NAME

Intuitive Calculus 11012  Examination 2  A
March 11, 2010  Richard M. Aron

Directions: Please answer questions A, B, C, and D in the space provided. Please write extremely neatly. The rest of the Examination questions are to be answered on the “scan-tron” papers, but you must show your work on this paper—even for the “scan-tron” part. Good luck!

Product Rule: if \( p(x) = f(x) \cdot g(x) \), then \( p'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x) \).
Quotient Rule: if \( q(x) = \frac{f(x)}{g(x)} \), then \( q'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2} \).

A. Let \( f(x) = x^4 - 4x^3 + 4x^2 \) on the interval \([0, 3]\).
(a). Find all critical numbers of \( f(x) \).
\[ 4x^3 - 12x^2 + 8x = 4x(x^2 - 3x + 2) = 0 \iff x = 0, 1, 2. \]

(b). Classify the critical numbers found in part (a). (That is, explain whether a particular critical number corresponds to a relative maximum or a relative minimum of \( f(x) \).)
\[ f''(x) = 12x^2 - 24x + 8. \]
\[ f''(0) = 8 > 0 \quad \text{min} \]
\[ f''(1) = -4 < 0 \quad \text{max} \]
\[ f''(2) = 8 > 0 \quad \text{min} \]

(c). Find the absolute extreme values of \( f(x) \) on the closed interval \([0, 3]\).
\[ f(0) = 0 \]
\[ f(1) = 4 \]
\[ f(2) = 0 \]
\[ f(3) = 81 - 108 + 36 = 9 \]

\[ \text{Total} \]
B. A car dealer can sell four cars per day at a price of $12,000. She estimates that for each $200 price reduction, she can sell two more cars per day. What price should she charge to maximize her revenue $R(x)$? How many cars will she sell each day? [Hint: Let $x$ = the number of $200 price reductions.]

Step 1. Let $p(x)$ be the price she charges for a car, after $x$ price reductions.

$$p(x) = 12000 - 200x$$

Step 2. Let $q(x)$ be the quantity of cars she sells in a day, after $x$ price reductions.

$$q(x) = 4 + 2x$$

Step 3. Calculate $R(x)$, the number of cars that are sold.

$$R(x) = (12000 - 200x)(4 + 2x)$$
$$= 48000 + 23200x - 400x^2$$

Step 4. Maximize $R(x)$, explaining why your answer gives a maximum.

$$R'(x) = 23200 - 800x$$
$$R'' = -800 < 0, \quad \text{max.}$$

$$x = \frac{-23200}{8} = 29$$

Step 5. What price should she charge to maximize her revenue? How many cars will she sell each day?

$$p(x) = 12000 - 200(29) = 6200$$

$$q(x) = 4 + 29 + 2 = 35$$
C. On the axes below, draw an extremely clear, careful sketch of the graph of a function $f$ which has the following properties: $f$ is continuous and differentiable, $f(0) = 0$, $f'(x) > 0$ for $x \in (0, 2)$, $f$ is concave down on $(-2, 0)$ and $f''(x) > 0$ on $(0, 2)$. Indicate any relative maxima or minima, and any inflection points.

\[ f''(x) > 0 \quad \text{on} \quad (0, 2) \]

D. (a). Evaluate $f'$ and $f''$ for $f(x) = 12\sqrt{x^3} - 9\sqrt{x}$.

\[ f'(x) = : \]
\[ \frac{1}{2} \cdot 12 \cdot x^{3/2} - \frac{9}{2} \cdot x^{1/2} \]
\[ \frac{1}{2} \cdot 18 \cdot x^{1/2} - \frac{9}{2} \cdot x^{-1/2} \]

(b). $f''(x) = :$
\[ \frac{1}{2} \cdot 18 \cdot x^{-1/2} + \frac{9}{2} \cdot x^{1/2} \]
The rest of this Examination is to be done using the “scantron” sheet. Please write your work on this paper.

1. Let \( f(x) = (2x + 1)^{10} \). Then \( f'(x) =: \)
(a). \( 10(2x + 1)^9 \).
(b). \( 20(2x + 1)^9 \).
(c). \( 10(2x)^9 \).
(d). None of the above answers.

2. Let \( y = f(x) \) be a differentiable function on \([-2, 2]\), whose graph is given below.

Which of the following is the sign diagram for \( f'' \)?
(a) \[
\begin{array}{cccc}
< & 0 & < & 0
\end{array}
\]
(b) \[
\begin{array}{cccc}
< & 0 & > & 0
\end{array}
\]
(c) \[
\begin{array}{cccc}
> & 0 & < & 0
\end{array}
\]
(d) \[
\begin{array}{cccc}
> & 0 & < & 0
\end{array}
\]
3. Let \( y = f(x) = 2x^3 - 3x^2 - 12x \), where \( x \) is in the closed interval \([-2, 4]\). Then the smallest and largest values that the function attains in this interval are:
(a). 0 and \( \infty \).
(b). -20 and 7.
(c). -\( \infty \) and \( \infty \).
(d). -20 and 32.

4. The point \( x = 0 \) is where the function \( y = f(x) = x^4 + 6 \) has:
(a). a relative maximum.
(b). a relative minimum.
(c). neither a relative maximum nor a relative minimum.
(d). Answer cannot be determined from the given information.

5. Let \( y = f(x) \) be a function such that \( f'(x) \) has the following sign diagram.

\[
\begin{array}{cccccc}
& -2 & -1 & 0 & 1 & 2 \\
\hline
\text{sign of } f'(x) & >0 & <0 & =0 & >0 & >0 \\
\end{array}
\]

Which of the following is true? \( f \) has:
(a). a relative maximum at \(-1\).
(b). a relative maximum at \(-1\) and a relative minimum at 0.
(c). relative maxima at \(-1\) and at 1.
(d). a relative minimum at \(-1\) and a relative maximum at 1.
6. The derivative of \( y = f(x) = (x^{10} + 3x^5 - 2x^3 - 4)^6 \) is:
(a) \( 10(x^9 + 15x^4 - 6x^5) \).
(b) \( 6(10x^9 + 15x^2 - 6x)^5 \).
(c) \( 6(x^{10} + 3x^5 - 6x)^5(10x^9 + 15x^4 - 6x) \).
(d) \( 10x^9 + 15x^2 - 6x^5 \).

7. The critical numbers for the function \( f(x) = x^3 - 9x \) are:
(a) 0, 3, and −3.
(b) \( \sqrt{3}, -\sqrt{3} \).
(c) 0.
(d) 3 and −3.

8. Which are the points of inflection in the following graph?

![Graph with points of inflection](image)

(a). Only −1 and 1.
(b). Only 0 and 1.
(c). Only 0.
(d). There are no inflection points in this graph.

9. Let \( f \) have critical points at −1, 0, and 2. If \( f''(x) = x^4 - 16x^2 + 9 \),
then \( f \) has a relative maximum at:
(a). 0 and 2.
(b). −1 and 2.
(c). 0.
(d). None of the above answers.
10. As \( x \to -\infty \), the function \( f(x) = \frac{3x^2 + 15x - 10}{4x^2 - 7} \) tends to
(a). 10/7.
(c). 3.
(d). -10.

11. Let
\[
y = f(x) = \frac{(x^2 + x)^3}{x + 2}.
\]
Then \( f'(x) = \)
(a).
\[
\frac{(x + 2)[3(x^2 + x)^2(2x + 1)] - (x^2 + x)^3}{(x + 2)^2}.
\]
(b).
\[
3(x^2 + x)^2(2x + 1).
\]
(c).
\[
\frac{(x^2 + x)^3 - (x + 2)[3(x^2 + x)^2(2x + 1)]}{(x + 2)^2}.
\]
(d). None of the above answers.

Use the following information for the next two problems:
A supermarket expects to sell 10,000 boxes of a certain pet food in a
year. The supermarket owner must pay $3.00 for each box. There is a
$10.00 delivery charge for each delivery from the manufacturer, and
there is a $2.00 annual storage charge per box of pet food. Let \( x \) be
the number of boxes of pet food that the supermarket owner
purchases from the manufacturer each time.

12. What are the annual storage costs for the dealer, in dollars?
(a). 3x.
(b). 3x + 10.
(c). 2x.
(d). \( x \).

13. What does it cost, in dollars, the supermarket owner each time
she places an order for this pet food?
(a). 3x.
(b). 3x + 10.
(c). 5x.
(d). 5x + 10.
Directions: Please answer questions A, B, C, and D in the space provided. Please write extremely neatly. The rest of the Examination questions are to be answered on the “scan-tron” papers, but you must show your work on this paper—even for the “scan-tron” part. Good luck!

Product Rule: if \( p(x) = f(x) \cdot g(x) \), then \( p'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x) \).

Quotient Rule: if \( q(x) = \frac{f(x)}{g(x)} \), then \( q'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2} \).

A. Let \( f(x) = x^4 + 4x^3 + 4x^2 \) on the interval \([-2, 1]\).

(a). Find all critical numbers of \( f(x) \).

\[
\frac{d}{dx} \left( x^4 + 4x^3 + 4x^2 \right) = 4x^3 + 12x^2 + 8x
\]

\[
0, -1, -2
\]

(b). Classify the critical numbers found in part (a). (That is, explain whether a particular critical number corresponds to a relative maximum or a relative minimum of \( f(x) \).)

\[
\frac{d^2}{dx^2} \left( x^4 + 4x^3 + 4x^2 \right) = 12x^2 + 24x + 8
\]

\[
\begin{align*}
q''(0) &= 8, \text{ min} \\
q''(-1) &= -4, \text{ max} \\
q''(-2) &= 8, \text{ min}
\end{align*}
\]

(c). Find the absolute extreme values of \( f(x) \) on the closed interval \([-2, 1]\).

\[
\begin{align*}
q(-2) &= 0 \\
q(-1) &= -1 \\
q(0) &= 0 \\
q(1) &= 9
\end{align*}
\]

\[
0 \& 9
\]
B. A car dealer can sell four cars per day at a price of $24,000. She estimates that for each $400 price reduction, she can sell one more car per day. What price should she charge to maximize her revenue $R(x)$? How many cars will she sell each day? [Hint: Let $x$ = the number of $400 price reductions.]

Step 1. Let $p(x)$ be the price she charges for a car, after $x$ price reductions.

$$p(x) = 24000 - 400x$$

Step 2. Let $q(x)$ be the quantity of cars she sells in a day, after $x$ price reductions.

$$q(x) = 4 + x$$

Step 3. Calculate $R(x)$.

$$R(x) = p(x)q(x) = (24000 - 400x)(4 + x)$$

$$= 96000 + 24000x - 400x^2$$

Step 4. Maximize $R(x)$, explaining why your answer gives a maximum.

$$R'(x) = 24000 - 800x$$

$$R'' = -800 < 0$$

$$\therefore\text{max}$$

Step 5. What price should she charge to maximize her revenue? How many cars will she sell each day?

$$x = \frac{24400}{800} = 30$$

$$p(30) = 41200$$

$$q(30) = 32$$
C. On the axes below, draw an extremely clear, careful sketch of the graph of a function \( f \) which has the following properties: \( f \) is continuous and differentiable, \( f(0) = 0, f'(x) < 0 \) for \( x \in (0, 2) \), \( f''(x) > 0 \) on \((-2, 0)\) and \( f \) is concave down on \((0, 2)\). Indicate any relative maxima or minima, and any inflection points.

\[ \text{\hspace{2cm}} \]

D.(a). Evaluate \( f' \) and \( f'' \) for \( f(x) = 4\sqrt{x^3} - 18\sqrt{x^2} \).

(a). \( f'(x) =: \]
\[ 6 x^{1/2} - 12 x^{-1/3} \]

(b). \( f''(x) =: \]
\[ 3 x^{-1/2} + 4 x^{-4/3} \]
The rest of this Examination is to be done using the “scantron” sheet. Please write your work on this paper.

1. Let \( f(x) = (3x - 5)^8 \). Then \( f'(x) = \):
   (a) \( 8(3x - 5)^7 \).
   (b) \( 24(2x)^7 \).
   (c) \( 24(3x - 5)^7 \).
   (d) None of the above answers.

2. Let \( y = f(x) \) be a differentiable function on \([-2, 4]\), whose graph is given below.

Which of the following is the sign diagram for \( f'' \)?

(a)

(b)

(c)

(d)
3. Let \( y = f(x) = x^3 + 3x^2 - 9x \), where \( x \) is in the closed interval \([-3, 4]\). Then the smallest and largest values that the function attains in this interval are:

(a). -5 and 76.
(b). 0 and \( \infty \).
(c). -5 and 27.
(d). -20 and 32.

4. The point \( x = 0 \) is where the function \( y = f(x) = x^4 - 8 \) has:

(a). a relative minimum.

(b). a relative maximum.

(c). neither a relative maximum nor a relative minimum.

(d). Answer cannot be determined from the given information.

5. Let \( y = f(x) \) be a function such that \( f'(x) \) has the following sign diagram.

\[
\begin{array}{cccccc}
& < & 0 & > & 0 & < \\
-1 & 0 & 1 & & & \\
\end{array}
\]

Which of the following is true? \( f \) has:

(a). a relative maximum at \(-1\).

(b). a relative maximum at \(-1\) and a relative minimum at \(0\).

(c). relative maxima at \(-1\) and \(1\).

(d). a relative minimum at \(-1\) and a relative maximum at \(1\).
6. The derivative of \( y = f(x) = (x^{12} + 4x^4 - 3x^2 - 5)^4 \) is:
(a) \( 12(x^{11} + 16x^3 - 9x)^3 \).
(b) \( 4(x^{12} + 4x^4 - 3x^2 - 5)^3(12x^{11} + 16x^3 - 6x) \).
(c) \( 4(x^{12} + 4x^4 - 3x^2 - 5)^3 \).
(d) \( 12x^{11} + 16x^3 - 6x \).

7. The critical numbers for the function \( f(x) = x^3 - 25x \) are:
(a) \( \frac{5\sqrt{3}}{3} \) and \( -\frac{5\sqrt{3}}{3} \).
(b) \( 0, 5, \) and \( -\frac{5\sqrt{3}}{3} \).
(c) \( 0 \) and \( -5 \).
(d) \( 0 \).

8. Which are the points of inflection in the following graph?

(a) Only \(-1\) and 1.
(b) Only 0 and 1.
(c) Only 1.
(d) There are no inflection points in this graph.

9. Let \( f \) have critical points at \(-1, 0, \) and 1. If \( f''(x) = x^3 - 8x^2 + 2 \),
then \( f \) has a relative maximum at:
(a) 0 and 1.
(b) \(-1\) and 1.
(c) 0.
(d) None of the above answers.

\[
\begin{align*}
  f''(0) & = 2 > 0 \quad \text{max} \\
  f''(1) & = -5 < 0 \quad \text{max} \\
  f''(-1) & = -7 < 0 \quad \text{max}
\end{align*}
\]
10. As \( x \to -\infty \), the function \( f(x) = \frac{4x^{2.5x-1}}{2x^2+3} \) tends to

(a) 2.
(b) -1/3.
(c) -1.
(d) 4.

11. Let

\[ y = f(x) = \frac{(x^2 - x)^3}{x + 1} \]

Then \( f'(x) = \)

(a) \[ \frac{(x^2 - x)^3 - 3(x + 1)(x^2 - x)^2(2x - 1)}{(x + 1)^2} \]

(b) \[ 3(x^2 - x)^2(2x - 1) \]

(c) \[ \frac{3(x + 1)(x^2 - x)^2(2x - 1) - (x^2 - x)^3}{(x + 1)^2} \]

(d) None of the above answers.

Use the following information for the next two problems:
A supermarket expects to sell 16,000 boxes of a certain pet food in a year. The supermarket owner must pay $2.00 for each box. There is a $20.00 delivery charge for each delivery from the manufacturer, and there is a $3.00 annual storage charge per box of pet food. Let \( x \) be the number of boxes of pet food that the supermarket owner purchases from the manufacturer each time.

12. What are the annual storage costs for the dealer, in dollars?

(a) \( \frac{3x}{2} \).
(b) \( 3x + 20 \).
(c) \( 2x \).
(d) \( 3x \).

13. What does it cost, in dollars, the supermarket owner each time he places an order for this pet food?

(a) \( 3x \).
(b) \( 3x + 20 \).
(c) \( 2x \).
(d) \( 2x + 20 \).