday conversion factors. Non-standard ones would be given to you on a test.)
> Step 2: Set up the conversion factor so that the old units will cancel away when multiplied by the conversion factor.
> Step 3: Do the final calculation.

Example 3.1 *Use the fact that* 1 *kilogram (kg) is approximately* 2.2 *pounds* *(lb) to determine the weight in kg of a* 200*-pound man. Round your final answer to the nearest tenth.*

Solution: In terms of Step 1, the conversion factor information is given. For Step 2, we will set up the conversion factor so that pounds cancel out and kilograms remain, thus making the conversion factor $\frac{1 \text{ kg}}{2.2 \text{ pound}}$. Step 3 is:

$$200 \text{ pound} \approx 200 \cancel{\text{lb}} \cdot \frac{1 \text{ kg}}{2.2 \cancel{\text{lb}}} = \frac{200}{2.2}\text{kg} \approx 90.9 \text{ kg}.$$

So, 200 pounds $\approx$ 90.9 kg. We used $\approx$ twice because the conversion factor $\frac{1 \text{ kg}}{2.2 \text{ pounds}}$ is not exact, and because we rounded 200 divided by 2.2 to the nearest tenth as requested. ∎

Notice how the last part of the example was a division problem? Setting up the conversion factor so the old units cancel out determines automatically whether the final arithmetic will be multiplication and/or division.

Example 3.2 *If it takes* 12 *minutes to burn* $5\frac{1}{3}$ *CDs, determine the number of minutes it would take burn* 20 *CDs.*

Solution: Again the conversion factor information is given. We we will think of 12 min as equivalent to $5\frac{1}{3}$ CDs. Next, because we want CDs to cancel out, our conversion factor will be $\frac{12 \text{ min}}{5\frac{1}{3} \text{ CD}}$. Since there are no instructions to round, we will work with fractions to avoid decimals and rounding. Here's the conversion step:

$$20 \text{ CD} = 20 \cancel{\text{CD}} \cdot \frac{12 \text{ min}}{5\frac{1}{3} \cancel{\text{CD}}} = \frac{20 \cdot 12}{5\frac{1}{3}} \text{ min}.$$

Next, to simplify $\frac{20 \cdot 12}{5\frac{1}{3}}$ first rewrite $5\frac{1}{3}$ as $\frac{16}{3}$ and then do the fraction work:

$$\frac{20 \cdot 12}{5\frac{1}{3}} = \frac{20 \cdot 12}{\frac{16}{3}} = 20 \cdot 12 \cdot \frac{3}{16} = 45.$$

Hence, 20 CD equates to 45 minutes. ∎

You may recall that one-step conversions like the last two examples can also be done using proportions. However when there are two or more conversions to be accomplished, as in the next example, it is generally easier to use the Multiplication Method.

Example 3.3 *Which speed is faster:* $8\ \frac{ft}{sec}$ *or* $8\ \frac{mi}{hr}$*?*

To answer this question we could convert $\frac{\text{ft}}{\text{sec}}$ to $\frac{\text{mi}}{\text{hr}}$ or vice versa. Let's go with $\frac{\text{ft}}{\text{sec}}$ to $\frac{\text{mi}}{\text{hr}}$ because we are more familiar with $\frac{\text{mi}}{\text{hr}}$. This will require two conversions from ft to mi and from sec to hr, but both can be done in one step.

Since we are to start with $\frac{\text{ft}}{\text{sec}}$ and end with $\frac{\text{mi}}{\text{hr}}$, our conversion factors will be $\frac{1\text{ mi}}{5280\text{ ft}}$ (to cancel away ft) and $\frac{3600\text{ sec}}{1\text{ hr}}$(to cancel away sec). So, the conversion is:

$$\begin{aligned} 8\ \frac{\text{ft}}{\text{sec}} &= 8\ \frac{\cancel{\text{ft}}}{\cancel{\text{sec}}} \cdot \frac{1\text{ mi}}{5280\ \cancel{\text{ft}}} \cdot \frac{3600\ \cancel{\text{sec}}}{1\text{ hr}} \\ &= \frac{8{\cdot}3600}{5280}\frac{\text{mi}}{\text{hr}} = \frac{8{\cdot}360}{528}\frac{\text{mi}}{\text{hr}} = \frac{1{\cdot}360}{66}\frac{\text{mi}}{\text{hr}} \\ &= \frac{60}{11}\frac{\text{mi}}{\text{hr}} = 5\frac{5}{11}\frac{\text{mi}}{\text{hr}}. \end{aligned}$$

Hence, $8\ \frac{\text{ft}}{\text{sec}}$ is $5\frac{5}{11}\frac{\text{mi}}{\text{hr}}$; so $8\ \frac{\text{mi}}{\text{hr}}$ is actually much faster than $8\ \frac{\text{ft}}{\text{sec}}$. ∎

Here's a question that will not allow any guessing or intuition since the units are complete non-sense!

Try This 3.1 *Suppose that* 3 *yobs are equivalent to* 8 *babs and* 14 *babs are equivalent* 23 *hons. How many yobs equal* 50 *hons?* Ⓢ

Before going on with conversions, please look through these tables of equivalent measures. You probably are already familiar with many of them. You can also look up equivalent measure information on your own at asknumbers.com.

Length	Length	Time
1 ft=12 in	1 km =1000 m	1 min=60 sec
1 yd=3 ft	1 m=100 cm	1 hr=3600 sec
1 mi=5280 ft	1 m=1000 mm	1 day=24 hr
1 km ≈ 0.62 mi	1 in = 2.54 cm	1 sec =1000 millisec

Volume	Volume	Weight
1 =1000 ml	1 G = 4 qt	1 kg ≈ 2.2 pound
1 cc=1 ml	1 qts = 2 pt	1 pound=16 oz
1 G ≈ 3.79 L	1 pt = 2 cup	1 oz =28.53 g
1 fl oz ≈ 29.57 ml	1 cup = 8 fl oz	1 ton =2000 pound

Examples with Area and Volume

The area for a region like a rectangle or triangle is the number of square units (or units^2) the region contains. Units for area are examples of two-dimensional units because area is generally determined by length times width with some adjustment due to shape! For example, the area of a rectangle is $l \cdot w$ while the area of a triangle is $\frac{1}{2} \cdot b \cdot h$. So, when ft or m used used to measure length, the units for area are ft·ft or square feet (ft^2) or m·m or square meters (m^2).

Similarly the volume of object that holds space like a box or a ball is the number of cubic units (or units^3) the object contains. Units for volume are examples of three-dimensional units because the volume of a region is fundamentally determined by length times width times height again with some adjustment due to shape. For example the volume of a typical box is $l \cdot w \cdot h$ but the volume of a ball (sphere) is $\frac{4}{3}\pi \cdot r \cdot r \cdot r$. So, when ft or m are used to measure length, the units for volume are ft·ft·ft or cubic feet (ft^3) or m·m·m or cubic meters (m^3).

Conversions that involve area and volume are not more difficult than others, but there are some common errors to be aware of.

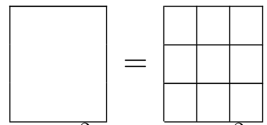
Example 3.4 *Determine the number of square yards there are in a rectangle that is* 6 *ft long by* 9 *ft wide?*

Solution: A common incorrect answer would be 18 yd^2 based on the intuition of $6 \cdot 9$ for the area then divided by 3 because there are 3 feet in a yard. The issue is that 1 yd^2 = 9 ft^2, not 3 ft^2. Hence, the correct conversion factor to change from ft^2 to yd^2 is $\frac{1 \text{ yd}^2}{9 \text{ ft}^2}$ and so:

$$54 \text{ ft}^2 = 54 \cancel{\text{ft}^2} \cdot \frac{1 \text{ yd}^2}{9 \cancel{\text{ft}^2}} = \left(\frac{54}{9}\right) \text{yd}^2 = 6 \text{ yd}^2.$$

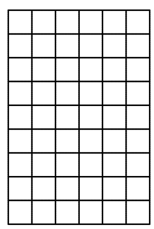
Another correct answer could be found by converting both ft measurements to yd and then multiplying because the area of a rectangle is $l \cdot w$. In this case it's easy to see that 6 ft = 2 yd and 9 ft = 3 yd. So, again the correct area is (2 yd) · (3 yd) = 6 yd^2. ∎

The last example can be understood entirely with simple geometry. First, it can be seen that 1 square yd is 9 square ft by dividing a square yard into 1 foot by 1 foot squares:

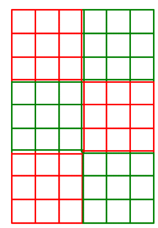


1 yd^2 9 ft^2

Then to see 54 ft^2 = 6 yd^2, take 6 ft by 9 ft rectangle and partition it into square yards–squares that are 3 ft long and 3 ft wide:



6 ft by 9 ft = 54 ft^2



54 ft^2 = 6 yd^2

Now, it is not practical to draw pictures to do conversions with area, especially when unusual lengths are involved. Here's a simple easy-to-remember procedure to derive conversion factors for square units:

How to Determine Conversion Factors for Area
Just square the corresponding conversion factor for length! For example, 1 yd=3 ft $\rightarrow$ (1 yd)^2=(3 ft)^2 $\rightarrow$ 1yd^2=9 ft^2.

Some other examples would be 1 ft = 12 in$\rightarrow$ 1 ft^2 = 144 in^2; and 1 m = 10 cm$\rightarrow$ 1 m^2 = 100 cm^2.

Example 3.5 *A can of paint will cover* 150 *sq ft. If a region extends over an area of* 38 *sq yd, how many cans of paint should be purchased to paint the region?*

Solution: The idea is to covert 38 square yards to square feet, this time using the conversion factor $\frac{9 \text{ ft}^2}{1 \text{ yd}^2}$ to cancel away the yd^2. The conversion process gives:

$$38 \text{ yd}^2 = 38 \cancel{\text{yd}^2} \cdot \frac{9 \text{ ft}^2}{1 \cancel{\text{yd}^2}} = (38 \cdot 9) \text{ ft}^2 = 342 \text{ ft}^2.$$

Hence, 3 cans of paint should be purchased since $2 < \frac{342}{150} < 3$. ∎

Example 3.6 *Nautical miles (nmi) are used at times in sea navigation to measure long lengths. If* 1 *mi is approximately* 0.87 *nmi and a sea rescue mission must cover an area of* 400 *nmi*^{2}*, would the search area be bigger or smaller than Stark County? According to the US Census Bureau, Stark County encompasses* 576 *mi*^{2}*.*

Solution: We need to make a conversion factor for square nmi to square mile and are given 1 mi $\approx$ 0.87 nmi. Squaring both sides gives 1 mi$^2 \approx$ 0.7569 nmi^2. So the approximate conversion factor will be $\frac{1\text{ mi}^2}{0.7569\text{ nmi}^2}$. Hence,

$$400\text{ nmi}^2 \approx 400\ \cancel{\text{nmi}^2}\cdot\frac{1\text{ mi}^2}{0.7569\ \cancel{\text{nmi}^2}} = \frac{400}{0.7569}\text{ mi}^2 \approx 528.47\text{ mi}^2.$$

Therefore, the search area is slightly smaller than Stark County. Let's just hope none of us are ever that sought-after needle in a hay stack. ∎

One last major type of measurement is volume. For example, in the rectangular box below that is 5 inches long, 3 inches high and 2 inches wide, the volume is literally (5 in)$\cdot$(3 in)$\cdot$(2 in) = 30 in^3:

This can be seen in the rectangular box to the right in that there are thirty 1-in by 1-in by 1-in cubes stacked on top, beside and behind each other to make up the volume of the box.

Cubic inch is a common unit of volume as is cubic centimeter (cc). There are also units for volume in everyday life that do not contain the word "cubic"–such as gallons (G) and liters (L). But they have cubic equivalents. For example, 1 G = 231 in^3 and 1 cc = 1 ml = 0.001 L.

In moving from area to volume, the only change is that to derive a cubic conversion factor from length, we will need to cube both sides of the equation.

How to Determine Conversion Factors for Volume
Just cube the corresponding conversion factor for length! For example, 1 yd=3 ft $\rightarrow$ (1 yd)^3=(3 ft)^3 $\rightarrow$ 1yd^3=27 ft^3.

Example 3.7 *How many cubic centimeters are in one cubic meter?*

Solution: Don't be hasty, the answer is not 100! Just start with the length conversion factor 1 m $=$ 100 cm and then cube both sides of the equation. Hence, $(1 \text{ m})^3 = (100 \text{ cm})^3 \rightarrow 1 \text{ m}^3 = 1,000,000 \text{ cm}^3$. ∎

To imagine a volume of 1 m^3, you could think of a large computer box. And if you think of a large computer box being filled with teaspoons of sugar, it's rather amazing. We know $1 \text{ m}^3 = 1,000,000 \text{ cm}^3$ and according to www.asknumbers.com, 1 teaspoon is about 5 cm^3. So a 1 m^3 box could hold about $200,000$ teaspoons of sugar. Wow that would be one "sweet" box!

Example 3.8 *A fish tank is shaped like a rectangular box that is $2\frac{1}{4}$ ft long, $1\frac{3}{4}$ ft wide and 1 ft tall. How many gallons of water would it take to fill the tank? Round the final answer to the nearest tenth.*

Solution: To solve this problem, we could convert the feet measurements to inches, then use the conversion factor $\frac{1\text{G}}{231 \text{ in}^3}$. Volume for a rectangular box is given by $V = l \cdot w \cdot h$. So $V = 27 \cdot 21 \cdot 12 = 6804 \text{ in}^3$. So:

$$6804 \text{ in}^3 = 6804 \cancel{\text{in}^3} \cdot \frac{1 \text{ G}}{231 \cancel{\text{in}^3}} = \frac{6804}{231}\text{G} = 29\frac{104}{231} \approx 29.5 \text{ G}. ∎$$

Homework Problems

When exact conversion factors are known, give exact answers. Otherwise round as indicated in the problem.

1. If a flight is 265 minutes long, how long is the flight in hours? If the plane departs at 12 PM EST, what would be the arrival time EST?

2. One kilometer (km) is approximately 0.62 mile (mi). If the distance from the Kent Campus to the Stark Campus is 29 miles, what would be that distance in kilometers (round to the nearest hundredth)? How many whole round-trips could be completed with a tank of gas if the service truck averages 500 km on a full tank?

3. One kilogram (kg) is about 2.2 pounds (lb). Which is heavier: a 50-lb child or a 23-kg dog?

4. A floor is 60 feet in length and 30 feet in width. How many square yard tiles should be purchased to cover the floor?

5. A circle has radius 33 inches. In the following problems, give an exact answer using π and an approximate answer (to the nearest tenth) using $\pi \approx 3.14$:

   (a) Determine the circumference of the circle in feet.

   (b) Determine the area of the circle in square feet.

   (c) If the radius is doubled to 66 inches, determine the new circumference (in feet) and area (in square feet) of the circle. (Use π to get the exact answers.)

6. A while ago the average price for 1 liter (L) of gasoline was about 1.26 euros in Paris, France. If the exchange rate between USD and euros was 1 USD = 0.82 euro and there about 3.79 liters in 1 gallon (G), what was the cost in USD to fill up a 25-gallon SUV in Paris?

7. Sharon can type 75 words in one minute. Determine how many 30-word paragraphs she could type in 8 hours of uninterrupted typing.

8. A professional pitcher throws a fastball 95 mile/hr and a cheetah at full sprint runs 90 ft/sec. Which speed is faster?

9. One fluid ounce is 2 tablespoons. If a homemade chili recipe calls for 5 tablespoons of mystery ingredient X per quart of chili, determine the number of fluid ounces of mystery X needed to make $10\frac{1}{2}$ gallons of chili.

10. How many cubic feet are in 1 cubic yard: 3, 9 or 27? Make a picture of a cubic yard that illustrates the answer.

11. A photo is about 0.25 mm thick and 6 cm long and 6 cm wide.

    (a) What would be the volume of a stack of 1000 photos in cm^3?

    (b) What would be the volume of a stack of 1000 photos in cm^3 if the photo dimensions are doubled to 0.5 mm thick, 12 cm long and 12 cm wide?

12. A standard sheet of paper is $8\frac{1}{2}$ inches by 11 inches.

    (a) Determine the area of one sheet of paper in square inches.

(b) Determine the area of one sheet of paper in square feet.

(c) Would a ream of paper (500 sheets) be enough to cover a floor that is the shape of a 20 ft by 20 ft square? (Cutting the paper to fit miscellaneous areas would be allowed.)

13. If there are 640 acres in 1 square mile, how many square feet are in one acre?

14. If the proper dose of a medication is 20 ml, what is the (approximate) equivalent doses in fluid ounces and tablespoons?

15. Re-solve Example 3.1 using the proportion $\frac{x}{200} \approx \frac{1}{2.2}$.

16. Re-solve Example 3.2 using the proportion $\frac{x}{20} \approx \frac{12}{5\frac{1}{3}}$.

17. Re-solve Example 3.3 using proportions. $\boxed{a}$

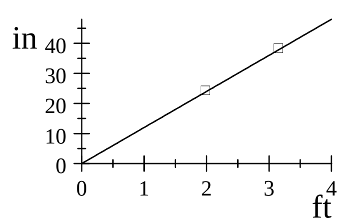
3.2 Conversion Factors as Slope

Let's go back to a very basic conversion factor like $\frac{12 \text{ inches}}{1 \text{ ft}}$. We can view this conversion factor as a slope because it implies that if we change the number of feet x by 1 ft, then the change in the number of inches y will be 12 inches. The fact that 12 represents a slope reoccurs in the formula $y = 12x$ which gives the number of inches y there are in feet x. Isn't $y = 12x$ simply the equation of a line with slope $m = 12$ and y-intercept at $(0, 0)$ (since $b = 0$)?

With the formula $y = 12x$ we can do many conversions, starting with x and getting y or vice versa. For example,

$$x = 2 \rightarrow y = 12 \cdot (2) = 24; \text{ or}$$
$$y = 38 \rightarrow 38 = 12x \rightarrow \frac{38}{12} = x \rightarrow x = \frac{19}{6} = 3\frac{1}{6}.$$

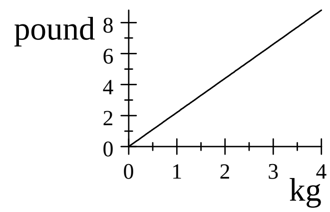
To the right is a graph of the line relating feet to inches using the points $(2, 24)$ and $(\frac{19}{6}, 38)$. Notice how we labeled the axes. This is very important in application problems where x and y have special meaning.



Graph of $y = 12x$

As another example, consider kilograms and pounds. We will work with the approximate conversion factor as $\frac{2.2 \text{ pounds}}{1 \text{ kg}}$ to take advantage of having a 1 in the denominator. If we think of x as kg and y as pounds, then $\frac{2.2 \text{ pounds}}{1 \text{ kg}}$ says that as x changes by 1 kg, y changes by approximately 2.2 pounds.

Hence a graph (right) of x kg compared to its equivalent y pounds would be a line with approximate slope 2.2. And since 0 kg $=$ 0 pound, the y-intercept is again at $(0, 0)$, so the equation of the line would be $y \approx 2.2x$.



Graph of $y \approx 2.2x$

The overall idea in the last two examples can be summarized with:

How to Make a Conversion Factor Equation
For each conversion factor, there is a conversion factor equation of the form $y = mx$. To determine the equation: Step 1: Think of the conversion factor (in the simplest form) as slope m. Step 2: Let y measure the amount in the units of the numerator of m. Let x measure the amount in the units of the denominator. Step 3: The equation relating y to x is the linear equation $y = mx$.

Example 3.9 *Suppose that* 8 *cars are produced for every* 750 *labor hours. Determine the number of cars produced when labor hours* = 500, 1000, 1500 *and* 2000. *Determine the number of labor hours needed to make* 50, 100, 150 *and* 200 *cars. Round any final decimal values to the nearest tenth.*

Solution: You may be inclined to solve this problem with proportions, and in principle you could. But it's not practical when there are many calculations to do. Instead, we will write a formula for x (number of cars) and y (number of labor hours) using the conversion factor $\frac{750 \text{ labor hours}}{8 \text{ cars}}$ as slope. Since $\frac{750}{8}$ reduces to $93\frac{3}{4}$ or 93.75, we can do all of the requested calculations using the formula $y = 93.75x$ since rounded decimal answers are acceptable. For example:

$$x = 50 \to y = 93.75 \cdot 50 = 4687.5; \text{ or}$$
$$y = 500 \to 500 = 93.75x \to x = \tfrac{500}{93.75} \approx 5.3.$$

The two tables below give the rest of the solutions:

x cars		y hours
≈ 5.3	$\leftarrow$	500
≈ 10.7	$\leftarrow$	1000
16	$\leftarrow$	1500
≈ 21.3	$\leftarrow$	2000

x cars		y hours
50	$\rightarrow$	4687.5
100	$\rightarrow$	9375
150	$\rightarrow$	14062.5
200	$\rightarrow$	18750.

∎

We have seen that when two units of measure are related by a conversion factor then they are automatically related by a linear equation. However, there are linear equations relating to two units of measure but no conversion factor. For example, give this problem a try.

Try This 3.2 *Let y=degrees Fahrenheit ($^\circ F$) and x=degrees Celsius ($^\circ C$).*

(a) What is the equation that relates y to x? In other words what is the equation that converts $^\circ C$ to $^\circ F$? (If you can't remember you can derive it by finding the equation of the line that goes through the points $(0, 32)$ and $(100, 212)$.)
(b) Is the slope of the equation a true conversion factor? For example, would just the slope correctly convert $0\ ^\circ C$ to $32\ ^\circ F$? What else is needed?
(c) Think about the graph of this linear equation. How could you tell from the graph that the slope by itself is not a true conversion factor? $\boxed{Ⓢ}$

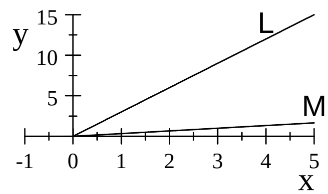
Homework Problems

1. Use the equivalence 1 tablespoon equals 3 teaspoons to determine an equation that relates the number of teaspoons y to the number of tablespoons x.

2. Use the equivalence 1 tablespoon equals 3 teaspoons to determine an equation that relates the number of tablespoons y to the number of teaspoons x.

3. Referring back to Problems 1-2:

(a) In the graph (right), which line segment corresponds to the equation in Problem 1 and which corresponds to the equation in Problem 2?
(b) How could you tell at a quick glance? Use the word "slope" in your answer.



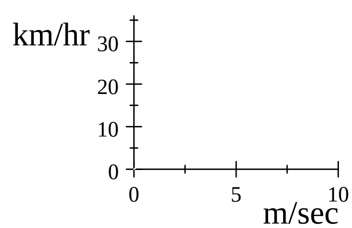
4. Suppose in a certain region for every 55000 people there are 12500 residential homes.

(a) Determine an equation that relates the numbers of homes y to people x.
(b) Determine the missing table values (right). Round final values to the nearest whole.
(c) Based on the table, how would you predict the size of y if x was 7000000?

x (people)	y (homes)
100000	
200000	
1000000	
	20000
	80000

5. In this problem, recall 1000 m = 1 km and 3600 sec = 1 hr.

(a) Solve: how many $\frac{\text{kilometers (km)}}{\text{hour (hr)}}$ does $1\ \frac{\text{meters (m)}}{\text{second (sec)}}$ equal?
(b) Letting y denote speed in $\frac{\text{km}}{\text{hr}}$ and x denote speed in $\frac{\text{m}}{\text{sec}}$, determine the linear equation relating y to x.
(c) Graph the line on the axes (right).



$\boxed{a}$

Chapter 4

Linear Systems of Equations

4.1 Solving Systems by Graphing

A system of equations in variables x and y is a collection of equations using variables x and/or y. To solve a system means to determine all points (x, y) that satisfy all equations in the system.

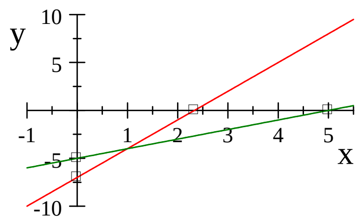
When a system of equations only contains linear equations, then it is called a linear system. For example, $\left\{\begin{smallmatrix} 3x - y = 7 \\ x - y = 5 \end{smallmatrix}\right\}$ is a linear system because $3x - y = 7$ and $x - y = 5$ are both linear equations (in General Form $Ax + By = C$). To solve the system would mean to determine all points that satisfy both the equations $3x - y = 7$ and $x - y = 3$. You could try to solve by guessing points, but it could take quite awhile. For example, $(3, 2)$ satisfies the first equation but not the second equation. Similarly, $(4, 1)$ satisfies the second equation but not the first. Actually, guessing can take forever because some systems have no solution! And even if you find one solution, guessing doesn't tell you if you've found all the solutions.

In this section we will solve a few linear systems of equations by graphing. This method is easy to understand, but is not always pragmatic.

Example 4.1 *Solve the system* $\left\{\begin{smallmatrix} 3x - y = 7 \\ x - y = 5 \end{smallmatrix}\right\}$ *by graphing.*

Solution: Since both equations are linear equations, we will only need two points on each line. Using the intercepts, here are the points and graph obtained:

	$3x - y = 7$	$x - y = 5$
x-intercept	$(\frac{7}{3}, 0)$	$(5, 0)$
y-intercept	$(0, -7)$	$(0, -5)$



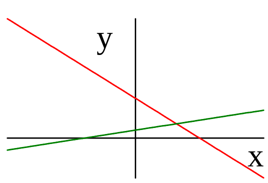
System $\left\{ \begin{smallmatrix} 3x - y = 7 \\ x - y = 5 \end{smallmatrix} \right\}$

The next step is to determine the point or points of intersection because such points satisfy both equations. From the graph, we can see there is only one point of intersection. What's unclear is the exact location of that point. It appears to be the point $(1, -4)$. Indeed $(1, -4)$ is correct since $3(1) - (-4) \stackrel{?}{=} 7$ and $1 - (-4) \stackrel{?}{=} 5$ are both true. ∎

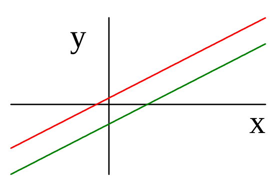
OK, let's summarize how to solve a system by graphing:

How to Solve a System by Graphing
Step 1: Graph each equation in the system. If the equations are linear, then only two points will be needed for each line. You could use intercepts and/or any other points. Step 2: Look for any points of intersection (solutions). Step 3: Check the solution by plugging into each equation.

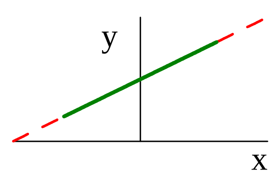
As we have seen with $\left\{ \begin{smallmatrix} 3x - y = 7 \\ x - y = 5 \end{smallmatrix} \right\}$, a system of two linear equations can have exactly one solution. More generally, if we visualize different pairs of lines, only three different outcomes are possible when solving a system of two linear equations:



One Solution



No Solution



Infinite Solutions

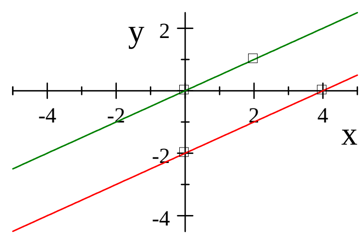
The next two examples show systems that have no solution (lines are parallel) or have infinitely many solutions (lines are equal).

Example 4.2 *Solve the system* $\left\{ \begin{smallmatrix} x & - & 2y & = & 4 \\ -2x & + & 4y & = & 0 \end{smallmatrix} \right\}$ *by graphing.*

Solution: As before, we will graph both lines using two points each. The intercepts give us:

	$x - 2y = 4$	$-2x + 4y = 0$
x-intercept	$(4, 0)$	$(0, 0)$
y-intercept	$(0, -2)$	$(0, 0)$

The second equation $-2x + 4y = 0$ has x- and y-intercept at the origin, so we will need an additional point. For example, $y = 1 \to -2x + 4 = 0 \to -2x = -4 \to x = 2$. The top line is $-2x + 4y = 0$ with y-intercept $(0, 0)$. So, $(2, 1)$ is also on the graph of $-2x + 4y = 0$. Here's a graph of the system (right):



System $\left\{ \begin{smallmatrix} x & - & 2y & = & 4 \\ -2x & + & 4y & = & 0 \end{smallmatrix} \right\}$

The two lines appear to be parallel, but remember graphs can be deceiving. One way we can determine if the lines really are parallel is to write both in Slope-Intercept Form. Remember that means rearranging each equation so y is isolated. Here are the steps for both lines:

$x - 2y = 4$	$-2x + 4y = 0$
$\leftrightarrow -2y = -x + 4$	$\leftrightarrow 4y = 2x$
$\leftrightarrow y = \frac{-1}{2}(-x + 4)$	$\leftrightarrow y = \frac{1}{4} \cdot (2x)$
$\leftrightarrow y = \frac{1}{2}x - 2$	$\leftrightarrow y = \frac{1}{2}x$
So, $m = \frac{1}{2}$ and $b = -2$.	So, $m = \frac{1}{2}$ and $b = 0$.

Hence the lines really are parallel since they have the same slope but different y-intercepts. Therefore the system has no solution. ∎

Example 4.3 *Solve the system* $\left\{ \begin{smallmatrix} x & - & 2y & = & 4 \\ -2x & + & 4y & = & -8 \end{smallmatrix} \right\}$ *(by graphing).*

Solution: Note this system is exactly the same as the previous example except for the change in the constant on the right hand side of the second equation. Now, if we go through the steps of finding intercepts for each line, here's the outcome:

	$x - 2y = 4$	$-2x + 4y = -8$
x-intercept	$(4, 0)$	$(4, 0)$
y-intercept	$(0, -2)$	$(0, -2)$

Since both lines pass through the same two points, the lines must be equal. So there is no need to actually make a graph. The lines intersect infinitely many times, and so the system has infinitely many solutions. ∎

In the last two examples, the final solutions really did not require graphs. As we will see in the next section, there are better ways to solve linear systems! Here's one more problem that will reinforce the idea that you do not need to graph a linear system to predict how many solutions it will have.

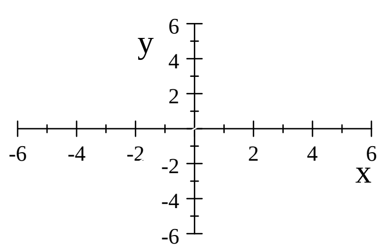
Try This 4.1 *After writing each line in Slope-Intercept Form, determine the number b for which the system* $\left\{\begin{smallmatrix} 2x - by = 6 \\ 4x + 2y = 7 \end{smallmatrix}\right\}$ *will have no solution.* Ⓢ

Homework Problems

1. Solve the system $\left\{\begin{smallmatrix} 3x - y = -6 \\ -x + 2y = -3 \end{smallmatrix}\right\}$ by graphing. Plot at least two points for each line.

$3x - y = -6$	
x	y

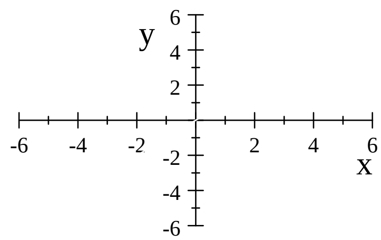
$-x + 2y = -3$	
x	y



2. Solve the system $\left\{\begin{smallmatrix} 3x + y = 1 \\ -2x + 2y = 10 \end{smallmatrix}\right\}$ by graphing. Plot at least two points for each line.

$3x + y = 1$	
x	y

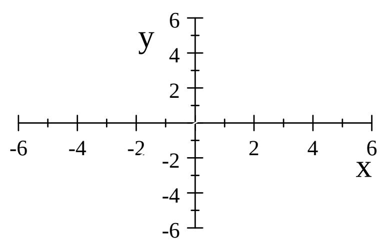
$-2x + 2y = 10$	
x	y



3. Solve the system $\left\{\begin{matrix} 3x - y = 0 \\ -6x + 2y = 6 \end{matrix}\right\}$ by graphing. Plot at least two points for each line.

$3x - y = 0$	
x	y

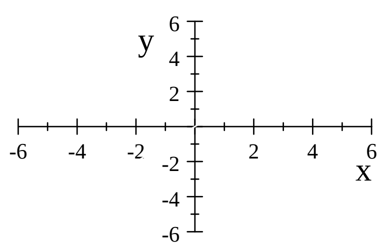
$-6x + 2y = 6$	
x	y



4. Solve the system $\left\{\begin{matrix} y = \frac{1}{2}x - 3 \\ -x + 2y = -6 \end{matrix}\right\}$ by graphing. Plot at least two points for each line.

$y = \frac{1}{2}x - 3$	
x	y

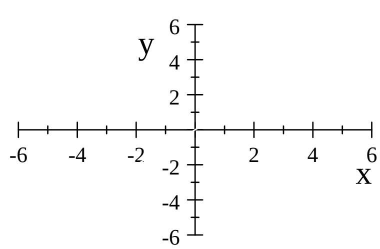
$-x + 2y = -6$	
x	y



5. Solve the system $\left\{\begin{matrix} x + 2y = 3 \\ y + 1 = 0 \end{matrix}\right\}$ by graphing. Plot at least two points for each line.

$x + 2y = 3$	
x	y

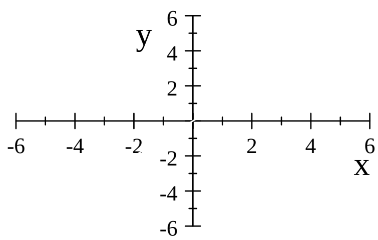
$y + 1 = 0$	
x	y



6. Solve the system $\left\{ \begin{smallmatrix} x = -3 \\ x + y = 1 \end{smallmatrix} \right\}$ by graphing. Plot at least two points for each line.

$x = -3$	
x	y

$x + y = 1$	
x	y



7. Suppose line L has equation $3x = -2y + 1$. Which equation below would be the equation of a line that is parallel to L?; which would be equal to line L?; and which would intersect line L once? Explain your reasoning.

   (a) $3y = -2x + 1$

   (b) $6x = -4y + 2$

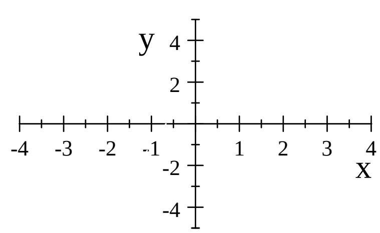
   (c) $-6x - 2 = 4y$

8. Determine the number c that will make the lines $y = \frac{-1}{2}x + c$ and $x = -2y + 7$ identical.

9. Graph the linear functions $f(x) = 3x + 2$ and $g(x) = -x - 2$. At what point do the graphs intersect?

$f(x) = 3x + 2$	
x	y

$g(x) = -x - 2$	
x	y



10. Using a graphing calculator, graph the line L with equation $3x + 2y = 8$ (you will need to isolate y first). Then, graph the line with equation $y = -1.45x + 3$. Do you think lines L and M are parallel based on the calculator's graph? Are the lines really parallel? Why or why not? $\boxed{a}$

4.2 Solving Systems Without Graphing

The big plus of solving by graphing is that it is visual: solutions correspond to intersection points. It will also be a way we will solve non-linear systems in Section 4.4. However, a major weakness is that graphs can be deceiving. Some lines that look parallel are not, or sometimes two lines clearly intersect but in a hard-to-guess location.

In this section we will discuss two other methods to solve systems that do not rely on graphs: Solving by Substitution and Solving by Addition-Elimination. We will illustrate both methods with the system $\left\{\begin{smallmatrix} 3x - y = 7 \\ x - y = 5 \end{smallmatrix}\right\}$ from the last section. Recall the solution for the system was the point $(1, -4)$.

An Example of Solving by Substitution

In this method the overall goal will be to substitute one of the variables in one of the equations into the other equation. The steps for solving by substitution are as follows:

How to Solve a System by Substitution
Step 1: Isolate one variable in one of the equations–whichever is easiest! Step 2: Substitute that result into the other equation. Step 3: Solve the resulting one variable equation and state the final solution.

Example 4.4 *Solve* $\left\{\begin{smallmatrix} 3x - y = 7 \\ x - y = 5 \end{smallmatrix}\right\}$ *by substitution.*

Solution: Solving for y in the 1st equation looks easy and gives $y = 3x - 7$. Next, substitute $y = 3x - 7$ into the 2nd equation and solve for x:

$$\begin{aligned} & x - y = 5 \\ & \quad \to x - (3x \ - 7) = 5 \\ & \quad \to -2x + 7 = 5 \\ & \quad \to -2x = -2 \\ & \quad \to x = 1. \end{aligned}$$

Hence, thesolution for the x-coordinate is $x = 1$. To determine the corresponding y-coordinate plug $x = 1$ into $y = 3x - 7$ which gives $y = -4$. So the solution to the system is the single point $(1, -4)$. ∎

An Example of Solving by Addition-Elimination

Next, we will solve the same system $\left\{\begin{smallmatrix} 3x & - & y & = & 7 \\ x & - & y & = & 5 \end{smallmatrix}\right\}$ by addition-elimination. In this method, the goal is to eliminate one of the variables by adding the equations together when the x- or y-coefficients are the same number but opposite in sign, e.g. 4 and -4. Here are the basic steps:

How to Solve a System by Addition-Elimination
Step 1: If necessary, rearrange the equations so the same variables line up in columns; Step 2: If necessary, multiply the first equation and/or the second equation by a non-zero number so that adding the equations cancels out one of the variables; Step 3: Add the equations together and replace the bottom equation with the sum; Step 4: Solve the resulting system and state your final solution.

Example 4.5 *Solve* $\left\{\begin{smallmatrix} 3x & - & y & = & 7 \\ x & - & y & = & 5 \end{smallmatrix}\right\}$ *by addition-elimination.*

Solution: The variables x and y are lined up in columns so Step 1 can be skipped. If we were to add the two equations together immediately, neither variable is eliminated. So for Step 2, we will need to adjust one or both of the equations with multiplication before doing the addition. For example, we could eliminate y by multiplying the 2nd equation by -1, then adding the equations. So, Steps 2-3 for $\left\{\begin{smallmatrix} 3x & - & y & = & 7 \\ x & - & y & = & 5 \end{smallmatrix}\right\}$ would be:

$$\begin{array}{ll} \text{multiply the 2nd equation by } -1: & \left\{\begin{smallmatrix} 3x & - & y & = & 7 \\ -x & + & y & = & -5 \end{smallmatrix}\right\} \\ \text{sum the two equations:} & 2x \ \ = 2 \\ \text{replace the 2nd equation with the sum:} & \left\{\begin{smallmatrix} 3x & - & y & = & 7 \\ 2x & & & = & 2 \end{smallmatrix}\right\}. \end{array}$$

So the system to solve is $\left\{\begin{smallmatrix} 3x & - & y & = & 7 \\ 2x & & & = & 2 \end{smallmatrix}\right\}$. Finally for Step 4, next $2x = 2 \to x = 1$. So, subbing $x = 1$ into the top equation gives

$$3(1) - y = 7 \to 3 - y = 7 \to -4 = y.$$

So the solution is the point $(1, -4)$. ∎

Use Whatever Method Seems Easiest!

As we have seen, the system $\left\{\begin{smallmatrix} 3x & - & y & = & 7 \\ x & - & y & = & 5 \end{smallmatrix}\right\}$ was easily solved by substitution and by addition-elimination. However, sometimes one method is easier than the other or sometimes a mixture of both methods would be best.

Example 4.6 *Solve* $\left\{\begin{smallmatrix} 3x & - & 2y & = & 7 \\ y & = & x & + & 9 \end{smallmatrix}\right\}$ *by whichever method is easiest.*

Solution: This problem is super easy with substitution because the 2nd equation already has y isolated. Subsituting $y = x + 9$ into the first equation gives:

$$\begin{aligned} &3x - 2y = 7 \\ &\quad \to 3x - 2(x + 9) = 7 \\ &\quad \to x - 18 = 7 \\ &\quad \to x = 25. \end{aligned}$$

Hence $x = 25$. Who would have guessed? Next, $x = 25 \to y = 25 + 9 = 34$. So, the system has one solution, the point $(25, 34)$. ∎

Example 4.7 *Solve* $\left\{\begin{smallmatrix} 3x & - & 2y & = & 7 \\ 4x & - & 5y & = & 3 \end{smallmatrix}\right\}$ *by whichever method is easiest.*

Solution: Substitution would be unwise for this system because there is no variable to solve for without getting involved with fractions. So using addition-elimination would be best. There's no clear advantage of eliminating either x or y.

Let's eliminate x by multiplying the top equation by 4 to make the coefficient for x be 12, and multiplying the bottom equation by -3 to make the bottom coefficient for x be -12. By the way 12 is a good choice since it is the LCM (least common multiple) of 3 and 4. Here are the steps:

multiply to make the x-coefficients 12 and -12: $\left\{\begin{smallmatrix} 4(3x & - & 2y) & = & 4\cdot(7) \\ -3(4x & - & 5y) & = & -3\cdot\,(3) \end{smallmatrix}\right\}$

simplify both equations: $\left\{\begin{smallmatrix} 12x & - & 8y & = & 28 \\ -12x & + & 15y & = & -9 \end{smallmatrix}\right\}$

sum the equations: $7y = 19$

replace 2nd equation with sum: $\left\{\begin{smallmatrix} 12x & - & 8y & = & 28 \\ & & 7y & = & 19 \end{smallmatrix}\right\}$.

So the system to solve is $\left\{\begin{smallmatrix} 12x & - & 8y = 28 \\ & & 7y = 19 \end{smallmatrix}\right\}$. The bottom equation gives $y = \frac{19}{7}$. But we still need to figure out the x-value. At this stage, we could (A) substitute $y = \frac{19}{7}$ into any of the equations that have an x and solve for x or (B) we could go back to the original system and eliminate y.

Usually (B) is a little easier when fractions are involved. So, starting over again with $\left\{\begin{smallmatrix} 3x & - & 2y = 7 \\ 4x & - & 5y = 3 \end{smallmatrix}\right\}$ we will now eliminate y:

multiply to make the y-coefficients 10 and -10:	$\left\{\begin{smallmatrix} 5(3x - 2y) = 5\cdot(7) \\ -2(4x - 5y) = -2\cdot(3) \end{smallmatrix}\right\}$
simplify both equations:	$\left\{\begin{smallmatrix} 15x - 10y = 35 \\ -8x + 10y = -6 \end{smallmatrix}\right\}$
sum the equations:	$7x \quad = 29$
replace 2nd equation with sum:	$\left\{\begin{smallmatrix} 15x - & 10y & = 35 \\ 7x & & = 29 \end{smallmatrix}\right\}$.

Hence, $7x = 29 \to x = \frac{29}{7}$. So the final answer is the point $(\frac{29}{7}, \frac{19}{7})$. ∎

Example 4.8 *Solve* $\left\{\begin{smallmatrix} 3x - \frac{3}{4}y = \frac{7}{2} \\ \frac{1}{2}x = y \end{smallmatrix}\right\}$ *by whichever method is easiest.*

Solution: Here you might be tempted to do substitution immediately since we are given $y = \frac{1}{2}x$. It's possible, but the fractions may turn out to be difficult to deal with. So, we will first use multiplications to clear the fractions by multiplying each equation by its LCD (least common denominator). Here's the work:

multiply by LCDs to clear fractions:	$\left\{\begin{smallmatrix} 4\cdot\left(3x - \frac{3}{4}y\right) = 4\cdot\left(\frac{7}{2}\right) \\ 2\cdot\left(\frac{1}{2}x\right) = 2\cdot(\,y) \end{smallmatrix}\right\}$
simplify both equations:	$\left\{\begin{smallmatrix} 12x - 3y = 14 \\ x = 2y \end{smallmatrix}\right\}$.

Now we will do substitution with $x = 2y$!

$$\begin{aligned} &12x - 3y = 14 \\ &\to 12(2y) - 3y = 14 \\ &\to 21y = 14 \\ &\to y = \tfrac{14}{21} = \tfrac{2}{3}. \end{aligned}$$

Finally $y = \frac{2}{3} \to x = 2 \cdot (\frac{2}{3}) = \frac{4}{3}$. So again there's one solution, the point $(\frac{4}{3}, \frac{2}{3})$. ∎

The No Solution and Infinite Solution Cases

Perhaps you have noticed that all of our examples in this section so far have had exactly one solution. However we know from Section 4.1 there could also be no solution (parallel lines) or infinitely many solutions (equal lines). We will cover these situations shortly. Before doing so, try this review problem from Fundamentals of Mathematics II, and then check your answers:

Try This 4.2 *Solve each of the following equations. Which equation has one solution, which has no solution and which has infinitely many solutions? Explain your decisions. (a)* $2(x+3) = 2x+3$*; (b)* $2(x+3) = x+6$*; (c)* $2(x+3) = 2(x-4)+14$. $\boxed{\text{S}}$

If you understand how to solve the *Try This* well, you are ready to go on.

Example 4.9 *Solve* $\left\{\begin{matrix} x - 2y = 0 \\ 3x - 6y = 9 \end{matrix}\right\}$ *using whichever method seems easiest.*

Solution: Substitution looks very easy since we have $x = 2y$ in the first equation. Substituting into the 2nd equation gives $3(2y) - 6y = 9$ which simplifiess to $0 = 9$, a contradiction! Hence $\left\{\begin{matrix} x - 2y = 0 \\ 3x - 6y = 9 \end{matrix}\right\}$ has no solution.

Example 4.10 *Solve* $\left\{\begin{matrix} -x + 2y = 3 \\ 3x - 6y = 9 \end{matrix}\right\}$ *using whichever method seems easiest.*

Solution: Here, probably addition-elimination is easiest. To eliminate x we only need to multiply the first equation by 3 then add the equations:

$$\left\{\begin{matrix} -x + 2y = 3 \\ 3x - 6y = -9 \end{matrix}\right\} \leftrightarrow \left\{\begin{matrix} -3x + 6y = 9 \\ 3x - 6y = -9 \end{matrix}\right\} \leftrightarrow \left\{\begin{matrix} -3x + 6y = 9 \\ 0 = 0 \end{matrix}\right\}$$

The $0 = 0$ is an identity and as in the past indicates there will be infinitely many solutions. The reason here is that in the system $\left\{\begin{matrix} -3x + 6y = 9 \\ 0 = 0 \end{matrix}\right\}$ all points satisfy the equation $0 = 0$ and infinitely many satisfy $3x - 6y = 9$ (a line). So, infinitely many points must satisfy both equations in the system. ∎

Homework Problems

1. Solve $\left\{\begin{matrix} -2x - 4y = 0 \\ 3x - y = 7 \end{matrix}\right\}$ by substitution.

2. Solve $\left\{\begin{matrix} -2x + y = 1 \\ x - 3y = 5 \end{matrix}\right\}$ by substitution.

3. Solve $\left\{\begin{matrix} -2x + 3y = -1 \\ x - 2y = 3 \end{matrix}\right\}$ by addition-elimination.

4. Solve $\left\{\begin{matrix} 2x + 3y = 14 \\ x - 2y = 0 \end{matrix}\right\}$ by addition-elimination.

5. Solve $\left\{\begin{matrix} -2x + 4y = 7 \\ x - 2y = -7 \end{matrix}\right\}$ by substitution.

6. Solve $\left\{\begin{matrix} -4x + 8y = 0 \\ 2x - 4y = 0 \end{matrix}\right\}$ by substitution.

7. Solve $\left\{\begin{matrix} 3x + 3y = -1 \\ 2x - 2y = 5 \end{matrix}\right\}$ by addition-elimination.

8. Solve $\left\{\begin{matrix} -2x + 3y = -1 \\ .5x - 2y = 5 \end{matrix}\right\}$ by addition-elimination.

9. Solve $\left\{\begin{matrix} -x + y = 5 \\ 2x - 2y = 9 \end{matrix}\right\}$ using whatever strategy seems easiest.

10. Solve $\left\{\begin{matrix} -3x + 2y = 0 \\ 2x - 3y = 4 \end{matrix}\right\}$ using whatever strategy seems easiest.

11. Solve $\left\{\begin{matrix} -\frac{1}{2}x + \frac{1}{4}y = \frac{-3}{4} \\ 2x - y = 3 \end{matrix}\right\}$ using whatever strategy seems easiest.

12. Solve $\left\{\begin{matrix} -\frac{1}{4}x + \frac{1}{2}y = \frac{9}{4} \\ .1x - .2y = -.9 \end{matrix}\right\}$ using whatever strategy seems easiest.

13. Use a graphing calculator to verify the solution to Problem 3. (You will need to isolate y in each equation in order to make the graph.)

14. Translate to a system and solve: Suppose the sum of two numbers is 72 and the difference is -93. Determine the numbers.

15. Translate to a system and solve: The sum of Jill's and John's weight is 350 pounds. John's weight decreased by 25% of Jill's weight is 180 pounds. Determine Jill's and John's weight.

16. Translate to a system and solve: One side of a square is 5 feet more than two times one side of an equilateral triangle. The perimeters of the square and triangle sum to 100 feet. Determine the side lengths of the square.

17. If two angles are complementary (sum to 90°) and one angle measures 5° less than two-thirds the other angle, determine the larger angle.

18. Twice the length of a rectangle equals four feet more than three times its width. If the total perimeter is 100 meters, determine the length of the rectangle.

19. In a triangle the measure of the biggest angle is two times the smallest angle, and twenty more than the other angle. Determine the measure of each angle angle. $\boxed{a}$

4.3 Some Applications of Systems

In this section we will cover a few applications of linear systems. The typical problem will involve "modeling" or describing a situation with two linear equations–a linear system–then solving the system. The problems are mixture types in terms of monetary value or distance. What typically happens is there are two quantities, and their amounts and corresponding values being considered. In some problems the amounts and/or values are mixed to form a total. In other problems they form an equation in some other way. You'll see the distinctions as we do examples.

Interest Value Mixture Problems

In this group of problems, money is being mixed in some way, and the formula $I = Prt$ will be necessary for these problems. Recall $I = Prt$ is used to calculate the simple interest I on an investment, where P is the initial number of dollars (principal) invested, r is the yearly interest rate and t is the number of years of investment. The word "simple" means that the interest is paid only once, so no compounding occurs like in a savings account where interest might be paid monthly.

The critical point to understanding the next two examples is the difference between principal and interest. For example, if someone invests x dollars at 6% simple interest and y dollars at 3% simple interest for one year, the total principal value would be $x + y$ whereas the total interest value would be $.06x + .03y$. The $.06x$ and $.03y$ come from $I = Prt$. Literally, the $.06x$ is $x \cdot (.06) \cdot (1)$ and the $.03y$ is $y \cdot (.03) \cdot (1)$.

Example 4.11 *Erin took an early retirement incentive of* \$500,000. *She will invest this money and live off the interest. Some of the money will be the principal for a savings account that pays* 3% *simple interest. The rest of the money will serve as the principal for a stock fund that pays on average* 10% *simple interest. If the goal is to achieve* \$35,000 *in interest per year for living expenses, how much principal should be invested at* 3% *and* 10% *respectively?*

Solution: In this problem there are actually two mixtures. First there are the two principals (amounts) which we know sum to \$500,000. So if we let x be principal at 3% and y be the principal at 10%, the 1st equation of the system is $x + y = 500000$.

Next, there is the mix of interests (values) The interest from the 3% principal would be $x \cdot (.03) \cdot (1)$ or just $.03x$. Similarly, the interest from the 10% principal is $y \cdot (.10) \cdot (1)$. Since the interests must sum to \$35,000, the 2nd equation of the system is $.03x + .10y = 35000$.

So the system to solve is $\left\{ \begin{smallmatrix} x + y = 500000 \\ .03x + .10y = 35000 \end{smallmatrix} \right\}$. It's easy to solve for x in the 1st equation so we will use the substitution. In fact, $x = 500000 - y$ and from the 2nd equation we get:

$$
\begin{aligned}
&.03x + .10y = 35000 \\
&\quad \to .03(500000 - y) + .10y = 35000 \\
&\quad \to 15000 + .07y = 35000 \\
&\quad \to .07y = 20000. \\
&\quad \to y = \tfrac{20000}{.07} \approx 285,714.29\ .
\end{aligned}
$$

So, for Erin to achieve a total of \$35,000 in interest she should invest \$285,714.29 in stock and the rest, \$214,285.71, in savings. ∎

By the way, here's the break down of Erin's interest income from the first example,

interest from savings:	$.03 \cdot (\$214,285.71) \approx \$6,428.57$
interest from stocks:	$.10 \cdot (\$285,714.29) \approx \$28,571.43.$

So, she's depending heavily on the stocks to achieve her \$35,000 goal. *We wish her well*!

Example 4.12 *Next Erin is nervous about the risk in the stock market. She would rather invest her early retirement incentive of* $500,000 *so that half of her interest income will come from the* 3%*-interest savings account and half will come from the stock fund which pays on the average* 10% *interest. What will be her yearly interest income?*

Solution: Again if we let x be the 3% principal and y be the 10% principal, these two mix to form a sum of 500,000. So the 1st equation of the system is still $x + y = 500,000$.

Next we need to consider the interest equation. In this problem the interests from the two investments equate rather than total. In other words since half of the interest comes from savings and half comes from stocks, those two amounts must be equal. Since the interest amount from savings is $.03x$ and the interest amount from stocks is $.10y$, the 2nd equation of the system is $.03x = .10y$.

So the system to solve is $\left\{\begin{smallmatrix} x + y = 500000 \\ .03x = .10y \end{smallmatrix}\right\}$. Similar to the last example, substitution gives:

$$\begin{aligned} &.03x = .10y \\ &\quad \to .03(500000 - y) = .10y \\ &\quad \to 15000 - .03y = .10y \\ &\quad \to 15000 = .13y \\ &\quad \to \tfrac{15000}{.13} = y. \end{aligned}$$

Therefore, $y = \frac{15000}{.13} \approx 115,384.62$. Hence, Erin should invest $115,384.62 in stocks and the rest, $384,615.38, in savings.

Note that based on how we set up in the second example, Erin's interest income from both accounts should be equal. Let's verify,

interest from savings:	$.03 \cdot (\$384,615.38) \approx \$11,538.46$
interest from stocks:	$.10 \cdot (\$115,384.62) \approx \$11,538.46$

So her total interest income will only be $23,076.92. That's a lot less interest than the $35,000 in the first example, *but she'll probably sleep better* investing more in savings and less in the stock market. ∎

Distance Mixture Problems

Finally let's do some examples involving distance. Here the key formula is the classic $d = r \cdot t$, where d is the distance an object travels traveling at a average rate r over a time period t. The units for d and t must agree with r's units. For example, if r is measured in $\frac{\text{mi}}{\text{hr}}$ then d must be measured in miles and t in hours. If someone gave us r in $\frac{\text{ft}}{\text{min}}$ and t in hours then we would need to convert t to min or r to $\frac{\text{ft}}{\text{hr}}$.

Example 4.13 *Roberta and Gene recently completed a two-day* 1050*-mile trip to Daytona. On first day of the trip they averaged* $60\ \frac{mi}{hr}$*. On the second day of the trip they averaged* $65\ \frac{mi}{hr}$*. If the trip took a total of* 17 *hours to complete, determine their driving time on each day of the trip.*

Solution: Despite the fact that they we are dealing with distances, rates and times, this problem is very much a mixture problem. In fact, both equations that describe this problem are sum equations dealing with amounts of time and distance traveled (resulting values).

If we let x be the driving time for day one of the trip and y be the driving time for day two, then the 1st equation is $x + y = 17$.

Next, the distances from each day sum to 1050. Using $d = r \cdot t$ to express the distances on day one and day two, the 2nd equation would be $60x + 65y = 1050$.

So the system to solve is $\left\{ \begin{smallmatrix} x + y = 17 \\ 60x + 65y = 1050 \end{smallmatrix} \right\}$. If we go with the substitution $x = 17 - y$ from the 1st equation, then from the 2nd equation it follows:

$$\begin{aligned} &60x \ + \ 65y \ = \ 1050 \\ &\ \to 60(17 - y) + 65y = 1050 \\ &\ \to 1020 + 5y = 1050 \\ &\ \to 5y = 30 \\ &\ \to y = 6. \end{aligned}$$

So, Roberta and Gene drove 6 hours on day two, and hence 11 hours on day one. ∎

Example 4.14 *Roberta and Gene are* 28 *miles apart and begin to bicycle towards each other at the same moment. Gene's rate is* $5\ \frac{mi}{hr}$ *greater than Roberta's rate, and after* 1.5 *hours they reach each other. Determine the distance each will travel before reaching the other.*

Solution: The key is that Gene's and Roberta's unknown distances will total to 28 miles. Now even though we are looking for distances, the more crucial unknowns are the rates of Gene and Roberta. After all, distance comes from $d = r \cdot t$.

So let x = Roberta's rate and y = Gene's rate. Then one equation is $y = x + 5$.

The other equation comes from the total distance being 28. Since the time for both Roberta and Gene is 1.5 hr, the other equation would be $x(1.5) + y(1.5) = 28$. So, $1.5x + 1.5y = 28$.

Therefore the system to solve and $\left\{\begin{smallmatrix} y = x + 5 \\ 1.5x + 1.5y = 28 \end{smallmatrix}\right\}$. With the Substitution Method and $y = x + 5$ from the 1st equation, from the 2nd equation we get:

$$\begin{aligned}
&1.5x + 1.5y = 28 \\
&\to 1.5x + 1.5(x + 5) = 28 \\
&\to 1.5x + 1.5x + 7.5 = 28 \\
&\to 3x = 20.5 \\
&\to x = \tfrac{20.5}{3} \\
&\to x = \tfrac{205}{30} = \tfrac{41}{6} = 6\tfrac{5}{6}.
\end{aligned}$$

In the last step, the decimal was made into a fraction to avoid a repeating decimal answer. So, Roberta's rate was $6\frac{5}{6}$ mph. Therefore, her distance was $\frac{41}{6} \cdot 1.5 = \frac{41}{6} \cdot \frac{3}{2} = \frac{41}{4} = 10\frac{1}{4}$ mi. That leaves a distance of $17\frac{3}{4}$mi for Gene. ∎

Finally, here's a question for you to try before doing the homework. (We can let our imaginations run wild, but hopefully things are OK with Roberta and Gene...)

Try This 4.3 *Roberta and Gene have the same starting point for a bike trip, but Roberta leaves* 30 *minutes before Gene. If Gene's rate is* 12 *mph and Roberta's rate is* 10 *mph, how long will it take Gene to reach Roberta?* $\boxed{\text{Ⓢ}}$

Homework Problems

1. Steve invests a \$20000 bonus, part in a stock account and the rest in a bond fund. If the stock account grew at a yearly simple interest rate of 12% and the bond fund grew at a yearly simple interest rate of 7% and the total interest earned in one year is \$2000, determine the amount invested in the stock fund.

2. Steve invests a \$20000 bonus, part in a stock account and the rest in a bond fund. If the stock account lost money at a yearly simple interest rate of 2% and the bond fund grew at a yearly simple interest rate of 7% and the total interest earned in one year is \$1000, determine the amount invested in the stock fund.

3. Repeat Problem 2, but this time imagine that Steve breaks even (e.g. total interest with loss and gain is 0).

4. A wholesaler's premium coffee costs \$6.00 per pound while generic coffee costs \$1.50 per pound. How many pounds of each type of coffee should be mixed together to make a 100 pounds of mixture that has total value \$501. Hint: fill in the rest of table below to set up the problem.

coffee type	pounds	price per pound	value
premium			
generic			

5. Johnny has a piggy bank with just nickels and dimes. If he dumps his coins into a change counter and the counter says he has a total of 300 coins with a total value of \$27.30, how many nickels were in his piggy bank. Hint: fill in the rest of table below to set up the problem.

coin type	quantity	value per coin	value
nickels			
dimes			

6. The Anderson's travel from their home to Disney, a total distance of 850 miles, over two days. If on the first day they averaged 55 miles per hour and on the second day 65 miles per hour and the total driving time was 15 hours, determine the time traveled on day one.

7. Sue began her trip to the beach in a big hurry averaging 70 miles per hour. After a ticket for speeding, she dropped her speed to 65 miles per hour on the average. If the total distance to the beach was 265 miles and the total driving time was 4 hours, how long did she drive at the faster rate?

8. Two hikers are 20 miles apart and begin to walk directly towards each other at noon. If one of the hikers walks 20% faster than the other and they meet at 4:15 PM, determine the rate of each hiker. $\boxed{a}$

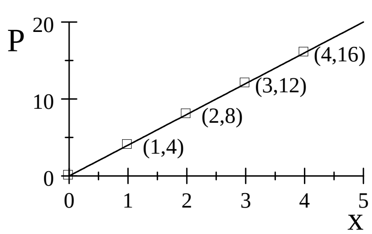
4.4 A Glimpse at Non-Linear Systems

Now that we have studied lines and linear systems in great detail, it's time to briefly consider the possibilities that non-linear equations and systems offer.

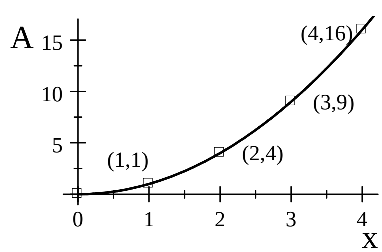
Linear Versus Non-Linear Equations and Graphs

Our first step is to understand the difference between linear equations and non-linear equations. Recall linear equations are usually written in General Form $Ax + By = C$ or Slope-Intercept Form $y = mx + b$; and so the implied powers of x and y are both 1. By "non-linear" equations we mean simplified equations that differ fundamentally from linear equations–e.g. they have an x^2, $|y|$, $\sqrt{x}$, etc.

As a good example of linear versus non-linear equations, let's compare the perimeter and area formulas for a square: $P = 4x$ which is a linear equation and $A = x^2$ which is a non-linear equation. In terms of their graphs, we already know we can expect a line from $P = 4x$. So hopefully that means we can expect a non-linear shape for the graph of $A = x^2$. Indeed by just plotting a few points, we can see the linear and non-linear shapes take form:



$P = 4x$

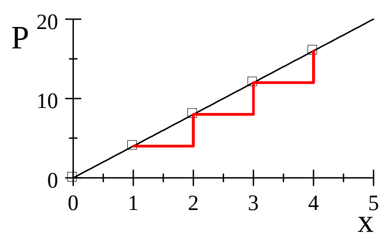


$A = x^2$

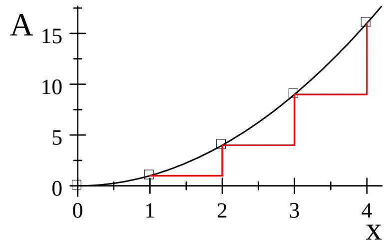
The equations $P = 4x$ and $A = x^2$ and their graphs are examples of a true general situation:

> Linear equations lead to linear graphs; non-linear equations lead to non-linear graphs (often curves).

The underlying reason comes from slope. Remember what really makes a graph linear is that the slope does not change from point to point. So it is really changing slope that makes a graph non-linear. This is illustrated in the two graphs below:



$P = 4x$ has constant slope



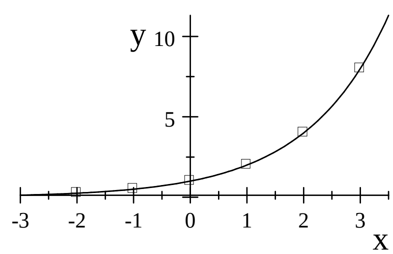
$A = x^2$ has changing slope

Two Important Non-Linear Examples: $\mathbf{y} = \mathbf{2}^x$ and $\mathbf{y} = \log_2(\mathbf{x})$

As we have seen, non-linear equations lead to non-linear graphs. Non-linear graphs are especially important because most shapes in reality are based on (non-linear) curves, not lines. Two important non-linear graphs come from the equations $y = 2^x$ and $y = \log_2(x)$

Let's start with $y = 2^x$. To make a graph of this equation we will follow standard procedure: we will make a table of points, in this case by picking x values and then computing the y-value. Now, because x plays the role of a power, we know x could be positive, negative or 0. Or in other words the domain of the expression 2^x is all real numbers. Below is a table of points and graph for $y = 2^x$:

x	$y = 2^x$
-2	$2^{-2} = \frac{1}{2^2} = \frac{1}{4}$
-1	$2^{-1} = \frac{1}{2^1} = \frac{1}{2}$
0	$2^0 = 1$
1	$2^1 = 2$
2	$2^2 = 4$
3	$2^3 = 8$



Graph of $y = 2^x$

Two important features of the graph of $y = 2^x$ are (A) how quickly it grows as the graph moving from the left to the far right, and (B) how slowly in falls moving from the right to the far left. In terms of (A), the graph grows so quickly to the far right it is said to have "exponential growth." In terms of (B), the graph falls so slowly to the far left it is said to have a "horizontal asymptote" which means it becomes nearly horizontal.

The equation $y = 2^x$ comes up in real life when you consider populations that double every certain unit of time. For example, if a bacteria population is known to double in size every day and if there is initially one bacteria cell, population of bacteria after 1 day would be $1 \cdot 2 = 2 = 2^1$; then $2 \cdot 2 = 4 = 2^2$ after 2 days; then $4 \cdot 2 = 8 = 2^3$ after 3 days, and in general 2^n after n days.

So, the graph of $y = 2^x$ is actually very common. Our brains are set up to think about the expression 2^x very easily. In particular, if someone gives us a value for x we can normally figure out what 2^x is.

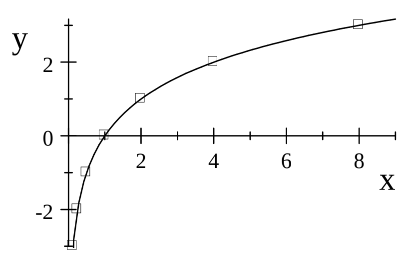
The expression $\log_2(x)$, called the log base 2 of x, is also very common but requires backwards thinking to compute. For example, to compute $\log_2(16)$ we need to think of 16as the result of 2 to a power. In other words $\log_2(16)$ is the power 2 is raised to produce 16. Does that make you think of 4? Indeed, $\log_2(16) = 4$. Similarly, $\log_2(1) = 0$ because 2 to the 0 power does produce 1. In general, we can define $\log_2(x)$ as follows:

How to Compute $\log_2(x)$
$\log_2(x)$ is the number a such that 2 to the a equals x.

Note that the domain of the expression $\log_2(x)$ includes only positive numbers x since 2 raised to any power can not be negative or 0. In other words $\log_2(-1)$ isn't possible for us because 2 to any real number power will never be -1.

OK, let's make a graph of $y = \log_2(x)$. We will make a table of points and graph using powers of 2 for x:

x	$y = \log_2(x)$
1	$\log_2(1) = 0$ since $2^0 = 2$
2	$\log_2(2) = 1$ since $2^1 = 2$
4	$\log_2(4) = 2$ since $2^2 = 4$
8	$\log_2(8) = 3$ since $2^3 = 8$
$\frac{1}{2}$	$\log_2(\frac{1}{2}) = -1$ since $2^{-1} = \frac{1}{2}$
$\frac{1}{4}$	$\log_2(\frac{1}{4}) = -2$ since $2^{-2} = \frac{1}{2^2} = \frac{1}{4}$
$\frac{1}{8}$	$\log_2(\frac{1}{8}) = -3$ since $2^{-3} = \frac{1}{2^3} = \frac{1}{8}$



Graph of $y = \log_2(x)$

The graph of $y = \log_2(x)$ also has two key features: (A) notice how slowly the graph grows moving to the far right. For example, by the time $x = 8$ the graph is only at a height of $y = 3$; and (B) notice how the graph becomes almost vertical as it approaches the y-axis from the right side. We say in that situation that the graph has a "vertical asymptote." If you continue to more advanced algebra or calculus courses, you will see other logarithms like the common logarithm $\log(x)$ or the natural logarithm $\ln(x)$ also share these two features as well.

Try This 4.4

(a) A certain population of bacteria doubles in size every day. If the initial size of the population was 12 *bacteria, what would be the size of the population after* 2 *days?;* 3 *days?;* 4 *days? What would be a formula for the size of the population after n days?*
(b) A wise investor invests \$50 *initially in a bond that typically doubles in value every* 6 *years. Assuming the money remains fully invested in the bond, how much money will be available after* 6 *years?; after* 12 *years?; after* 18 *years? What would be a formula for the available money after n years?*
(c) Determine the value of $\log_2(32)$ *and* $\log_2(64)$. *Between what two consecutive whole numbers is* $\log_2(300)$*?* $\boxed{\text{S}}$

Non-Linear Systems

Non-linear systems of equations have at least one equation that is not linear. The most interesting fact about non-linear systems is that they can have any number of solutions. This is quite different than linear systems which can only have no solution, one solution or infinitely many solutions. In addition the graphs of non-linear systems yield a variety of interesting shapes and regions.

Overall, our most useful strategies for solving non-linear systems will be substitution and/or graphing. Graphing is particularly helpful in knowing if all solutions have been found.

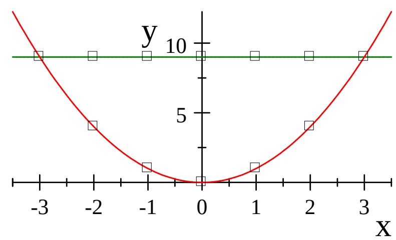
Example 4.15 *Solve the system* $\left\{ \begin{smallmatrix} y = x^2 \\ y = 9 \end{smallmatrix} \right\}$ *by substitution and by graphing.*

Solution: Solving by substitution is very easy. Subbing $y = x^2$ into $y = 9$ yields $x^2 = 9$. By mental math, we can see $x = 3$ is a correct solution since $(3)^2 = 9$. Though it is easy to overlook, $x = -3$ is also a solution since $(-3)^2 = 9$. So, there are two solutions to the system: $(3, 9)$ and $(-3, 9)$, e.g. $(\pm 3, 9)$.

Of course graphing the system should yield the same solutions. A table of points for both equations and the graph of the system is given next:

x		$y = x^2$
-3	$\rightarrow$	9
-2	$\rightarrow$	4
-1	$\rightarrow$	1
0	$\rightarrow$	0
1	$\rightarrow$	1
2	$\rightarrow$	4
3	$\rightarrow$	9

x		$y = 9$
-3	$\rightarrow$	9
-2	$\rightarrow$	9
-1	$\rightarrow$	9
0	$\rightarrow$	9
1	$\rightarrow$	9
2	$\rightarrow$	9
3	$\rightarrow$	9



Graph of $\left\{ \begin{smallmatrix} y \,=\, x^2 \\ y \,=\, 9 \end{smallmatrix} \right\}$

In this problem, we can actually see the points of intersection from the tables of points and graph. Notice how the region enclosed by the line and parabola has an interesting bowl shape. ∎

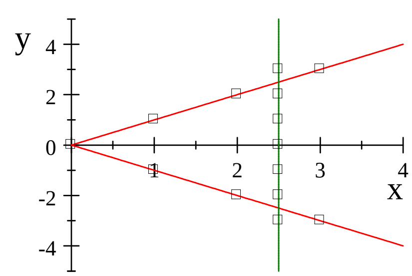
Example 4.16 *Solve the system* $\left\{ \begin{smallmatrix} x \,=\, |y| \\ x \,=\, 2.5 \end{smallmatrix} \right\}$ *by substitution and by graphing.*

Solution: Subbing $x = 2.5$ into $x = |y|$ gives $2.5 = |y|$. Again we have to think carefully about what y can be. To say $|y| = 2.5$ means by definition that y is a number whose distance to 0 is 2.5. So, again there are two possible values, $y = 2.5$ and $y = -2.5$. Since $x = 2.5$ and $y = \pm 2.5$, there are again two solutions the system: $(2.5, \pm 2.5)$.

Here's a graphing solution:

$x = 2.5$		y
2.5	$\leftarrow$	-3
2.5	$\leftarrow$	-2
2.5	$\leftarrow$	-1
2.5	$\leftarrow$	0
2.5	$\leftarrow$	1
2.5	$\leftarrow$	2
2.5	$\leftarrow$	3

$x = \|y\|$		y
3	$\leftarrow$	-3
2	$\leftarrow$	-2
1	$\leftarrow$	-1
0	$\leftarrow$	0
1	$\leftarrow$	1
2	$\leftarrow$	2
3	$\leftarrow$	3



Graph of $\left\{ \begin{smallmatrix} x \,=\, |y| \\ x \,=\, 2.5 \end{smallmatrix} \right\}$

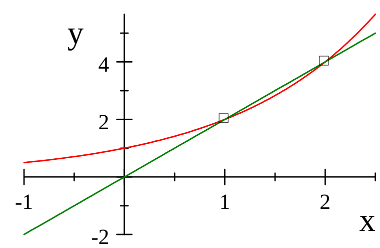
Because we only used integer values for y, the tables do not show the points of intersection at $(2.5, 2.5)$and $(2.5, -2.5)$. However, we could probably make that guess based on the graph and then check that both points satisfy both equations. Notice this time how the graphs enclose a triangular region? ∎

OK, one last example. Hopefully, these non-linear systems are interesting and seem easy to you. If not, please ask your instructor for some help.

Example 4.17 *Use substitution and mental math to solve the system* $\left\{\begin{smallmatrix} y = 2^x \\ y = 2x \end{smallmatrix}\right\}$. *Then use a graphing calculator as a way to verify your answers.*

Solution: Substituting $y = 2x$ into $y = 2^x$ gives the equation $2x = 2^x$ for us to solve. This equation is hard to solve with algebra, but simple enough to solve with mental math. Can you think of a number x that satisfies $2x = 2^x$? For example, $x = 0$ does not give a solution because $2 \cdot (0) = 0$ but $2^0 = 1$. But, $x = 1$ does solve $2x = 2^x$ because $2 \cdot (1) = 2$ and $2^1 = 2$. Since $x = 1 \rightarrow y = 2$, $(1, 2)$ is a solution to the system $\left\{\begin{smallmatrix} y = 2^x \\ y = 2x \end{smallmatrix}\right\}$. But is $(1, 2)$ the only solution?

Using a graphing calculator with a short range ($x = -1$ to $x = 2.5$) for the x-axis and a longer range for the y-axis ($y = -2$ to $y = 6$) should produce a graph similar to the one on the right. Remember $y = 2x$ is the line!

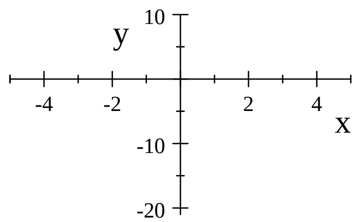


Graph of $\left\{\begin{smallmatrix} y = 2^x \\ y = 2x \end{smallmatrix}\right\}$

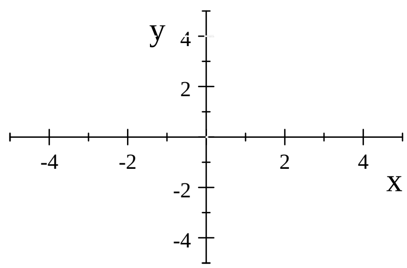
Notice that the graphs intersect again when $x = 2$. Since $x = 2 \rightarrow y = 4$, $(2, 4)$ is another solution. The graphs actually never intersect again before the point $(1, 2)$ and after the point $(2, 4)$. The exact reasons are studied in calculus, but the slow growth of $y = 2^x$ before the point $(1, 2)$ and its fast growth after the point $(2, 4)$ are the major reasons. ∎

Homework Problems

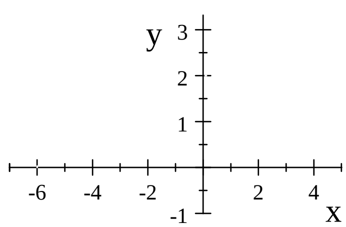
1. Solve the system $\left\{\begin{smallmatrix} y = 4 - x^2 \\ y = -5 \end{smallmatrix}\right\}$ by (a) graphing and (b) substitution.



2. Solve the system $\left\{\begin{smallmatrix} x = |y| \\ y = 4 \end{smallmatrix}\right\}$ by (a) graphing and (b) substitution.



3. Solve the system $\left\{\begin{smallmatrix} y = \sqrt{x+6} \\ y = 2 \end{smallmatrix}\right\}$ by (a) graphing and (b) substitution.



4. Solve the system $\left\{\begin{smallmatrix} 160x = y^2 \\ x = 10 \end{smallmatrix}\right\}$ by substitution only.

5. Solve the system $\left\{\begin{smallmatrix} y = x^3 + x \\ y = x - 8 \end{smallmatrix}\right\}$ by substitution only.

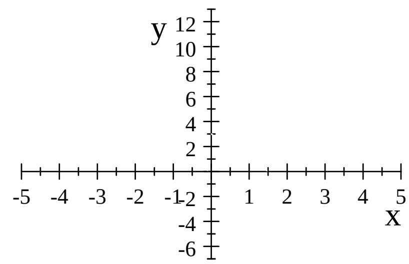
6. Solve the system $\left\{\begin{smallmatrix} x^2 + y^2 = 8 \\ y = x \end{smallmatrix}\right\}$ by substitution only.

7. Solve the system $\left\{\begin{smallmatrix} x^2 - y^2 = 9 \\ x = 2y \end{smallmatrix}\right\}$ by substitution only.

8. Solve the system $\left\{\begin{smallmatrix} x^2 - y^2 = 9 \\ x = 5 \end{smallmatrix}\right\}$ by substitution only.

9. By graphing, determine where the functions $f(x) = |x|$ and $g(x) = -2x + 3$ intersect.

x	$f(x) = \|x\|$	$g(x) = -2x + 3$
-4		
-2		
0		
2		
4		

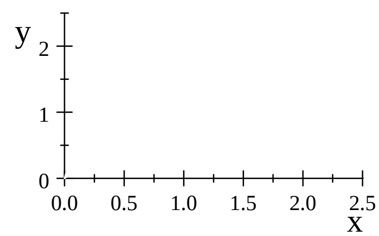


10. Verify your answer to Problem 5 using a graphing calculator.

11. Complete the table of points below and then solve the system $\left\{\begin{smallmatrix} y = \sqrt{x} \\ y = x \end{smallmatrix}\right\}$ by graphing.

x	$y = \sqrt{x}$
0	
$.25$	
$.49$	
1	
2.25	
4	

x	$y = x$
0	
$.25$	
$.49$	
1	
2.25	
4	



12. Complete the table below by evaluating $f(x) = x^3$ and $g(x) = 3^x$ with the specified values of x. As x gets larger, which function has larger value, f or g?

x	1	2	3	4	5	6	7
$f(x)$							
$g(x)$							

13. Complete the table below by evaluating $h(x) = \log_2(x)$ and $i(x) = \sqrt{x}$ with the specified values of x. As x gets larger, which function has larger value, h or i?

x	1	4	16	64	256	1024	4096
$h(x)$							
$i(x)$							

$\boxed{a}$

Chapter 5

Radical Expressions

In this chapter we will study the algebra of radical expressions. By "radical expressions" (or "radicals" for short) we mean expressions that involve square root, cube root, fourth root, etc. and by "algebra" we mean standard rules and procedures for simplifying radicals.

5.1 Exact Versus Approximate

We will start with square root $\sqrt{\ }$ (same as $\sqrt[2]{\ }$) and cube root $\sqrt[3]{\ }$ calculations when the radicand is a "nice" number. Recall the radicand is the expression on the inside of the radical symbol. The integer $n \geq 2$ describing the radical type is called the index.

Example 5.1 *Without a calculator, simplify the following radicals:* $\sqrt{81}$, $\sqrt[3]{-64}$, $\sqrt{.01}$, $\sqrt[3]{\frac{-8}{125}}$, $\sqrt{-16}$ *and* $-\sqrt{81}$.

Solution: Correct reasoning and answers are as follows:
$\sqrt{81} = 9$, since $(9)^2 = (9) \cdot (9) = 81$ and $9 \geq 0$;
$\sqrt[3]{-64} = -4$, since $(-4)^3 = (-4) \cdot (-4) \cdot (-4) = -64$;
$\sqrt{.01} = .1$, since $(.1)^2 = (.1) \cdot (.1) = .01$ and $.1 \geq 0$;
$\sqrt[3]{\frac{-8}{125}} = \frac{-2}{5}$, since $(\frac{-2}{5})^3 = (\frac{-2}{5}) \cdot (\frac{-2}{5}) \cdot (\frac{-2}{5}) = \frac{-8}{125}$;
$\sqrt{-16}$ does not exist as a real number (any real number squared is ≥ 0); and
$-\sqrt{81} = -9$, since $-\sqrt{81} = -1 \cdot \sqrt{81} = -1 \cdot (9) = -9$. ∎

Hopefully those calculations remind you that odd-indexed roots like cube root $\sqrt[3]{\ }$ are less complicated than the even-indexed radicals like square root $\sqrt{\ }$ and fourth root $\sqrt[4]{\ }$. With odd-indexed roots you can input negative numbers and get negative results. Whereas with even-indexed radicals, inputting negative numbers is a serious problem and outputs are specified to be non-negative. The official definition of $\sqrt[n]{\ }$ is:

$\sqrt[n]{x} = y$ means $(y)^n = x$.
(but read the fine print)

Fine print: For even-indexed roots, like square root and fourth root, x and y must be non-negative.

There are relatively few numbers for which we can calculate an exact root easily. For example $\sqrt{16} = 4$, but what about $\sqrt{17}$? Using a calculator to approximate a radical is fine when we are asked to or when we are curious about its numerical size. But keep in mind approximately equal ($\approx$) is not the same as equal ($=$).

For example, while it is true that $\sqrt{17} \approx 4.1$, it is not true that $\sqrt{17} = 4.1$ since $(4.1)^2 = 16.81$. We could round $\sqrt{17}$ to the nearest hundredth and get $\sqrt{17} \approx 4.12$, but still it is not true that $\sqrt{17} = 4.12$ since $(4.12)^2 = 16.9744$. In fact $\sqrt{17}$ is an irrational number, so it continues forever as a decimal number (without repeating a finite pattern). Therefore rounding will always cause some error with $\sqrt{17}$.

The next problem illustrates another hazard with rounding:

Try This 5.1 *In this problem you will be asked approximate* $\frac{\sqrt{27}-\sqrt{5}}{4}$ *two different ways.*

(a) Calculate $\frac{\sqrt{27}-\sqrt{5}}{4}$ *by rounding* $\sqrt{27}$ *and* $\sqrt{5}$ *to the nearest tenth first, then subtract, then divide and round to the nearest tenth again.*
(b) Calculate $\frac{\sqrt{27}-\sqrt{5}}{4}$ *in one step, then round to the nearest tenth at the end.*
(c) What two values did you get? Which value do you guess is more accurate? Why? $\boxed{\text{Ⓢ}}$

Homework Problems

1. Use a calculator to round $\sqrt{19}$ to the nearest tenth. Does your answer squared equal 19?

2. Use a calculator to round $\sqrt[3]{19}$ to the nearest tenth. Does your answer cubed equal 19?

3. You should not need a calculator to answer these questions. In each pair below, which number is bigger:

   (a) $\sqrt{20}$ or 20? (b) $\sqrt{20}$ or $\sqrt[3]{20}$?
   (c) $\sqrt{.4}$ or .4? (d) $\sqrt{.4}$ or $\sqrt[3]{.4}$?

4. In one or two sentences, explain why $\sqrt{-16}$ is not possible with real numbers.

5. Without using a calculator simplify $\left(4 \cdot \sqrt{6}\right)^2$.

6. Using a calculator, in one step calculate $\left(4 \cdot \sqrt{6}\right)^2$. Did you get exactly the same value as in Problem 5?

7. Without using a calculator simplify $\left(\frac{-1}{6} \cdot \sqrt{3}\right)^2$.

8. Using a calculator, in one step calculate $\left(\frac{-1}{6} \cdot \sqrt{3}\right)^2$. Did you get exactly the same value as in Problem 7?

9. An army sergeant has 800 soldiers she can use to make a square formation (soldiers line up inside and along border of the square). What is the greatest number of soldiers she can use for this formation? What percent of the soldiers would be unused?

10. What is the exact side length of a square with area 1250 square meters? What is the approximate answer rounding to the nearest tenth?

11. The shorter sides (legs) of a right triangle are 2 and 4 cm long. Determine the exact length of the hypotenuse and its approximate value rounded to the nearest tenth.

12. A right triangle has a leg that is 5 cm long and a hypotenuse that is 13 cm long. Determine the exact length of the missing leg.

13. The radius of a circle is 5 cm. If a square is to have approximately the same area as the circle what should the side length of the square rounded to the nearest tenth.

14. Simplify the following values or explain why they do not exist as real numbers:

    (a) $\sqrt[3]{-1000}$ (b) $\sqrt[4]{625}$ (c) $\sqrt{\frac{36}{121}}$

    (d) $\frac{\sqrt{900}}{2\cdot\sqrt{225}}$ (e) $\sqrt[9]{-1}$ (f) $\sqrt[10]{-1}$

15. Simplify the following values or explain why they do not exist as real numbers:

    (a) $\sqrt[3]{-64}$ (b) $\sqrt[4]{-16}$ (c) $\sqrt{\sqrt{81}}$

    (d) $\sqrt{3^2+4^2}$ (e) $\sqrt[3]{\frac{8}{27}}+\sqrt[4]{\frac{1}{16}}$ (f) $\left(\sqrt{16}-3\cdot\sqrt{49}\right)^2$ $\boxed{a}$

5.2 The Factor Rules for Radicals

If we are going to avoid approximating radicals with a calculator, we better have some methods to simplify them. There are only two major rules for simplifying radicals. Both rules are for factors (expressions being multiplied/divided), not for terms (expressions being added/subtracted).

The Factors Rules for Radicals
$\sqrt[n]{a\cdot b}=\sqrt[n]{a}\cdot\sqrt[n]{b}$ and
$\sqrt[n]{\frac{a}{b}}=\frac{\sqrt[n]{a}}{\sqrt[n]{b}}$ $(b\neq 0)$
Fine print: if n is even then a and b must both be non-negative.

The first rule also works for more than two factors, e.g. $\sqrt[n]{a\cdot b\cdot c}=\sqrt[n]{a}\cdot\sqrt[n]{b}\cdot\sqrt[n]{c}$ or $\sqrt[n]{a\cdot b\cdot c\cdot d}=\sqrt[n]{a}\cdot\sqrt[n]{b}\cdot\sqrt[n]{c}\cdot\sqrt[n]{d}$, again assuming each factor inside the radical is non-negative in n is even.

The most basic use of the first factor rule is to reduce the size of the radicand as follows:

How to Simplify $\sqrt[n]{c}$
Factor c into a product $a \cdot b$ such that $\sqrt[n]{a}$ and/or $\sqrt[n]{b}$ can be calculated exactly. Then, apply the rule $\sqrt[n]{a \cdot b} = \sqrt[n]{a} \cdot \sqrt[n]{b}$.

Example 5.2 *Simplify* $\sqrt{24}$.

Solution: We can't calculate the exact value of $\sqrt{24}$ because 24 is not a perfect square. However, we can simplify $\sqrt{24}$ since 24 has 4, a perfect square, as a factor. Here's the work:

$$\begin{aligned}\sqrt{24} &= \sqrt{4 \cdot 6}, \text{ (}24 = 4 \cdot 6 \text{ where 4 is a perfect square)}\\ &= \sqrt{4} \cdot \sqrt{6}, \text{ (first factor rule for radicals)}\\ &= 2 \cdot \sqrt{6}, \text{ (}\sqrt{4} = 2\text{).}\end{aligned}$$

Therefore, $\sqrt{24} = 2 \cdot \sqrt{6}$. ∎

Example 5.3 *Simplify* $\sqrt[3]{24}$.

Solution: $$\begin{aligned}\sqrt[3]{24} &= \sqrt[3]{8 \cdot 3}, \text{ (}24 = 8 \cdot 3 \text{ where 8 is a perfect cube)}\\ &= \sqrt[3]{8} \cdot \sqrt[3]{3}, \text{ (first factor rule for radicals)}\\ &= 2 \cdot \sqrt[3]{3}, \text{ (}\sqrt[3]{8} = 2\text{).}\end{aligned}$$

Therefore, $\sqrt[3]{24} = 2 \cdot \sqrt[3]{3}$. ∎

Example 5.4 *Simplify* $\sqrt{\frac{98}{9}}$.

Solution: $$\begin{aligned}\sqrt{\tfrac{98}{9}} &= \tfrac{\sqrt{98}}{\sqrt{9}}, \text{ (second factor rule)}\\ &= \tfrac{\sqrt{49 \cdot 2}}{3}, \text{ (}98 = 49 \cdot 2 \text{ and } \sqrt{9} = 3\text{)}\\ &= \tfrac{\sqrt{49} \cdot \sqrt{2}}{3}, \text{ (first factor rule)}\\ &= \tfrac{7 \cdot \sqrt{2}}{3}, \text{ (}\sqrt{49} = 7\text{).}\end{aligned}$$

Therefore, $\sqrt{\frac{98}{9}} = \frac{7 \cdot \sqrt{2}}{3}$. ∎

In simplifying radicals, the process ends when the radicand can not be broken down further. Take a minute to see if you can write down carefully what this means for square root.

Try This 5.2 *Give three examples of non-prime integers* $n > 25$ *such that* $\sqrt{n}$ *can not be simplified. Describe all positive integers* $n > 1$ *such that* $\sqrt{n}$ *can not be simplified.* $\boxed{\text{Ⓢ}}$

The next examples show how to add and subtract radical expressions. As always terms must be like to combine together. For example with $\frac{1}{3}+\frac{3}{4}$, the fractions can't be added until they become like terms with an LCD, e.g. $\frac{1}{3}+\frac{3}{4}=\frac{4}{12}+\frac{9}{12}=\frac{13}{12}$. For radical terms to be like terms, the individual terms must have the same index and radicand. In the next two examples the terms are not like because the radicands are not equal. Our first step will be to simplify each radical to see if they become like.

Example 5.5 *Simplify* $\sqrt{24}+\sqrt{54}$.

Solution: $\sqrt{24}+\sqrt{54} = \sqrt{4\cdot 6}+\sqrt{9\cdot 6}$, $(24 = 4\cdot 6$ and $54 = 9\cdot 6)$
$= 2\cdot\sqrt{6}+3\cdot\sqrt{6}$, (first factor rule applied twice)
$= 5\cdot\sqrt{6}$, (like radicals terms combined).
Therefore, $\sqrt{24}+\sqrt{54} = 5\cdot\sqrt{6}$. ∎

Example 5.6 *Simplify* $\sqrt[3]{16x}-5\cdot\sqrt[3]{2x}+\sqrt[3]{4x}$.

Solution: $\sqrt[3]{16x}-5\sqrt[3]{2x}+\sqrt[3]{4x} = \sqrt[3]{8\cdot 2x}-5\sqrt[3]{2x}+\sqrt[3]{4x}$, $(16 = 8\cdot 2)$
$= 2\cdot\sqrt[3]{2x}-5\cdot\sqrt[3]{2x}+\sqrt[3]{4x}$, (first factor rule)
$= -3\cdot\sqrt[3]{2x}+\sqrt[3]{4x}$, (combine like terms).
Therefore, $\sqrt[3]{16x}-5\sqrt[3]{2x}+\sqrt[3]{4x} = -3\cdot\sqrt[3]{2x}+\sqrt[3]{4x}$. ∎

Finally, let's do some examples of multiplication with radicals. Notice in the next example that the factor rule $\sqrt{a\cdot b}=\sqrt{a}\cdot\sqrt{b}$ is used backwards as $\sqrt{a}\cdot\sqrt{b}=\sqrt{a\cdot b}$ and forwards as $\sqrt{a\cdot b}=\sqrt{a}\cdot\sqrt{b}$. Now that's one hard-working rule!

Example 5.7 *Simplify* $\sqrt{6}\cdot\sqrt{10}$.

Solution: $\sqrt{6}\cdot\sqrt{10} = \sqrt{60}$, (factor rule backwards)
$= \sqrt{4\cdot 15}$, (factor 60 with largest perfect square)
$= 2\cdot\sqrt{15}$.
Therefore, $\sqrt{6}\cdot\sqrt{10} = 2\cdot\sqrt{15}$. ∎

Example 5.8 *Simplify* $(5x\cdot\sqrt{3x})^2$.

Solution: $(5x\cdot\sqrt{3x})^2 = 5^2\cdot x^2\cdot(\sqrt{3x})^2$, (rule of exponents)
$= 25x^2\cdot(3x)$, ($\sqrt{3x}$ is the number that when squared equals $3x$)
$= 75x^3$.
Therefore, $(5x\cdot\sqrt{3x})^2 = 75x^3$. ∎

Example 5.9 *Simplify* $(5+\sqrt{3})^2$.

Solution: $(5+\sqrt{3})^2 = (5+\sqrt{3})(5+\sqrt{3})$, (no nice rule of exponents with terms!!)
$= 25 + 5\cdot\sqrt{3} + 5\cdot\sqrt{3} + \sqrt{9}$, (multiply out)
$= 25 + 10\cdot\sqrt{3} + 3$, (collect like radical terms)
$= 28 + 10\cdot\sqrt{3}$.
Therefore, $(5+\sqrt{3})^2 = 28 + 10\cdot\sqrt{3}$. ∎

Rewriting Fractions with Radicals

So far we have done examples of addition, subtraction and multiplication with radicals. What about division or fractions with radicals? In this section, we will study some fractions involving having just one term in the numerator and denominator, at least one of which is a radical. The example below for $\frac{2}{\sqrt{6}}$, $\frac{\sqrt{10}}{\sqrt{8}}$ and $\frac{6}{\sqrt[3]{9}}$ are typical of the type of fractions we will study at this time.

The following procedure and examples show how to rewrite fractions with a single radical term so that any radicals end up in the numerator or denominator, whichever is preferred. Normally, the numerator will be the preferred location for radicals because dividing into a radical is easier to think about compared to dividing by a radical. For example, if you know $\sqrt{2}\approx 1.4$ then it's easy to estimate $\frac{\sqrt{2}}{2}$ with .7. But because dividing by 1.4 is tough to do mentally, it would be hard to estimate $\frac{7}{\sqrt{2}}$.

How to Rewrite One Term Radical Fractions
Multiply numerator and denominator by the radical that will combine with and then reduce the unwanted radical.

Example 5.10 *Rewrite* $\frac{2}{\sqrt{6}}$ *so the square root is in the numerator only.*

Solution: We will multiply numerator and denominator by $\sqrt{6}$ since $\sqrt{6}\cdot\sqrt{6} = 6$.

$\frac{2}{\sqrt{6}} = \frac{2\cdot\sqrt{6}}{\sqrt{6}\cdot\sqrt{6}}$, (multiply numerator and denominator by $\sqrt{6}$)
$= \frac{2\cdot\sqrt{6}}{\sqrt{36}}$, (factor rule)
$= \frac{2\cdot\sqrt{6}}{6}$, $(\sqrt{36} = 6)$
$= \frac{\sqrt{6}}{3}$, (reduce common factor of 2).
Therefore, $\frac{2}{\sqrt{6}} = \frac{\sqrt{6}}{3}$. ∎

Example 5.11 *Rewrite* $\frac{\sqrt{10}}{\sqrt{8}}$ *so the square root is in the numerator only.*

Solution: We will multiply numerator and denominator by $\sqrt{2}$ since $\sqrt{8}\cdot\sqrt{2}=\sqrt{16}=4$.

$\frac{10}{\sqrt{8}}=\frac{\sqrt{10}\cdot\sqrt{8}}{\sqrt{2}\cdot\sqrt{2}}$, (multiply numerator and denominator by $\sqrt{2}$)
$=\frac{\sqrt{20}}{\sqrt{16}}$, (factor rule)
$=\frac{\sqrt{4\cdot 5}}{4}$, ($\sqrt{16}=4$, factor 20 with a square number)
$=\frac{2\cdot\sqrt{5}}{4}$, (factor rule)
$=\frac{\sqrt{5}}{2}$, (reduce common factor of 2).
Therefore, $\frac{\sqrt{10}}{\sqrt{8}}=\frac{\sqrt{5}}{2}$. ∎

Example 5.12 *Rewrite* $\frac{6}{\sqrt[3]{9}}$ *so the cube root is in the numerator.*

Solution: We will multiply numerator and denominator by $\sqrt[3]{3}$ since $\sqrt[3]{9}\cdot\sqrt[3]{3}=\sqrt[3]{27}=3$.

$\frac{6}{\sqrt[3]{9}}=\frac{6\cdot\sqrt[3]{3}}{\sqrt[3]{9}\cdot\sqrt[3]{3}}$, (multiply numerator and denominator by $\sqrt[3]{3}$)
$=\frac{6\cdot\sqrt[3]{3}}{\sqrt[3]{27}}$, (factor rule)
$=\frac{6\cdot\sqrt[3]{3}}{3}$, ($\sqrt[3]{27}=3$)
$=2\cdot\sqrt[3]{3}$, (reduce common factor of 2).
Therefore, $\frac{6}{\sqrt[3]{9}}=2\cdot\sqrt[3]{3}$. ∎

We can also rewrite fractions with radicals so that there are radicals in the denominator only. Let's redo an example where we eliminate a radical from the numerator.

Example 5.13 *Rewrite* $\frac{\sqrt{10}}{\sqrt{8}}$ *so the square root is in the denominator only.*

Solution: With the focus now on eliminating the radical from the numerator, we will multiply numerator and denominator by $\sqrt{10}$ since $\sqrt{10}\cdot\sqrt{10}=10$.

$\frac{\sqrt{10}}{\sqrt{8}}=\frac{\sqrt{10}\cdot\sqrt{10}}{\sqrt{8}\cdot\sqrt{10}}$, (multiply numerator and denominator by $\sqrt{10}$)
$=\frac{\sqrt{100}}{\sqrt{80}}$, (factor rule)
$=\frac{10}{\sqrt{16\cdot 5}}$, $\sqrt{100}=10$, factor 80 with a square number
$=\frac{10}{4\cdot\sqrt{5}}$, (factor rule)
$=\frac{5}{2\cdot\sqrt{5}}$, (reduce common factor of 2).
Therefore, $\frac{10}{\sqrt{8}}=\frac{5}{2\cdot\sqrt{5}}$. ∎

Homework Problems

1. Simplify $\sqrt{54}$ and $\sqrt[3]{54}$.
2. Simplify $\sqrt{250}$ and $\sqrt[3]{250}$.
3. Simplify $\sqrt{1000x}$ and $\sqrt[3]{1000x}$.
4. Simplify $\sqrt{108x}$ and $\sqrt[3]{108x}$.
5. Simplify $\sqrt{\frac{25x}{64}}$ and $\sqrt[3]{\frac{25x}{64}}$.
6. Simplify $\sqrt{\frac{125x^2}{64}}$ and $\sqrt[3]{\frac{125x^2}{64}}$.
7. Simplify $\frac{\sqrt{3x}\cdot\sqrt{6}}{15}$.
8. Simplify $\sqrt{5}\cdot\sqrt{10}$.
9. Simplify $\left(\sqrt{12}\cdot\sqrt{10}\right)^2$.
10. Simplify $\left(\sqrt{3}\cdot\sqrt{5}\right)^4$.
11. Simplify $\left(\sqrt{6}+\sqrt{2x}\right)\left(\sqrt{6}-\sqrt{2x}\right)$.
12. Simplify $\left(\sqrt{6}+\sqrt{2x}\right)^2$.
13. Simplify $-2\cdot\sqrt{28}+3\cdot\sqrt{63}$.
14. Simplify $\sqrt[3]{8x}+2\cdot\sqrt[3]{-27x}$.
15. Simplify $\sqrt{\frac{8}{9}}-3\cdot\sqrt{2}$.
16. Simplify $\sqrt{\frac{5}{16}}-\sqrt{\frac{20}{121}}$.
17. Give a counterexample to show $\sqrt{a+b}=\sqrt{a}+\sqrt{b}$ is false.
18. Rewrite $\frac{2\cdot\sqrt{3}}{5\cdot\sqrt{10}}$ so: (a) there is no square root in the denominator; (b) there is no square root in the numerator.
19. Rewrite $\frac{4\cdot\sqrt{x}}{x\cdot\sqrt{2}}$ so: (a) there is no square root in the denominator; (b) there is no square root in the numerator.

20. Rewrite $\frac{2 \cdot \sqrt[3]{3}}{5 \sqrt[3]{10}}$ so: (a) there is no cube root in the denominator; (b) there is no cube root in the numerator.

21. Rewrite $\frac{x \cdot \sqrt[3]{2}}{6 \cdot \sqrt[3]{x}}$ so: (a) there is no cube root in the denominator; (b) there is no cube root in the numerator.

22. Determine the area of a triangle which has base $5 - \sqrt{3}$ feet and height $5 + \sqrt{3}$ feet.

23. A circle has area 500 square feet. Determine the radius of the circle. Express the final exact answer with a radical in the numerator only, then round the final answer to the nearest tenth. $\boxed{a}$

5.3 Simplifying $\sqrt[n]{x^m}$ When $m \geqslant n$

This section revolves around two main topics: simplifying $\sqrt[n]{x^n}$ and simplifying $\sqrt[n]{x^m}$ when the power m is larger than the index n. Both are pretty simple to do, but common errors can arise without proper care.

Does $\sqrt[n]{x^n} = x$?

As you will see the answer is no overall since it depends on whether index n is even or odd. We will consider two special cases as an illustration: does $\sqrt[2]{x^2} = x$?; and does $\sqrt[3]{x^3} = x$?

Let's start with the question, does $\sqrt{x^2} = x$? Unfortunately the answer is false because the rule does not work for negative values of x. For example, $\sqrt{4^2} \stackrel{?}{=} 4$ and $\sqrt{0^2} \stackrel{?}{=} 0$ are both true, but $\sqrt{(-2)^2} \stackrel{?}{=} -2$ is false since $\sqrt{(-2)^2} = \sqrt{4} = 2$. Similarly, $\sqrt{(-5)^2} \neq -5$ since $\sqrt{(-5)^2} = \sqrt{25} = 5$.

What is needed to make the rule $\sqrt{x^2} = x$ correct is a way to guarantee the right hand side is non-negative. Does that hopefully make you think of using absolute value on the right side? The valid rule for simplifying $\sqrt{x^2}$ is $\sqrt{x^2} = |x|$.

Now let's examine the question does $\sqrt[3]{x^3} = x$? Let's try a positive number, a negative number and 0 to see if the rule works. For example $\sqrt[3]{(5)^3} \stackrel{?}{=} 5$ is true since $\sqrt[3]{(5)^3} = \sqrt[3]{125} = 5$; and $\sqrt[3]{(-4)^3} \stackrel{?}{=} -4$ is true since $\sqrt[3]{(-4)^3} = \sqrt[3]{-64} = -4$; and $\sqrt[3]{(0)^3} \stackrel{?}{=} 0$ is again true since $\sqrt[3]{(0)^3} = \sqrt[3]{0} = 0$.

In general, it can be proved that for all numbers x, $\sqrt[3]{x^3} = x$ (without an absolute value). The main reason is that we do not insist that the output for cube root be non-negative like even roots.

The following theorem summarizes and generalizes this discussion in that all even-indexed roots tend to behave like square roots and all odd-indexed roots tend to behave like cube roots:

How to Simplify $\sqrt[n]{x^n}$
If n is even, $\sqrt[n]{x^n} = \lvert x\rvert$.
If n is odd, $\sqrt[n]{x^n} = x$.

Simplifying $\sqrt[n]{x^m}$ When $m > n$?

At this point, an important fact to recall is the "product rule" for exponents $(x^m)^n = x^{mn}$. For example, $(x^4)^2 = x^{4\cdot 2} = x^8$. Similarly $(x^3)^3 = x^{3\cdot 3} = x^9$. The product rule is key to simplifying square and cube roots of powers.

Example 5.14 *What squared equals x^6? What squared equals x^8? But what do $\sqrt{x^6}$ and $\sqrt{x^8}$ equal?*

Solution: Actually there are two answers for the first two questions each:

$$(\pm x^3)^2 = (\pm 1)^2 x^{3\cdot 2} = x^6 \text{ and } (\pm x^4)^2 = (\pm 1)^2 x^{4\cdot 2} = x^8.$$

So $\pm x^3$ squared equals x^6 and $\pm x^4$ squared equals x^8. But when it's time to do the square roots, there will only be one answer each because of the answer must be positive:

$\sqrt{x^6} = |x^3|$, (keep absolute value since x^3 could be negative).

$\sqrt{x^8} = |x^4| = x^4$, (drop absolute value since x^4 can't be negative).

Therefore, $\sqrt{x^6} = |x^3|$ and $\sqrt{x^8} = x^4$ are the final answers. ∎

Example 5.15 *What cubed equals x^6? What cubed equals x^9?What about x^{600}? What do $\sqrt[3]{x^6}$ and $\sqrt[3]{x^9}$ equal?*

Solution: With cube root, there is only one answer each for the first two questions. For example $(x^2)^3 = x^{2\cdot 3} = x^6$, but $(-x^2)^3 = -x^6$. Similarly $(x^3)^3 = x^{3\cdot 3} = x^9$, but $(-x^3)^3 = -x^9$.

So, it follows that $\sqrt[3]{x^6} = x^2$ and $\sqrt[3]{x^9} = x^3$. No absolute value is needed because cube root is an odd-indexed radical. ∎

Here are the key observations from the last examples that you should remember:

How to Simplify $\sqrt[2]{x^{2n}}$ and $\sqrt[3]{x^{3n}}$
To take the square root of an even power $2n$, just divide the power by 2 and use absolute value. So $\sqrt[2]{x^{2n}} = \vert x^n \vert$. Drop the absolute value if the resulting power n is even.
To take the cube root of a multiple of 3 power $3n$, just divide the power by 3. So $\sqrt[3]{x^{3n}} = x^n$. No absolute value is needed.

Now we will proceed with more examples of simplifying square roots and cube roots of a variety of powers. In simplifying $\sqrt[n]{x^m}$, when the power m is not a multiple of 2 for square root or a multiple of 3 for cube root, we will break x^m into two or more factors, at least one of which can be simplified.

Example 5.16 *Simplify (a) $\sqrt{x^3}$, (b) $\sqrt{x^4}$ and (c) $\sqrt{x^6}$.*

Solution: For part (a) note that x must be non-negative; else x^3 could be negative and the square root would not make sense. So,

$\sqrt{x^3} = \sqrt{x^2 \cdot x}$, ($x^3$ has an even power factor)
$= \sqrt{x^2} \cdot \sqrt{x}$, (first factor rule)
$= |x| \cdot \sqrt{x}$, $(2/2 = 1)$
$= x \cdot \sqrt{x}$, $(x \geq 0)$.
Therefore, $\sqrt{x^3} = x \cdot \sqrt{x}$. ∎

For part (b) x could be any number since $x^4 \geq 0$. So,
$\sqrt{x^4} = |x^2|$, (4/2=2)
$= x^2$, (absolute value can be dropped since $x^2 \geq 0$).
Therefore, $\sqrt{x^4} = x^2$. ∎

For part (c) again x could be any number since $x^6 \geq 0$. So,
$\sqrt{x^6} = |x^3|$, (6/2=3).
The absolute value can not be dropped since x^3 could be negative.
Therefore, $\sqrt{x^6} = |x^3|$. ∎

Example 5.17 *Simplify* $\sqrt{18x^7}$.

Solution: Here $x \geq 0$ is a must since for negative x, $18x^7$ would be negative and so $\sqrt{18x^7}$ would not make sense.

$\sqrt{18x^7} = \sqrt{9x^6 \cdot 2x}$, (factor $18x^7$ with perfect squares)
$= 3|x^3| \cdot \sqrt{2x}$, ($\sqrt{9} = 3$, $\sqrt{x^6} = |x^3|$)
$= 3x^3 \cdot \sqrt{2x}$, (since $x \geq 0$ the absolute value is not needed).
Therefore, $\sqrt{18x^7} = 3x^3 \cdot \sqrt{2x}$. ∎

Now, here are some examples with $\sqrt[3]{\ }$. As usual, cube root is less complicated in terms of simplifying.

Example 5.18 *Simplify (a)*$\sqrt[3]{x^5}$, *(b)* $\sqrt[3]{x^6}$ *and (c)* $\sqrt[3]{24x^{11}}$.

Solution: $\sqrt[3]{\ }$ is an odd-indexed root, we don't have to worry about using absolute value.

For part (a) $\sqrt[3]{x^5} = \sqrt[3]{x^3 \cdot x^2}$, (factor x^5 with a perfect cube)
$= \sqrt[3]{x^3} \cdot \sqrt[3]{x^2}$, (first factor rule)
$= x \cdot \sqrt[3]{x^2}$, ($\sqrt[3]{x^3} = x$).
Therefore, $\sqrt{x^5} = x \cdot \sqrt[3]{x^2}$. ∎

For part (b) $\sqrt[3]{x^6} = x^2$, $(6/3 = 2)$. ∎

And for part (c) $\sqrt[3]{24x^{11}} = \sqrt[3]{8 \cdot 3 \cdot x^9 \cdot x^2}$, (factor $24x^{11}$ with a perfect cubes)
$= \sqrt[3]{8} \cdot \sqrt[3]{x^9} \cdot \sqrt[3]{3x^2}$, (factor rule)
$= 2x^3 \cdot \sqrt[3]{3x^2}$.
Therefore, $\sqrt[3]{24x^{11}} = 2x^3 \cdot \sqrt[3]{3x^2}$. ∎

Give this problem a try to see if you can successfully generalize to fourth roots. Don't forget fourth root is an even-indexed radical!

Try This 5.3 *Simplify: (a)* $\sqrt[4]{x^5}$, *(b)* $\sqrt[4]{x^6}$ *and (c)* $\sqrt[4]{64x^{11}}$. $\boxed{\text{Ⓢ}}$

Solving Pure nth Degree Equations With nth Roots

Pure nth degree equations are equations which can be written in the form $x^n = c$ where n is a positive integer bigger than 1 and c is a real number. They have this name since x^n is the only variable term in the equation. A pure 2nd degree equation is $x^2 = 10$; so is $4x^2 - 25 = 0$ since it can be rearranged to $x^2 = \frac{25}{4}$. Similarly, $x^3 = 54$ and $2x^3 + 5000 = 0$ are pure 3rd degree equations. Solving pure nth degree equations is very natural.

How to Solve Pure nth Degree Equations
Just isolate the variable then take the nth root of both sides.
Fine print: remember if n is even, then $\sqrt[n]{x^n} = \lvert x\rvert$; but if n is odd, then $\sqrt[n]{x^n} = x$

Example 5.19 *Solve the equation* $x^2 = 64$.

Solution: The x^2 is already isolated. Taking the square root of both sides gives:

$$\begin{aligned} &\sqrt{x^2} = \sqrt{64} \\ &\leftrightarrow |x| = 8 \\ &\leftrightarrow x = \pm 8. \end{aligned}$$

So there are two solutions, $x = \pm 8$. ∎

Example 5.20 *Solve the equation* $x^3 + 81 = 0$.

Solution: Here, x^3 needs isolated first: $x^3 + 81 = 0 \leftrightarrow x^3 = -81$. Next, taking the cube root of both sides gives:

$$\begin{aligned} &\sqrt[3]{x^3} = \sqrt[3]{-81} \\ &\leftrightarrow x = \sqrt[3]{-81} \\ &\leftrightarrow x = \sqrt[3]{-27 \cdot 3} = \sqrt[3]{-27} \cdot \sqrt[3]{3} = -3 \cdot \sqrt[3]{3}. \end{aligned}$$

Therefore, there's just one solution, $x = -3 \cdot \sqrt[3]{3}$. ∎

Example 5.21 *Solve the equation* $16x^2 + 1 = 0$

Solution: Start by isolating x^2: $16x^2 + 1 = 0 \leftrightarrow 16x^2 = -1 \leftrightarrow x^2 = \frac{-1}{16}$.

At this stage we can conclude there are no real solutions since any number to the fourth power is non-negative, e.g. $x^2 = \frac{-1}{16}$ is impossible. Or if we take square root of both sides of $x^2 = \frac{-1}{16}$, we obtain a $\sqrt{\frac{-1}{16}}$ which is not a real number. Either way the conclusion is "no real solution." ∎

Homework Problems

1. In which of the following expressions could x be a negative: (a) $\sqrt{x^8}$; (b)$\sqrt{x^9}$; (c) $\sqrt{x^{10}}$; (d) $\sqrt[3]{x^8}$; (e) $\sqrt[3]{x^9}$; (f) $\sqrt[3]{x^{10}}$? Keep this in mind as you do the next two problems!
2. Simplify: (a) $\sqrt{x^8}$, (b)$\sqrt{x^9}$ and (c) $\sqrt{x^{10}}$.
3. Simplify: (a)$\sqrt[3]{x^8}$, (b) $\sqrt[3]{x^9}$ and (c) $\sqrt[3]{x^{10}}$.
4. Simplify: (a) $\sqrt{27x^4}$ and (b) $\sqrt[3]{27x^4}$.
5. Simplify: (a) $\sqrt{120x^5}$ and (b) $\sqrt[3]{120x^5}$.
6. Simplify $\sqrt{\frac{27x^9}{64}}$and $\sqrt[3]{\frac{27x^9}{64}}$.
7. Simplify $\sqrt{\frac{x^{10}}{16}}$and $\sqrt[3]{\frac{x^{10}}{8}}$.
8. Simplify: (a) $\sqrt[3]{8 \times 10^{-3}}$ and (b) $\sqrt[3]{8 \times 10^6}$.
9. Simplify: (a) $\sqrt{1.21 \times 10^4}$ and (b) $\sqrt{1.21 \times 10^{-6}}$.
10. Solve $x^2 - 125 = 0$.
11. Solve $9x^2 - 64 = 0$.
12. Solve $3x^3 + 24 = 0$.
13. Solve $.01x^3 - 150 = 0$.
14. Solve $.01x^3 - 10 = 0$.

15. Solve $\frac{3}{4}x^2 = 600$.

16. Solve $\frac{3}{16}x^2 - \frac{27}{49} = 0$.

17. Without actually solving the equations completely, state how many real solutions each of the following equations would have:

a. $x^2 = -1$	b. $x^3 = -15$
c. $-4x^2 = -3$	d. $4x^2 = 0$
e. $-x^5 - 10 = 0$	f. $2x - 1 = 0$

$\boxed{a}$

5.4 Rational Powers

In this section we will explore how to represent $\sqrt[n]{x^m}$ as a power of x. In other words we will try to determine a power r for which $\sqrt[n]{x^m} = x^r$.

Let's start with the simplest radical, $\sqrt{x}$. We know $(\sqrt{x})^2 = \sqrt{x} \cdot \sqrt{x} = x$ for all $x \geq 0$. So, if we were to think of $\sqrt{x}$ as x^r, then $x^r \cdot x^r = x$. So literally $x^{2r} = x^1$. Equating the powers $2r$ and 1, the logical choice for r is $\frac{1}{2}$. This leads to the following rule:

$$\boxed{\sqrt[2]{x} = x^{\frac{1}{2}} \text{ (Fine print: } x \text{ must be non-negative).}}$$

The power $\frac{1}{2}$ is called a rational power because $\frac{1}{2}$ is a rational number. It just gives us another way to represent square root. This was especially useful in the days when typing symbols like $\sqrt{}$ was difficult. Today many formulas still use the $\frac{1}{2}$ power in place of $\sqrt{}$.

Example 5.22 *Simplify* $16^{\frac{1}{2}}$; $4^2 \cdot 36^{\frac{-1}{2}}$; *and* $-100^{\frac{1}{2}} + 100^{\frac{-1}{2}}$.

Solution: $16^{\frac{1}{2}}$ is just by definition $\sqrt{16}$. Therefore $16^{\frac{1}{2}} = 4$. Next, $4^2 \cdot 36^{\frac{-1}{2}} = \frac{4^2}{36^{\frac{1}{2}}} = \frac{16}{6} = \frac{8}{3}$. Finally, $-100^{\frac{1}{2}}$ is by order of operations $-(100^{\frac{1}{2}})$, e.g. the power goes before the minus sign. Hence,

$$-100^{\frac{1}{2}} + 100^{\frac{-1}{2}} = -\sqrt{100} + \frac{1}{\sqrt{100}} = -10 + \frac{1}{10} = \frac{-100+1}{10} = \frac{-99}{10}. \blacksquare$$

As it turns out, for each radical type there is a logical rational power. For square root, the rule is $\sqrt[2]{x} = x^{\frac{1}{2}}$. Notice how the index becomes the denominator of the power. The rule for nth root follows the same pattern:

$$\boxed{x^{\frac{1}{n}} = \sqrt[n]{x} \text{ (Fine print: if } n \text{ is even the } x \text{ must be non-negative.)}}$$

Rational powers are used quite a bit in calculus. For now, they are important because they give a way to calculate roots on any calculator with a power key. For example, to calculate $\sqrt[4]{4096}$ we would think of $\sqrt[4]{4096}$ as $4096^{(1/4)}$. The parentheses around $\frac{1}{4}$ are necessary to keep the fraction power together in the calculation. To calculate $\sqrt[4]{4096}$ on the graphcalc.com calculator we would type: 4096^(1/4) ENTER, and the answer will be 8 since $8 \cdot 8 \cdot 8 \cdot 8 = 4096$.

All rational powers, including powers like $\frac{2}{3}$ and $\frac{-4}{3}$, are defined as follows:

$$x^{\frac{m}{n}} = \sqrt[n]{x^m} = (\sqrt[n]{x})^m$$

For a rational power $\frac{m}{n}$ the denominator n corresponds to the index of the radical while the numerator corresponds to a power which can be written inside or outside the radical.

Fine print: $\frac{m}{n}$ should be in simplified form. And, as usual, if n is even then x must be non-negative!

Example 5.23 *Simplify* $4^{-\frac{1}{2}}$ *and* $8^{\frac{2}{3}}$.

Solution: In radical form $4^{-\frac{1}{2}}$is $\sqrt[2]{4^{-1}}$. Since $4^{-1} = \frac{1}{4}$ it follows that $4^{\frac{-1}{2}} = \sqrt[2]{4^{-1}} = \sqrt[2]{\frac{1}{4}} = \frac{1}{2}$. Another way to solve this problem would be to say $4^{-\frac{1}{2}}$ is $\left(\sqrt[2]{4}\right)^{-1}$. So then $4^{-\frac{1}{2}} = \left(\sqrt[2]{4}\right)^{-1} = 2^{-1} = \frac{1}{2}$.

Next, in radical form $8^{\frac{2}{3}}$ is $\sqrt[3]{8^2}$. Hence, $8^{\frac{2}{3}} = \sqrt[3]{8^2} = \sqrt[3]{64} = 4$. This problem can also be solved by $8^{\frac{2}{3}} = \left(\sqrt[3]{8}\right)^2 = 2^2 = 4$.

Basically, according to the above rule we can choose to have the power m inside or outside the radical symbol, whichever seems easier or more convenient. ∎

Try This 5.4 *One way to see that* $x^{\frac{m}{n}} = \sqrt[n]{x^m}$ *is the argument* $x^{\frac{m}{n}} = x^{\frac{m}{1}\cdot\frac{1}{n}} = \left(x^{\frac{m}{1}}\right)^{\frac{1}{n}} = (x^m)^{\frac{1}{n}} = \sqrt[n]{x^m}$. *It's also true that* $x^{\frac{m}{n}} = \left(\sqrt[n]{x}\right)^m$. *Switch the* $\frac{m}{1} \cdot \frac{1}{n}$ *to* $\frac{1}{n} \cdot \frac{m}{1}$ *in the latter argument to show* $x^{\frac{m}{n}} = \left(\sqrt[n]{x}\right)^m$. $\boxed{Ⓢ}$

Homework Problems

1. Write $x^{\frac{1}{5}}$as a radical. Could x be a negative number?

2. Write $x^{\frac{1}{6}}$ as a radical. Could x be a negative number?

3. Write $x^{\frac{3}{4}}$ as a radical. Could x be a negative number?

4. Write $x^{\frac{5}{3}}$ as a radical. Could x be a negative number?

5. Without using a calculator, simplify: (a) $100^{\frac{1}{2}}$; (b) $100^{\frac{3}{2}}$; and (c) $100^{\frac{-5}{2}}$. (hint: put the power on the outside!)

6. Without using a calculator, simplify: (a) $27^{\frac{1}{3}}$; (b) $27^{\frac{2}{3}}$; and (c) $27^{\frac{-2}{3}}$. (hint: put the power on the outside!)

7. Without using a calculator, simplify: (a) $.0016^{\frac{1}{4}}$; (b) $.0016^{\frac{3}{4}}$; and (c) $.0016^{\frac{-5}{4}}$.

8. Without using a calculator, simplify: (a) $\left(\frac{-1}{32}\right)^{\frac{1}{5}}$; (b) $\left(\frac{-1}{32}\right)^{\frac{2}{5}}$; and (c) $\left(\frac{-1}{32}\right)^{\frac{-3}{5}}$.

9. The rules of exponents work the same for fraction powers as they do for integer powers. Based on that fact, combine $x^{\frac{2}{3}} \cdot x^{\frac{4}{3}}$ into a single power of x. Does x need to be non-negative?

10. The rules of exponents work the same for fraction powers as they do for integer powers. Based on that fact, combine $\left(x^{\frac{3}{2}}\right)^4$ into a single power of x. Does x need to be non-negative?

11. Jimmy entered $4^{\frac{3}{2}}$ into his calculator incorrectly and got 32 for an answer. What did he most likely do wrong?

12. On a graphing calculator, graph the equation $y = x^{\frac{1}{5}}$. Remember: the power $\frac{1}{5}$ should be typed as $(1/5)$. How would you describe the shape of the graph?

13. On a graphing calculator, graph the equations $y = x^{\frac{1}{2}}$ and $y = x^{\frac{1}{4}}$ together. Remember to type parentheses around the powers. Which graph is higher at first? Which one is ultimately higher? How would you know the answer to the last question without looking at a graph? $\boxed{a}$

Appendix A

Solutions/Answers

A.1 Try This Solutions

Section 1.1 Try This

Solution: Actually all the points, except point E, satisfy the equation. Points A, B, C and D satisfy the equation $x^2 + 4y^2 = 4$ because:

$(-2)^2 + 4 \cdot (0)^2 \stackrel{?}{=} 4$ is true since $(-2)(-2) + 4\,(0)\,(0) = 4 + 0 = 4$;

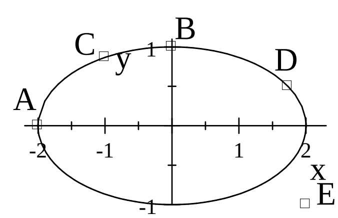
$(0)^2 + 4 \cdot (1)^2 \stackrel{?}{=} 4$ is true since $(0)(0) + 4\,(1)\,(1) = 0 + 4 = 4$;

$(-1)^2 + 4 \cdot (\frac{\sqrt{3}}{2})^2 \stackrel{?}{=} 4$ is true since $(-1)(-1) + 4\left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) =$
$1 + 4 \cdot \frac{3}{4} = 1 + 3 = 4$;

$(\sqrt{3})^2 + 4 \cdot (\frac{1}{2})^2 \stackrel{?}{=} 4$ is true since $(\sqrt{3})(\sqrt{3}) + 4\left(\frac{1}{2}\right) \cdot \left(\frac{1}{2}\right) =$
$3 + 4 \cdot \frac{1}{4} = 3 + 1 = 4$.

Point E does not satisfy the equation since $(2)^2 + 4 \cdot (-1)^2 = 4 + 4 = 8 \neq 4$.

To the right, points $A - E$ have been plotted. (We estimated $\sqrt{3}$ with 1.7 and $\frac{\sqrt{3}}{2}$ with .85.) Notice that points A to D lie on the ellipse, but E does not.

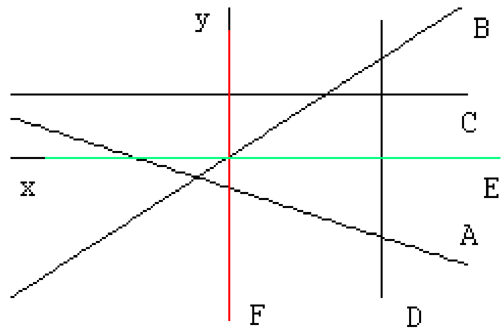


Ellipse $x^2 + 4y^2 = 4$

Click here to return to Section 1.1.

Section 2.1 Try This

Solution: If you say one x-intercept and one y-intercept, you are correct for many but not all lines. Here is a graph of six different lines that shows all the various possibilities of intercepts:



For lines like line A or B that slope down or up , there is an x-intercept and y-intercept. For a line B, the origin actually serves as both the x-intercept and the y-intercept. Horizontal line C also has a y-intercept but no x-intercept. Similarly vertical line D has an x-intercept and no y-intercept. Finally, horizontal line E, which is the x-axis, has infinitely many x-intercepts and no y-intercepts while vertical line F, which is actually the y-axis, has infinitely many y-intercepts and no x-intercepts. So a line can have 0, 1 or infinitely many x-intercepts. Similarly, a line can have 0, 1 or infinitely many y-intercepts. The total number of intercepts for a line could be 1, 2 or infinitely many!

What's important to remember is that lines will have one x-intercept and one y-intercept unless the lines are vertical or horizontal. As you will see later, it's always a good idea to think of vertical and horizontal lines as special situations unlike the typical line that rises or fall across the plane.

Click here to return to Section 2.1.

Section 2.2 Try This

Solution: For part (a) of the question, $x = 0 \to y = -5$. So, $(0, -5)$ is the y-intercept. Next $y = 0 \to x = \frac{5}{4}$. So, $(\frac{5}{4}, 0)$ is the x-intercept. For a third point, if we let $x = -1$ then $y = -9$, so the point $(-1, 9)$ is also on the line L.

Let $A = (-1, -9)$, $B = (0, -5)$ and $C = (\frac{5}{4}, 0)$. Now we will calculate slope three different ways:

- Using points A and B, the slope works out to be $m = \frac{(-5)-(-9)}{(0)-(-1)} = \frac{-5+9}{0+1} = \frac{4}{1} = 4$;

- Using points A and C, the slope works out to be $m = \frac{(0)-(-9)}{(\frac{5}{4})-(-1)} = \frac{0+9}{\frac{5}{4}+1} = \frac{9}{\frac{9}{4}} = 9 \cdot \frac{4}{9} = 4$; and

- Finally, using points B and C, the slope works out to be $m = \frac{(0)-(-5)}{(\frac{5}{4})-(0)} = \frac{0+5}{\frac{5}{4}} = \frac{5}{\frac{5}{4}} = 5 \cdot \frac{4}{5} = 4$;

- Notice that the same slope value of 4 resulted from each pair of points.

As for part (b), the slope value 4 occurs in the equation $y = 4x - 5$ as the coefficient of x.

As for part (c), the y-coordinate of the y-intercept $(0, -5)$ appears in the linear equation $y = 4x - 5$ as the constant term.

Click here to return to Section 2.2.

Section 2.3 Try This

Solution: According the Point-Slope Form, the equation of line L would be $y - b = m(x - 0)$. So, we get $y - b = mx$ or plain old $y = mx + b$. The $y = mx + b$ form is called the Slope-Intercept Form of the line L because m corresponds to its slope and b corresponds to the y-intercept.

Click here to return to Section 2.3.

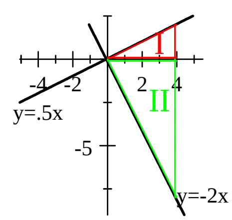
Section 2.4 Try This

Solution:

(a) The height of Triangle I is 2 and the base is 4. Next, $c^2 = (2)^2 + (4)^2 = 20$. So, $c = \sqrt{20}$.

(b) The height of Triangle II is 8 and the base is 4. $c^2 = (8)^2 + (4)^2 = 80$. So, $c = \sqrt{80}$.

(c) Triangle III, the combined triangle, had side lengths $\sqrt{20}$, $\sqrt{80}$ and 10.



(d) $(10)^2 = \left(\sqrt{20}\right)^2 + \left(\sqrt{80}\right)^2 \leftrightarrow 100 = 20 + 80$, which is true. So, the sides of the combined triangle adhere to the Pythagorean Theorem.

(e) Triangle III is a right triangle with the 90 degree angle where the two lines meet. So, the lines $y = \frac{1}{2}x$ and $y = 2x$ are perpendicular.

Click here to return to Section 2.4.

Section 3.1 Try This

Solution: Whether or not the units make any sense to use, the conversion process should be routine. Remember just set up the conversion factors so the unwanted factors cancel away. In this case, we get:

$$50 \text{ hons} = 50\ \cancel{\text{hons}} \cdot \frac{14\ \cancel{\text{babs}}}{23\ \cancel{\text{hons}}} \cdot \frac{3 \text{ yobs}}{8\ \cancel{\text{babs}}} = \frac{50 \cdot 14 \cdot 3}{23 \cdot 8} \text{ yobs} = \frac{525}{46} \text{ yobs} = 11\frac{19}{46} \text{ yobs.}$$

Click here to return to Section 3.1.

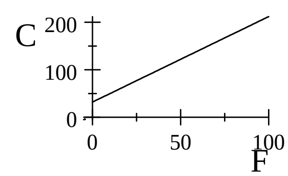
Section 3.2 Try This

Solution:

(a) In case you forgot, the slope of the line would be $\frac{212-32}{100-0} = \frac{180}{100} = \frac{9}{5}$. So, using the point $(0, 32)$, the equation of the line would be $y - 32 = \frac{9}{5}(x - 0) \leftrightarrow y = \frac{9}{5}x + 32$.

(b) The slope $\frac{9}{5}$ is not a true conversion factor since $\frac{9}{5} \cdot 0 = 0$, not 32. The $+32$ in the equation $y = \frac{9}{5}x + 32$ is needed to get the correct value.

(c) This line would not correspond to a conversion factor since its y-intercept is not at the origin.



Click here to return to Section 3.2.

Section 4.1 Try This

Solution: A linear system with two equations will have no solution when the lines are parallel. So we need to determine number(s) b so that the lines have the same slope but different y-intercepts. Getting the 2nd equation into Slope-Intercept Form, $4x + 2y = 7$ becomes $y = \frac{-4}{2}x + \frac{7}{2} = -2x + \frac{7}{2}$. Doing the same for the 1st equation, $2x - by = 6$ becomes $y = \frac{-2}{-b} + \frac{6}{-b} = \frac{2}{b} - \frac{6}{b}$. So the two lines are $y = -2x + \frac{7}{2}$ and $y = \frac{2}{b}x - \frac{6}{b}$. For lines to be parallel, the slope -2 and $\frac{2}{b}$ must be equal. Therefore, $b = -1$. Also notice that when $b = -1$ the lines will have different y-intercepts, one at $(0, \frac{7}{2})$ and the other at $(0, 6)$. So the lines will definitely be parallel when $b = -1$. So, $b = -1$ is the only value that makes the system $\left\{\begin{smallmatrix} 2x\ -\ by\ =\ 6 \\ 4x\ +\ 2y\ =\ 7 \end{smallmatrix}\right\}$ have no solution.

Click here to return to Section 4.1.

Section 4.2 Try This

Solution: Solving the first equation, $2(x+3) = 2x+3 \to 2x+6 = 2x\ +3 \to 2x = 2x-3 \to 0 = -3$. This is a "contradiction." When you solve an equation correctly and get a contradiction, the contradiction indicates the solution you are try to find does not exist. Hence, the first equation has no solution.

Solving the second equation $2(x+3) = x+6 \leftrightarrow 2x+6 = x+6 \leftrightarrow 2x = x$. Now, don't cross out the x's–subtract x from both sides. This gives $x = 0$. Hence the second equation has one solution.

By process of elimination, this means the last equation $2(x\ +3) = 2(x-4)+14$ must have infinitely many solutions. Indeed, $2(x\ +3) = 2(x-4)+14 \leftrightarrow 2x+6 = 2x-\ 8+14 \leftrightarrow 2x+6 = 2x+6$.

At this stage, you should recognize both sides are identically equally, so if we went ever further, we'd end up with $0 = 0$. Equations like $2x+6 = 2x+6$ or $0 = 0$ are "identities." Identities are equations which are true for all numbers x. What this means then is that the original equation $2(x\ +3) = 2(x-4)+14$ always holds no matter what number x is. For example just picking the number $x = -1000$ at random, $2(-1000\ +3) \stackrel{?}{=} 2(-1000-4)+14$ is true since $2(-997) = -1994$ and $2(-1004)+14 = -2008+14 = -1994$. So, $2(x\ +3) = 2(x-4)+14$ has infinitely many solutions.

Click here to return to Section 4.2.

Section 4.3 Try This

Solution: The key fact is that when Gene catches up to Roberta their distances will be equal, but not their times of course since Roberta had a head start. With x=Gene's time and y=Roberta's time, then $y = x+.5$ and $10x = 12y$. So with substitution, $12x = 10(x+.5)$. Therefore, $2x = 5$; so $x = 2.5$ hr which makes $y = 3$ hr.

Click here to return to Section 4.3.

Section 4.4 Try This

Solution: (a) To see pattern we will start with day 1. After the day 1, there will be $24 = 12 \cdot 2 = 12 \cdot 2^1$ bacteria; after day 2 there will be $48 = 24 \cdot 2 = 12 \cdot 2 \cdot 2 = 12 \cdot 2^2$ bacteria; after day 3 there will be $96 = 48 \cdot 2 = 12 \cdot 2 \cdot 2 \cdot 2 = 12 \cdot 2^3$ bacteria; and finally after day 4 there will be $192 = 96 \cdot 2 = 12 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 12 \cdot 2^4$ bacteria. Noting the pattern on the power of 2, there will be $12 \cdot 2^n$ bacteria after n days.

(b) At the end of 6 years, there will be $100 = 50 \cdot 2 = 50 \cdot 2^1$ dollars; at the end of 12 years, there will be $200 = 100 \cdot 2 = 50 \cdot 2 \cdot 2 = 50 \cdot 2^2$ dollars; and finally after 18 years there will be $400 = 200 \cdot 2 = 50 \cdot 2 \cdot 2 \cdot 2 = 50 \cdot 2^3$ dollars. Here we can't say $50 \cdot 2^n$ because the years need to go up by 6 to increase the power of 2 by 1. The correct answer would be $50 \cdot 2^{\frac{n}{6}}$.

(c) $\log_2(32) = 5$ because $2^5 = 32$ and $\log_2(64) = 64$ because $2^6 = 64$. Now $2^7 = 128$, $2^8 = 256$ and $2^9 = 512$. So, $\log_2(300)$ is between 8 and 9.

Click here to return to Section 4.4.

Section 5.1 Try This

Solution: (a) On the graphcalc.com calculator the typing would look like $\sqrt{}27$ ENTER ≈ 5.2, $\sqrt{}5$ ENTER ≈ 2.2. Then $(5.2 - 2.2)/4$ ENTER is .75 which rounds to the tenth as .8.

(b) On the graphcalc.com calculator the typing would look like $(\sqrt{}27 - \sqrt{}5)/4$ then ENTER. The result is .7400211... which rounds to the tenth as .7.

(c) Letting the calculator do as much as possible before rounding would tend to give the better result. So (b) would be the logical guess. In this case the answer in part (b) is a little closer (about .01 units) to the real value of about .74 than the answer is part (a).

Click here to return to Section 5.1.

Section 5.2 Try This

Solution: There are many possible answers to the first part. For example, $\sqrt{26}$, $\sqrt{30}$ and $\sqrt{33}$ are the first three examples. Integers like 26, 30, and 33 are neither prime nor can they be factored into a product where at one factor is a perfect square. An integer like 27 was skipped because it can be factored with a perfect square, e.g. $27 = 9 \cdot 3$. In general $\sqrt{n}$ can not be simplified if n can not be factored with a perfect square number.

Click here to return to Section 5.2.

Section 5.3 Try This

Solution: $\sqrt[4]{x^5} = \sqrt[4]{x^4 \cdot x} = \sqrt[4]{x^4} \cdot \sqrt[4]{x} = |x| \cdot \sqrt[4]{x} = x \cdot \sqrt[4]{x}$. The absolute value can be removed because x must be non-negative at the beginning of the problem; otherwise x^5 would be negative leading to $\sqrt[4]{x^5}$, the 4th root of a negative.

$\sqrt[4]{x^6} = \sqrt[4]{x^4 \cdot x^2} = \sqrt[4]{x^4} \cdot \sqrt[4]{x^2} = |x| \cdot \sqrt[4]{x^2}$. The absolute value can not be removed because x^6 is always non-negative making $\sqrt[4]{x^6}$ an OK root no matter if x is positive or negative.

$\sqrt[4]{64x^{11}} = \sqrt[4]{16 \cdot 4 \cdot x^8 \cdot x^3} = \sqrt[4]{16x^8} \cdot \sqrt[4]{4x^3} = \sqrt[4]{16} \cdot \sqrt[4]{x^8} \cdot \sqrt[4]{4x^3} = 2\,|x^2| \cdot \sqrt[4]{4x^3} = 2x^2 \cdot \sqrt[4]{4x^3}$. The absolute value can be removed because x^2 is automatically non-negative.

Click here to return to Section 5.3.

Section 5.4 Try This

Solution: Switching the order of $\frac{m}{1} \cdot \frac{1}{n}$ *to* $\frac{1}{n} \cdot \frac{m}{1}$ gives:

$$x^{\frac{m}{n}} = x^{\frac{1}{n} \cdot \frac{m}{1}} = \left(x^{\frac{1}{n}}\right)^{\frac{m}{1}} = \left(x^{\frac{1}{n}}\right)^{m} = (\sqrt[n]{x})^m$$

which proves that $x^{\frac{m}{n}}$ also equals $(\sqrt[n]{x})^m$.

Click here to return to Section 5.4.

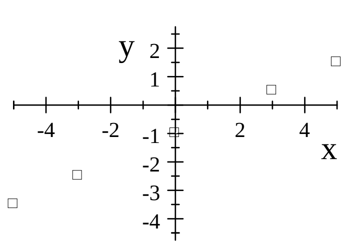
A.2 Answers to Homework Problems

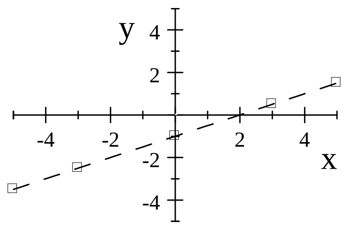
Section 1.1 Answers

1) d and e; 2) a, d and e; 3) $\frac{-5}{3}$;
4) $\frac{-1}{2}$; 5) $\frac{1}{2}$;

6)

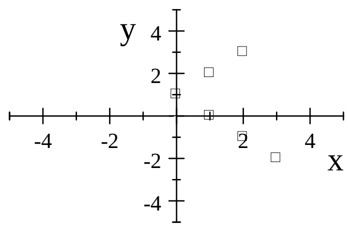
x		y
5	$\rightarrow$	$\frac{3}{2}$
3	$\rightarrow$	$\frac{1}{2}$
0	$\rightarrow$	-1
-3	$\rightarrow$	$\frac{-5}{2}$
-5	$\rightarrow$	$\frac{-7}{2}$

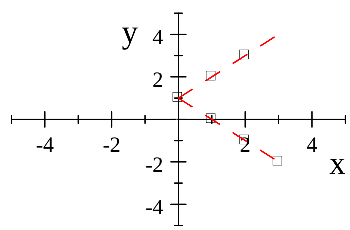


7) Linear; 

8)

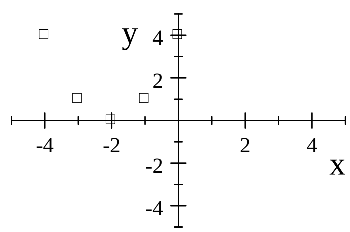
x		y
3	$\leftarrow$	-2
2	$\leftarrow$	-1
1	$\leftarrow$	0
0	$\leftarrow$	1
1	$\leftarrow$	2
2	$\leftarrow$	3

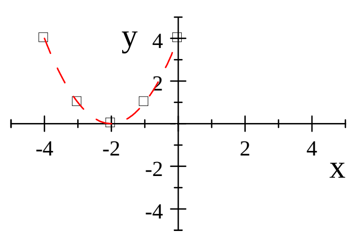


9) Sideways V-Shaped; 

10)

x		y
-4	$\rightarrow$	4
-3	$\rightarrow$	1
-2	$\rightarrow$	0
-1	$\rightarrow$	1
0	$\rightarrow$	4

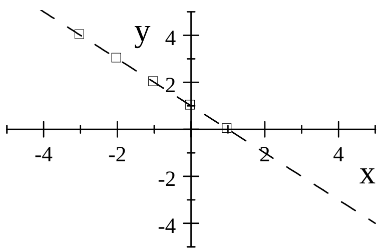


11) U-Shaped; 

12) $f(-2) = 22$; $f(-1) = 9$; $f(1) = -5$; $f(2) = -6$;
13) $f(-2) = -3$; $f(-1) = 0$; $f(1) = 0$; $f(\sqrt{2}) = -1$;
14) $f(-1) = \frac{-21}{10}$; $f(\frac{-1}{2}) = \frac{-11}{40}$; $f(\frac{1}{2}) = \frac{9}{40}$; $f(1) = \frac{19}{10}$;
15) $f(-1) = -2$; $f(\frac{-1}{2}) = \frac{-3}{8}$; $f(\frac{1}{2}) = \frac{-1}{8}$; $f(1) = 0$;
16) Different people can use different x-values, but the graph should be

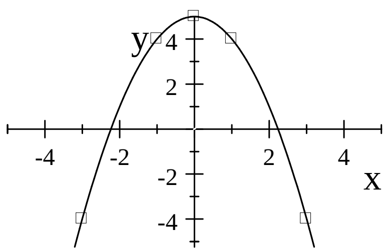
the same overall.

x		y
-3	$\rightarrow$	4
-2	$\rightarrow$	3
-1	$\rightarrow$	2
0	$\rightarrow$	1
1	$\rightarrow$	0



17) Different people can use different x-values, but the graph should be the same. Using $x = -3$, -1, 0, 1 and 3 gives:

x		y
-3	$\rightarrow$	$g(-3) = -4$
-1	$\rightarrow$	$g(-1) = 4$
0	$\rightarrow$	$g(0) = 5$
1	$\rightarrow$	$g(1) = 4$
3	$\rightarrow$	$g(3) = -4$



18) $h(-1) = 6$, $h(0) = 6$ and $h(1) = 6$. Because $h(x)$ always equals 6 (no matter what the input x is), the function could be described as being constant function or as being constant in value;

19) $i(1) = 2 \cdot (1)^3 + 3 \cdot (1)^2 - 8 \cdot (1) + 3 = 2 + 3 - 8 + 3 = 0$;
$i(-3) = 2 \cdot (-3)^3 + 3 \cdot (-3)^2 - 8 \cdot (-3) + 3 = -54 + 27 + 24 + 3 = 0$;
$i(\frac{1}{2}) = 2 \cdot (\frac{1}{2})^3 + 3 \cdot (\frac{1}{2})^2 - 8 \cdot (\frac{1}{2}) + 3 = 2 \cdot (\frac{1}{8}) + 3 \cdot (\frac{1}{4}) - 4 + 3 = \frac{1}{4} + \frac{3}{4} - 1 = 0$;
Getting an output of 0 has been a coincidence. For example $i(2) = 2 \cdot (2)^3 + 3 \cdot (2)^2 - 8 \cdot (2) + 3 = 16 + 12 - 8 + 3 = 23 \neq 0$;

20) No, you remember square-rooting negative numbers is a problem. For example $\sqrt{-4}$ is not a real number (no real number times itself is -4). Yes any number can be squared since any number can be multiplied times itself. Yes any number can be divided by 3. Division is only a problem when dividing by 0;

21) The last two functions $g(x) = x^2$ and $h(x) = \frac{x}{3}$ have domain the set of all real numbers since any number can be squared and any number can be divided by 3. The domain of function $f(x) = \sqrt{x}$ is the only set of all numbers greater than or equal to 0 (square-rooting negative numbers leads to non-real values, e.g.$\sqrt{-4}$;

22) The domain would be all numbers greater than or equal to 10;

23) For example, let $j(x) = x^2$, $a = 1$ and $b = 2$. Then $j(1+2) = j(3) = (3)^2 = 9$; but $j(1) = (1)^2 = 1$ and $j(2) = (2)^2 = 4$ and so $j(1) + j(2) =$

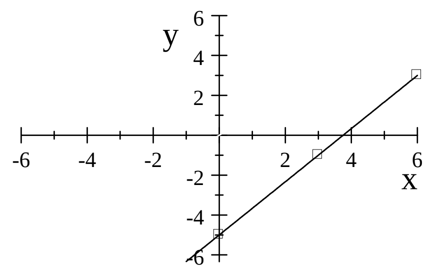
$1 + 4 = 5 \neq 9 = j(1 + 2)$. So, $j(a + b) = j(a) + j(b)$ is false as a general statement;

Click here to return to Section 1.1.

Section 2.1 Answers

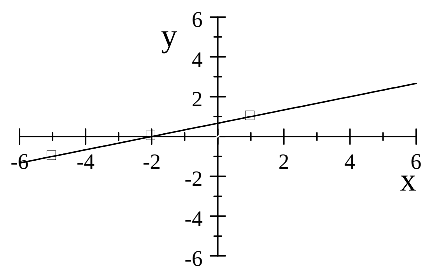
1) (points can vary)

x		y
0	$\rightarrow$	-5
3	$\rightarrow$	-1
6	$\rightarrow$	3



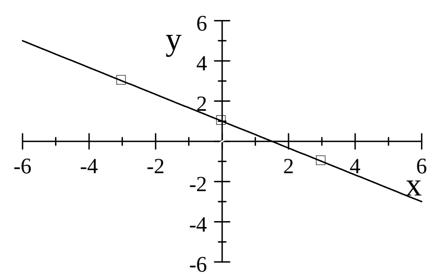
2) (points can vary)

x		y
1	$\leftarrow$	1
-2	$\leftarrow$	0
-5	$\leftarrow$	-1



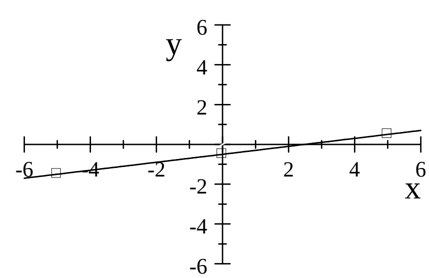
3) $y = \frac{-2}{3}x+1$; (points can vary)

x	y
0	1
-3	3
3	-1



4) $y = \frac{1}{5}x - \frac{1}{2}$; (points can vary)

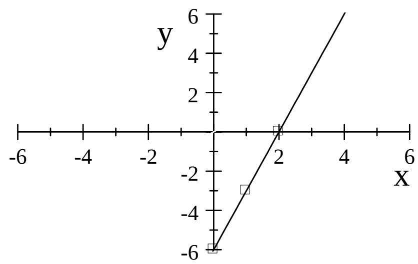
x		y
0	$\rightarrow$	$\frac{-1}{2}$
5	$\rightarrow$	$\frac{1}{2}$
-5	$\rightarrow$	$\frac{-3}{2}$



5) In each, the powers of x and y are 1. Or, each can be written as $Ax + By = C$;

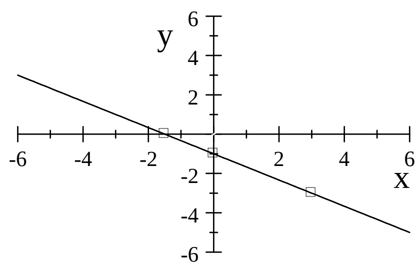
6) The x-intercept is $(2, 0)$; the y-intercept is $(0, -6)$;

x		y
0	$\rightarrow$	-6
2	$\leftarrow$	0
1	$\rightarrow$	-3



7) The x-intercept is $(\frac{-3}{2}, 0)$; the y-intercept is $(0, -1)$;

x		y
0	$\rightarrow$	-1
$\frac{-3}{2}$	$\leftarrow$	0
3	$\rightarrow$	-3



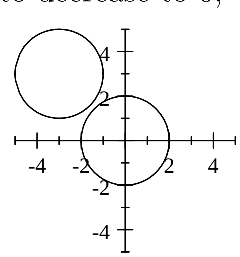
8) The x-intercept would really be $(3, 0)$;

9) a) 852; b) 43; c) y-intercept is $(0, 12)$ which means when initially (after 0 days) 12 people have heard the rumor; d) the y-intercept is $(\frac{-1}{10}, 0)$ and it does not make sense since a negative number of days does not normally apply;

10) Number of M-Mart greater than the number of Z-Mart stores up to and including the start of the year 1999. In subsequent years, the number of M-Mart stores is less than the number of Z-Mart stores;

11) The t-intercept is $(0, 3500)$ which means in 1990 there were 3500 M-Mart stores. The m-intercept is $(140, 0)$ which means it takes 140 years from 1990 (until year 2130) for the number of M-mart stores to decrease to 0;

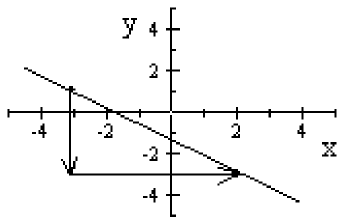
12)
a. there could be 0, 1 or 2 x-intercepts; same for y-intercepts;
b. there could be a total of 0, 1, 2, 3 or 4 intercepts;
c. See the graph to the right;



Click here to return to Section 2.1.

Section 2.2 Answers

1) The slope is $\frac{-4}{5}$ as seen in terms of rise over run:



2) $m = \frac{-6}{5}$; 3) Undefined slope; vertical line;
4) $m = \frac{\sqrt{2}}{2}$; 5) $m = \frac{-4}{3}$;

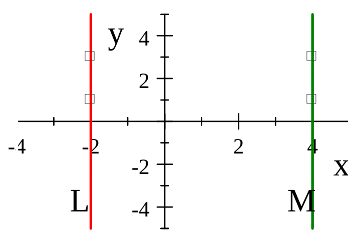
6) Points can vary, for example

$x = -2 + 3y$		
x		y
-2	$\leftarrow$	0
1	$\leftarrow$	1

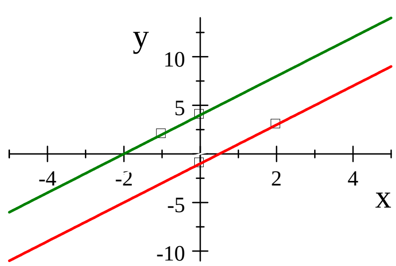
. But slope for this line should work out to be $\frac{1}{3}$;
7) The rearranged equation is $y = \frac{1}{3}x + \frac{2}{3}$. The slope is $\frac{1}{3}$;
8) Isolating y as in Problem 7 is normally easier!;
9) The equation of the line is $y = -2$; it's a horizontal line.;
10) $x = -2$;

11)

line L		line M	
x	y	x	y
-2	1	4	1
-2	3	4	3



12) $y = -4$, a horizontal line;
13) Line L has slope 2; line M also has slope 2; so the lines are parallel as they appear:

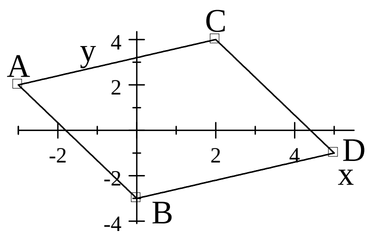


14) b, c and d are false;

15) Sally's average rate of increase is greater at \$350 per year (compared to \$300 per year);

16) 17.5 million;

17)



The slope of $\overline{AB}$ and $\overline{CD}$ have slope $\frac{-5}{3}$. $\overline{BD}$ and $\overline{AC}$ have slope $\frac{2}{5}$. So opposite sides are parallel.

Therefore, $ABDC$ is a parallelogram;

18) $w = 1$;

19a) $(1, 1100)$; $(2, 1210)$; $(3, 1331)$;

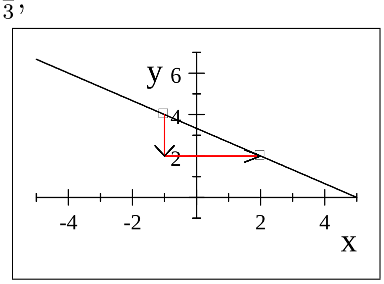
19b) The cost does not rise at a steady slope, e.g. points $(1, 1100)$ and $(2, 1210) \to m = 110\frac{\text{dollars}}{\text{year}}$; but points $(2, 1210)$ and $(3, 1231) \to m = 121\frac{\text{dollars}}{\text{year}}$;

Click here to return to Section 2.2.

Section 2.3 Answers

1) $y = -2x - 7$; 2) $y = \frac{-2}{3}x + \frac{8}{3}$;

3)
a. see graph to the right;
b. $y = \frac{-2}{3}x + \frac{10}{3}$;
c. $x = 3 \to y = \frac{4}{3}$;
d. $y = 6 \to x = -4$;



4) a. $y = 2x - 2$; b. $x = \frac{5}{4} \to y = \frac{1}{2}$; c. $y = -2.3 \to x = -.15$;

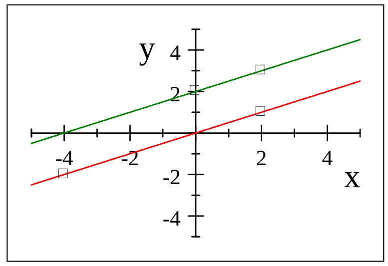
5) $y = \frac{1}{2}x$;

6a) $y = \frac{1}{2}x + 2$;

6b-c)

line N	
x	y
-4	-2
2	1

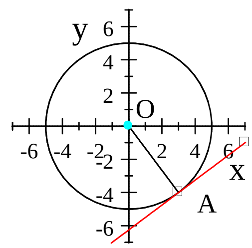
line O	
x	y
2	3
0	2



(parallel lines);

7) $y = 3x - \frac{3}{2}$;

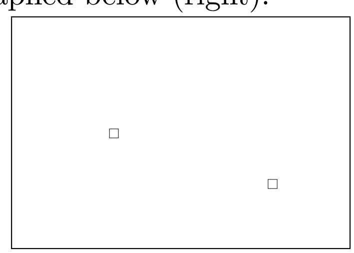
8)
a. For example from $(3, -4)$ the slope leads to the point $(7, -1)$; see graph to the right.
b. Line Q should look tangent to the circle;
c. $\frac{-4}{3}$.
d. It's the negative reciprocal;

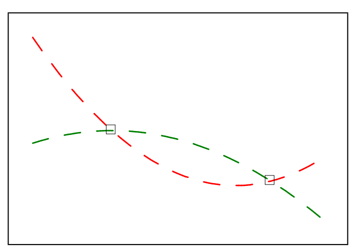


9) Remember F is like y and C is like x in this problem. Based on the information, $(0, 32)$ is the F-intercept and the slope is $\frac{9}{5}$ since F rise 9 units when C runs 5 units. So, the equation is $F - 32 = \frac{9}{5}(C - 0)$ which is the same as $F = \frac{9}{5}C + 32$;

10) $y = \frac{1}{15}x + \frac{5}{3}$; $\frac{13}{3}$ ounces;

11) For the two points below (left), there would be infinitely many different U-shaped graphs that pass through the two points. Two examples are graphed below (right):





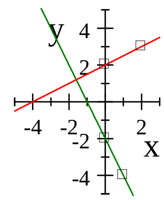
Click here to return to Section 2.3.

Section 2.4 Answers

1) $y = \frac{1}{2}x + 2$; slope is $\frac{1}{2}$; $y-$intercept is $(0, 2)$;
2) $y = -2x - 2$; slope is -2; $y-$intercept is $(0, -2)$;

For graph of $y = \frac{1}{2}x + 2$, start at $(0, 2)$ and go up 1 and right 2;

3) For the graph of $y = -2x - 2$, start at and go down 2 and right 1;
The lines (right) are perpendicular;



4) 1; -1;

5) The line for Fund A has greater slope. So Fund A will have greater value in the long run. The line for Fund B has a higher y-intercept, so Fund B will have greater value in the short run;

6) 2; 7) Identical lines;
8) Parallel lines; 9) Perpendicular lines;

10) The slopes of the adjacent sides are negative reciprocals making the adjacent sides perpendicular:

the slope of $\overline{AB} = \frac{-2}{7}$; the slope of $\overline{BD} = \frac{7}{2}$; the slope of $\overline{DC} = \frac{-2}{7}$; the slope of $\overline{CA} = \frac{7}{2}$;

11a) 297.5 and 747.5; the plumber makes \$297.50 for 3.5 hours and \$747.50 for 9.5 hours;

11b) 75 is the slope of the linear function. It represents the amount the bill goes up per hour (the hourly fee).

The 35 corresponds to the y-intercept $(0, 35)$ and literally means there is a fee of \$35 even if 0 hours of work are done;

12) Each function would correspond to a line with positive slope; the functions have different slopes and y-intercepts; the graph of f(x) would be lowest when $x = 0$ because it has the smallest y-intercept; the graph of $h(x)$ would be highest when $x = 1000$ because it has the highest slope (rises the fastest).

13a) For line L, the x-intercept is $(\frac{1}{2}, 0)$ and the y-intercept is $(0, \frac{-3}{2})$. For line M, the x-intercept is $(3, 0)$ and the y-intercept is $(0, -4)$.

13b) To find the x-intercept cover up the B term and solve for x. To find the the y-intercept, cover up the A term and solve for y.

Click here to return to Section 2.4.

Section 3.1 Answers

1) $4\frac{5}{12}$ hours; arrival time is 4:25 PM;

2) about 46.77 km; 5 whole round trips;

3) The dog is heavier (the boy weighs slightly less);

4) 200 square yard tiles;

5a) Circumference is $\frac{11}{2}\pi$ ft or about 17.3 ft;

5b) Area is $\frac{121}{16}\pi$ ft^2 or about 23.7 ft^2;

5c) Circumference is 11π ft or about 34.5 ft; Area is $\frac{121}{4}\pi$ ft^2 or about 95.0 ft^2;

6) Ouch!: $145.59;

7) 1200 paragraphs;

8) The fastball is faster (cheetah's speed is about 61.4 mph);

9) 105 fl oz;

10) 1 cubic yd = 27 cubic feet, just picture a normal-sized Rubik's cube;

11) a. 900 cm^3; b. 7200 cm^3;

12) a. $\frac{187}{2} = 93.5$ in^2; b. $\frac{187}{288} \approx .65$ ft^2; c. $\frac{187}{288} \cdot 500 = \frac{23\,375}{72} \approx 324.65$ ft^2 (not enough since the floor has 400 ft^2 of area);

13) 43560 ft^2;

14) 20 ml $\approx$ 0.67628 fluid ounce; 20 ml $\approx$ 1.35256 tablespoons;

15) $\frac{x}{200} \approx \frac{1}{2.2} \to 2.2x \approx 200 \to x \approx \frac{200}{2.2} \approx 90.\overline{9}$. Therefore 90.9 kg $\approx$ 200 pounds;

16) $\frac{x}{20} \approx \frac{12}{5\frac{1}{3}} \to \frac{16}{3}x \approx 240 \to x \approx \frac{3}{16} \cdot 240 = 45$.

17) By proportions or elementary reasoning, 8 ft $= \frac{8}{5280}$ mi $= \frac{1}{660}$mi and 1 sec $= \frac{1}{3600}$hr. So, $\frac{8}{1}\frac{\text{ft}}{\text{sec}} = \frac{\frac{1}{660}}{\frac{1}{3600}}\frac{\text{mi}}{\text{hr}} = \frac{3600}{660}\frac{\text{mi}}{\text{hr}} < 6\frac{\text{mi}}{\text{hr}}$;

Click here to return to Section 3.1.

Section 3.2 Answers

1) $y = 3x$ where x=tablespoons and y=teaspoons;

2) $y = \frac{1}{3}x$ where x=teaspoons and y=tablespoons;

3) In Problem 2, the equation is $y = \frac{1}{3}x$ which corresponds to line M. In Problem 1 the equation is $y = 3x$ which corresponds to line L. One way to tell which line belongs to which equation is to use the slopes. The line with greater slope (L) must correspond to the equation $y = 3x$;

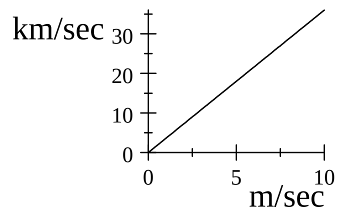
4a) $y = \frac{5}{22}x$;

4b)

x (people)	y (homes)
100000	22727
200000	45455
1000000	227270
88 000	20000
352000	80000

4c) Multiply the y value for $x = 1000000$ by 7, e.g. $7 \cdot 227270 = 1590\,890$;

5) a. $1\ \frac{\text{m}}{\text{sec}} \approx 3.6\ \frac{\text{km}}{\text{hr}}$;
b. The equation of the line is $y = 3.6x$;
c. The graph of the line is to the right;



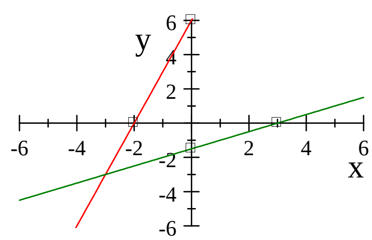
Click here to return to Section 3.2.

Section 4.1 Answers

1) E.g.,

$3x - y = -6$	
x	y
0	6
-2	0

$-x + 2y = -3$	
x	y
0	$\frac{-3}{2}$
3	0



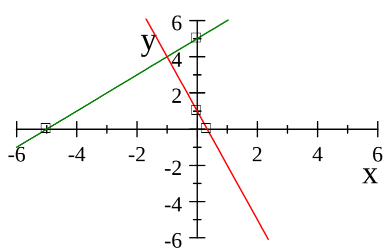
.

The solution appears to be the point $(-3, -3)$. Substituting $(-3, -3)$ into both equations as a double-check, $3(-3) - (-3) \stackrel{?}{=} -6$ and $-(-3) + 2(-3) \stackrel{?}{=} -3$ are both true. So, $(-3, -3)$ is the solution;

2) E.g.,

$3x + y = 1$	
x	y
0	1
$\frac{1}{3}$	0

$-2x + 2y = 10$	
x	y
0	5
-5	0

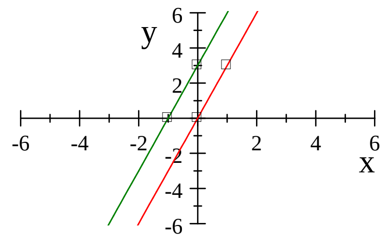


The solution appears to be the point $(-1, 4)$. Substituting $(-1, 4)$ into both equations as a double-check, $3(-1) + 4 \stackrel{?}{=} 1$ and $-2(-1) + 2(4) \stackrel{?}{=} 10$ are both true. So, $(-1, 4)$ is the solution;

3) E.g.,

$3x - y = 0$	
x	y
0	0
1	3

$-6x + 2y = 6$	
x	y
0	3
-1	0



.

The lines are probably parallel and so the system probably has no solution. As a double-check we can use slope. Getting y by itself, $3x - y = 0 \to y = 3x$; and $-6x + 2y = 6 \to y = 3x + 3$. Hence, both lines have slope 3 but different y-intercepts and so must be parallel. Therefore the system definitely has no solution!

4) E.g. ,

$y\ = \frac{1}{2}x - 3$	
x	y
0	-3
6	0

$-x + 2y = -6$	
x	y
0	-3
6	0

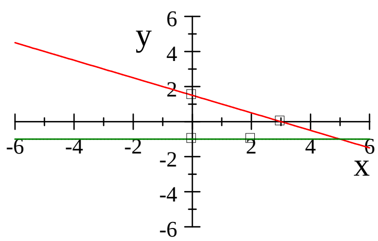
.

Since both lines have the same x- and y- intercepts they must be equal lines. Therefore, there would be infinitely many solutions the system;

5) E.g.,

$x + 2y = 3$	
x	y
3	0
0	$\frac{3}{2}$

$y + 1 = 0$	
x	y
0	-1
2	-1

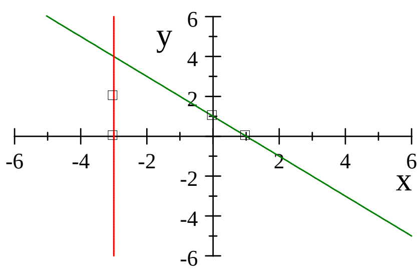


The solution appears to be the point $(5, -1)$. Substituting $(5, -1)$ into both equations as a double-check, $5 + 2(-1) \stackrel{?}{=} 3$ and $-1 + 1 \stackrel{?}{=} 0$ are both true. So, $(5, -1)$ is the solution;

6) E.g.,

$x = -3$	
x	y
-3	0
-3	2

$x + y = 1$	
x	y
0	1
1	0



The solution appears to be the point $(-3, 4)$, which is correct since it satisfies both equations;

7) Line L has slope $\frac{-3}{2}$ and y-intercept $(0, \frac{1}{2})$;

a. $3y = -2x + 1 \leftrightarrow y = \frac{-2}{3}x + \frac{1}{3}$ has slope $\frac{-2}{3} \neq \frac{-3}{2}$ so must intersect L once;

b. $6x = -4y + 2 \leftrightarrow y = \frac{-3}{2}x + \frac{1}{2}$ has slope $\frac{-3}{2}$ and y-intercept $(0, \frac{1}{2})$ is identical to line L;

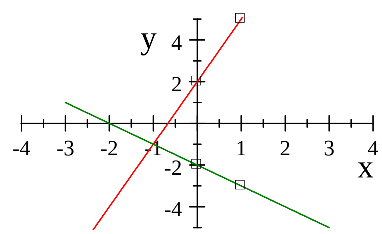
c. $-6x - 2 = 4y \leftrightarrow y = \frac{-3}{2}x - \frac{1}{2}$ has slope $\frac{-3}{2}$ but y-intercept $(0, \frac{-1}{2})$ so must be parallel to L;

8) $c = \frac{7}{2}$;

9) E.g.,

$f(x) = 3x + 2$	
x	y
0	2
1	5

$g(x) = -x - 2$	
x	y
0	-2
1	-3

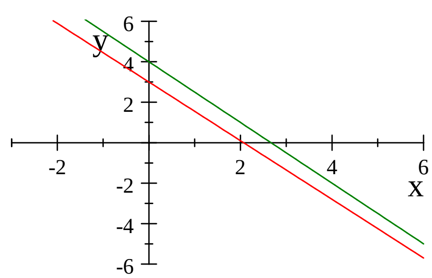


The point $(-1, -1)$ appears to be the intersection point. This can be verified as correct with function evaluation:

$f(-1) = 3 \cdot (-1) + 2 = -3 + 2 = -1$; and

$g(-1) = -(-1) - 2 = 1 - 2 = -1$;

10) The lines may or may not look parallel to you:



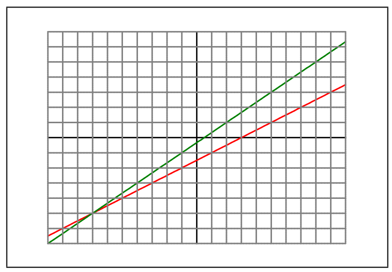
To tell for sure, we can use slope. Line L with y isolated has equation $y = -1.5x + 4$. Line M with y isolated has equation $y = -1.45x + 3$. Since the lines have unequal slope (remember $y = mx + b$), they are not really parallel.

Click here to return to Section 4.1.

Section 4.2 Answers

1) the point $(2, -1)$;
2) the point $(\frac{-8}{5}, \frac{-11}{5})$;
3) the point $(-7, -5)$;
4) the point $(4, 2)$;
5) no solution;
6) infinitely many solutions;
7) the point $(\frac{13}{12}, \frac{-17}{12})$;
8) the point $(-\frac{26}{5}, \frac{-19}{5})$;
9) no solution;
10) the point $(\frac{-8}{5}, \frac{-12}{5})$;
11) infinitely many solutions;
12) infinitely many solutions;

13) Using the graphcalc.com calculator with the grid option on, here's the graph:



14) The numbers are -10.5 and 82.5 (in that order);

15) Jill's weight is 136 pounds. John's is 214 pounds;

16) The square is $\frac{215}{11}$feet long;

17) The larger angle measures 57 degrees;

18) $2l = 4 + 3w$ and $2l + 2w = 100$ yield ($w = \frac{96}{5}$ m) and $l = \frac{154}{5}$ m;

19) With x= smallest angle measure, $2x$=largest angle measure, y =middle angle measure, then $y + 20 = 2x$ and $x + 2x + y = 180$ which yield small angle $40°$, big angle equal $80°$ and middle angle equal $60°$;

Click here to return to Section 4.2.

Section 4.3 Answers

1) With x= principal invested in the bond and y= principal invested in stocks, $x + y = 20000$ and $.07x + .12y = 2000$ which yield ($x = 8000$ dollars) and $y = 12000$ dollars.

2) With x= principal invested in the bond and y= principal invested in stock, then $x + y = 20000$ while $0.07x + (-0.02)y = 1000$ which yield ($x = \$15555.56$) and $y = \$4444.44$.

3) With $x=$ principal invested in the bond and $y=$ principal invested in stock, $x + y = 20000$ and $.07x + (-.02)y = 0$ which yield ($x \approx \$4444.44$) and $y = \$15555.56$.

4) With $x=$ number of pounds of premium coffee and $y=$ number of pounds of generic coffee, the filled in table would be:

coffee type	pounds	price per pound	value
premium	x	6	$6x$
generic	y	1.5	$1.5y$

So, $x + y = 100$ while $6x + 1.5y = 501$ which yield $x = 78$ and $y = 22$.

5) With $x=$ number of nickels and $y=$ number of dimes, the filled in table would be:

coin type	quantity	value per coin	value
nickels	x	.05	$.05x$
dimes	y	.10	$.10y$

So, $x+y = 300$ while $.05x+.10y = 27.30$ which yield $x = 54$ and $y = 246$.

6) With $x=$ driving time on day one and $y=$ driving time on day two, then $x + y = 15$ while $55x + 65y = 850$ which yield $x = 12.5$ hr and $y = 2.5$ hr.

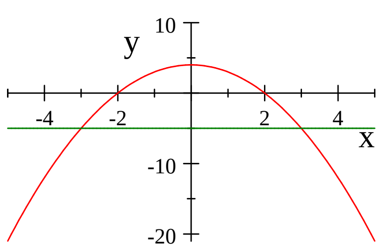
7) With $x=$ driving time at the fast speed and $y=$ driving time at the slower speed, then $x + y = 4$ while $70x + 65y = 265$ which yield $x = 1$ hr and $y = 3$ hr.

8) With $x=$ rate of slower hiker and $y=$ rater of faster hiker, then $y = 1.2x$ while $4.25x + 4.25y = 20$ which yield $x = 2.139$ mph and $y = 2.5668$ mph.

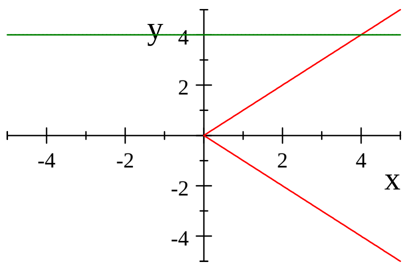
Click here to return to Section 4.3.

Section 4.4 Answers

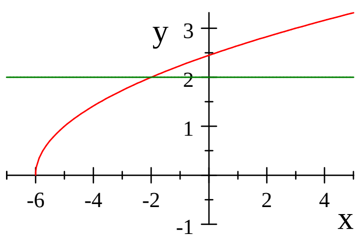
1) There are two solutions: the points $(\pm 3, -5)$. A graph of the system looks like:



2) There is only one: the point $(4,4)$. A graph of the system looks like:



3) There is only one solution: the point $(-2,2)$. A graph of the system looks like:



4) There are two solutions: the points $(10,40)$ and $(10,-40)$;

5) The only solution is the point $(-2,-10)$;

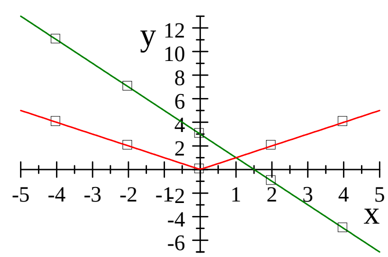
6) There are two solutions: the points $(-2,-2)$ and $(2,2)$;

7) There are two solutions: the points $(2\sqrt{3},\sqrt{3})$ and $(-2\sqrt{3},-\sqrt{3})$;

8) There are two solutions: the points $(5,4)$ and $(5,-4)$;

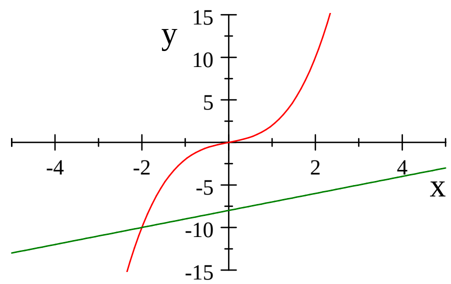
9)

x	$f(x)=\|x\|$	$g(x)=-2x+3$
-4	4	11
-2	2	7
0	0	3
2	2	-1
4	4	-5



The solution appears to be the point $(1,1)$. This can be verified with function evaluation: $f(1)=|1|=1$ while $g(x)=-2\cdot(1)+3=-2+3=1$;

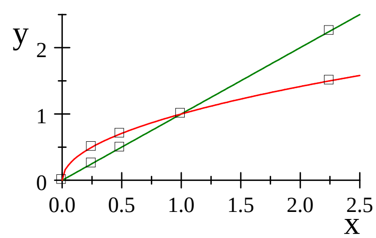
10) A graphing calculator should produce a graph something like what's below. Notice that there does appear to be only one solution at the point $(-2,-10)$;



11) The system has two solutions: $(0,0)$ and $(1,1)$. Here's a table of points and graph:

x	$y=\sqrt{x}$
0	0
.25	.5
.49	.7
1	1
2.25	1.5
4	2

x	$y=x$
0	0
.25	.25
.49	.49
1	1
2.25	2.25
4	4



Graph of $\left\{ \begin{smallmatrix} y=\sqrt{x} \\ y=x \end{smallmatrix} \right\}$

12)

x	1	2	3	4	5	6	7
$f(x)$	1	8	27	64	125	216	343
$g(x)$	3	9	27	81	243	729	2187

As x gets larger, the value for g becomes much larger than the value for f;

13)

x	1	4	16	64	256	1024	4096
$h(x)$	0	2	4	6	8	10	12
$i(x)$	1	2	4	8	16	32	64

As x gets larger the value for i becomes much larger than the value for h;

Click here to return to Section 4.4.

Section 5.1 Answers

1) On the graphcalc.com calculator press the $\sqrt{x}$ then type 19 then press ENTER. The answer is 4.35889894354068. Rounding to the nearest tenth $\sqrt{19} \approx 4.4$. $\sqrt{19} \neq 4.4$ since $(4.4)^2 = 19.36$;

2) To compute $\sqrt[3]{19}$ on the graphcalc.com calculator, type 19 then **right click** on the $\sqrt{x}$ button and select the $x\hat{}(1/3)$ option. (Soon we will talk about why $x\hat{}(1/3)$ represents cube root of x). So type 19ˆ(1/3) ENTER gives 2.66840164872194 which rounds to 2.7. But 2.7ˆ3 ENTER gives 19.683;

3a) $\sqrt{20} < 20$ since $\sqrt{20}$ is between 4 and 5 in value ($4^2 = 16$ and $5^2 = 25$);

3b) $\sqrt[3]{20} < \sqrt{20}$ since $\sqrt[3]{20}$ is between 2 and 3 ($2^3 = 8$ and $3^3 = 27$) in value and $\sqrt{20}$ is between 4 and 5 in value;

3c) $\sqrt{.4} > .4$ since $\sqrt{.4}$ is between .6 and .7 in value ($.6^2 = .36$ and $.7^2 = .49$);

3d) $\sqrt[3]{.4} > \sqrt{.4}$ since $\sqrt[3]{.4}$ is between .7 and .8 in value ($.7^3 = .343$ and $.8^3 = .512$) and $\sqrt{.4}$ is between .6 and .7 in value;

4) if $\sqrt{-16} = x$ then $-16 = x^2$. The problem is that with real numbers x^2 is non-negative so $-16 = x^2$ is impossible. So, $\sqrt{-16}$ is impossible using real numbers;

5) 96;

6) On the graphcalc.com calculator, typing $(4 * \sqrt{}6)\hat{}2$ ENTER does produce the correct value 96, but it's kind of lucky in that the calculator does not know how to do this exactly. But doing the calculation in one step greatly increases the chances the calculator will get the correct answer.

7) $\frac{1}{12}$;

8) On the graphcalc.com calculator, it's important to enter negative numbers within parentheses. So -1 will be (-1). So now typing $((-1)/6 * \sqrt{}3)\hat{}2$ ENTER produces .08333333333333. This value is almost correct. The exact value is $\frac{1}{12}$ which is .0833.... (the 3's never end);

9) 784 soldiers used; 2% of the soldiers will be unused;

10) $\sqrt{1250}$ m; 35.4 m;

11) The hypotenuse is exactly $\sqrt{20}$cm long, or approximately 4. 5 cm;

12) $\sqrt{144}$ cm or 12 cm;

13) 8. 9 cm;

14) a. -10; b. 5; c. $\frac{6}{11}$; d. 1; e. -1; f. does not exists (any real number to the tenth is non-negative; hence not -1);

15a) -4;

15b) does not exist (as a real number); reason: any number to the fourth power is non-negative (so can't equal -16).

15c-f) 3; 5; $\frac{7}{6}$; 289;

Click here to return to Section 5.1.

Section 5.2 Answers

1) $3\cdot\sqrt{6}$; $3\cdot\sqrt[3]{2}$; 2) $5\cdot\sqrt{10}$; $5\cdot\sqrt[3]{2}$;
3) $10\cdot\sqrt{10x}$; $10\cdot\sqrt[3]{x}$; 4) $6\cdot\sqrt{3x}$; $3\cdot\sqrt[3]{4x}$;
5) $\frac{5\cdot\sqrt{x}}{8}$; $\frac{\sqrt[3]{25x}}{4}$; 6) $\frac{5\cdot\sqrt{5}\cdot|x|}{8}$; $\frac{5\cdot\sqrt[3]{x^2}}{4}$;
7) $\frac{\sqrt{2x}}{5}$; 8) $5\cdot\sqrt{2}$;
9) 120; 10) 225;
11) $6-2x$; 12) $6+2x+4\cdot\sqrt{3x}$;
13) $5\cdot\sqrt{7}$; 14) $-4\cdot\sqrt[3]{x}$;
15) $\frac{-7\cdot\sqrt{2}}{3}$; 16) $\frac{3\cdot\sqrt{5}}{44}$;
17) There are many possible counterexamples. For example$\sqrt{9+16}=\sqrt{25}=5$, but $\sqrt{9}+\sqrt{16}=3+4=7$;
18) a. $\frac{\sqrt{30}}{25}$; b. $\frac{6}{5\cdot\sqrt{30}}$; 19) a. $\frac{2\cdot\sqrt{2x}}{x}$; b. $\frac{4}{\sqrt{2x}}$;
20) a. $\frac{\sqrt[3]{300}}{25}$; b. $\frac{6}{5\cdot\sqrt[3]{90}}$; 21) a. $\frac{\sqrt[3]{2x^2}}{6}$; b. $\frac{x}{3\cdot\sqrt[3]{4x}}$;
22) 11 square feet; 23) $\frac{10\cdot\sqrt{5\pi}}{\pi}$ feet ≈ 12.6 feet

Click here to return to Section 5.2.

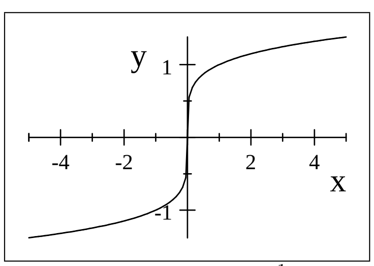
Section 5.3 Answers

1) negative x values OK in all parts except (b);
2) a. x^4; b. $x^4\cdot\sqrt{x}$; c. $|x^5|$;
3) a. $x^2\cdot\sqrt[3]{x^2}$; b. x^3; c. $x^3\cdot\sqrt[3]{x}$;
4) a. $3x^2\cdot\sqrt{3}$; b. $3x\cdot\sqrt[3]{x}$;
5a) $2x^2\cdot\sqrt{30x}$ (absolute value not needed for the x^2 which is automatically non-negative);
5b) $2x\cdot\sqrt[3]{15x^2}$;
6) a. $\frac{3x^4\cdot\sqrt{3x}}{8}$; b. $\frac{3x^3}{4}$;
7a) $\frac{|x^5|}{4}$ (absolute value needed because x and therefore x^5 could be non-negative);
7b) $\frac{x^3\cdot\sqrt[3]{x}}{2}$;
8) a. $2\times10^{-1}=.2$; b. $2\times10^2=200$;
9) a. 110; b. .0011; 10) $\pm5\cdot\sqrt{5}$;
11) $\pm\frac{8}{3}$; 12) -2;
13) $10\cdot\sqrt[3]{15}$; 14) 10;
15) $\pm20\cdot\sqrt{2}$; 16) $\pm\frac{12}{7}$;

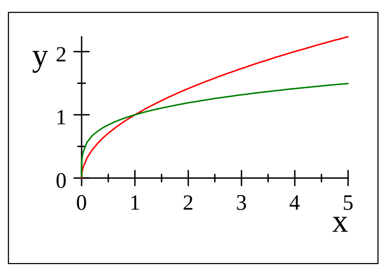
17) a. none; b. 1; c. 2; d. 1; e. 1; f. 1;
Click here to return to Section 5.3.

Section 5.4 Answers

1) $\sqrt[5]{x}$; x could be negative; 2) $\sqrt[6]{x}$; x can't be negative;
3) $\sqrt[4]{x^3}$; x can't be negative; 4) $\sqrt[5]{x^3}$ or $\left(\sqrt[5]{x}\right)^3$; x can be negative;
5) a. $\sqrt{100} = 10$; b. $\left(\sqrt{100}\right)^3 = 10^3 = 1000$;
5c) $\left(\sqrt{100}\right)^{-5} = 10^{-5} = \frac{1}{100000}$;
6) a. 3; b. 9; c. $\frac{1}{9}$;
7) a. .2; b. .008; c. 3125;
8) a. $\frac{-1}{2}$; b. $\frac{1}{4}$; c. -8;
9) x^2; x does not need to be positive;
10) x^6; going back to the original $x^{\frac{3}{2}}$, since $x^{\frac{3}{2}}$ is $\sqrt[2]{x^3}$, x can not be negative;
11) He did not use parentheses around the power, e.g. since power is higher priority than division, without parentheses 4ˆ3/2 is $\frac{4^3}{2}$ which is 32;
12) The graph of $y = x^{\frac{1}{5}}$ (below left) should have somewhat of an S-shape.
13) The graph of $y = \sqrt{x}$ starts off lower but then ultimately grows higher than $y = \sqrt[4]{x}$ (see graph below on right). Taking a fourth root x will tend to produce a smaller number y than square root because there will be four factors of y to make x rather than just two.



Graph of $y = x^{\frac{1}{5}}$



$y = \sqrt{x}$ vs. $y = \sqrt[4]{x}$

Click here to return to Section 5.4.

Appendix B

More on Simplifying Radicals

The Two Term Situation

Now let's consider the case in which a fraction will be rewritten and two terms are involved. For example, how could we rewrite $\frac{2}{\sqrt{3}-\sqrt{5}}$ so that any square root appear in the numerator only? The method is to multiply numerator and denominator by the "conjugate" of the terms to be eliminated. The conjugate of a two term expression (with square root only) is the same expression with the addition/substraction reversed. So the conjugate of $\sqrt{3}-\sqrt{5}$ is $\sqrt{3}+\sqrt{5}$. Similarly the conjugate of $x+\sqrt{5}$ is $x-\sqrt{5}$.

An expression and it's conjugate have the property that when they are multiplied together, the radicals end up canceling out. Watch out for this in the next two examples.

Example B.1 *Rewrite* $\frac{2}{\sqrt{3}-\sqrt{5}}$ *so there is square root in the numerator.*

Solution: $\frac{2}{\sqrt{3}-\sqrt{5}} = \frac{2\cdot(\sqrt{3}+\sqrt{5})}{(\sqrt{3}-\sqrt{5})(\sqrt{3}+\sqrt{5})}$, (multiply numerator and denominator by conjugate)

$= \frac{2\cdot(\sqrt{3}+\sqrt{5})}{\sqrt{9}+\sqrt{15}-\sqrt{15}-\sqrt{25}}$, (distribute denominator)

$= \frac{2\cdot(\sqrt{3}+\sqrt{5})}{3-5}$, (simplify denominator)

$= \frac{2\cdot(\sqrt{3}+\sqrt{5})}{-2}$, (simplify denominator)

$= -(\sqrt{3}+\sqrt{5})$, (reduce common factor of 2, move minus to numerator).

Therefore, $\frac{\sqrt{5}-\sqrt{6}}{\sqrt{3}-\sqrt{5}} = -(\sqrt{5}-\sqrt{6})$. ∎

Example B.2 *Rewrite* $\frac{2}{x+\sqrt{5}}$ *so that the square root is in the numerator.*

Solution: $\frac{2}{x+\sqrt{5}} = \frac{2\cdot(x-\sqrt{5})}{(x+\sqrt{5})\cdot(x-\sqrt{5})}$, (multiply numerator and denominator by conjugate)
$= \frac{2(x-\sqrt{5})}{x^2-\sqrt{5}x+\sqrt{5}x-\sqrt{25}}$, (distribute denominator)
$= \frac{2(x-\sqrt{5})}{x^2-5}$, (simplify denominator).
Therefore, $\frac{2}{x+\sqrt{5}} = \frac{2(x-\sqrt{5})}{x^2-5}$. ∎

Try This B.1 *Verify that the conjugate of* $\sqrt[3]{a} - \sqrt[3]{b}$ *is* $\sqrt[3]{a^2} + \sqrt[3]{ab} + \sqrt[3]{b^2}$ *by multiplying the expressions together and verifying all radicals cancel out.*

Solution:
$\left(\sqrt[3]{a} - \sqrt[3]{b}\right)\left(\sqrt[3]{a^2} + \sqrt[3]{ab} + \sqrt[3]{b^2}\right)$
$= \sqrt[3]{a}\cdot\sqrt[3]{a^2} + \sqrt[3]{a}\cdot\sqrt[3]{ab} + \sqrt[3]{a}\cdot\sqrt[3]{b^2} - \sqrt[3]{b}\cdot\sqrt[3]{a^2} - \sqrt[3]{b}\cdot\sqrt[3]{ab} - \sqrt[3]{b}\cdot\sqrt[3]{b^2}$
$= \sqrt[3]{a^3} + \sqrt[3]{a^2b} + \sqrt[3]{ab^2} - \sqrt[3]{a^2b} - \sqrt[3]{ab^2} - \sqrt[3]{b^3}$
$= \sqrt[3]{a^3} - \sqrt[3]{b^3}$
$= a - b.$

Homework Problems

Rationalize the denominator and simplify.

1. $\frac{4}{2+\sqrt{5}}$ (answer: $-4(2 - \sqrt{5})$)
2. $\frac{2}{3-\sqrt{2}}$ (answer: $\frac{2(3+\sqrt{2})}{7}$)
3. $\frac{\sqrt{3}}{\sqrt{15}-\ 2}$ (answer: $\frac{\sqrt{3}(\sqrt{15}+\ 2)}{11}$)
4. $\frac{5}{\sqrt{2}-\sqrt{6}}$ (answer: $\frac{-5(\sqrt{2}+\sqrt{6})}{4}$)
5. $\frac{\sqrt{3}+1}{3-\sqrt{3}}$ (answer: $\frac{4(\sqrt{3}+1)}{3}$)
6. $\frac{\sqrt{5}-1}{\sqrt{5}+1}$ (answer: $\frac{-\sqrt{5}+3)}{2}$)

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