Mathematics 11012 Intuitive Calculus
Practice Final Examination R. M. Aron

A. Let $f(x) = \frac{x^2}{x^2+1}$. Find $f'(x)$. (The answer need not be simplified.)

$$\frac{(x^2+1)2x - (x^2)(2x)}{(x^2+1)^2}$$

B. Consider the curve given by the function $g(x) = x^3 - 3x + 2$.

(a). Find the tangent line to this curve at $x = -2$.

$g'(x) = 3x^2 - 3$. $g'(-2) = 3(-2)^2 - 3 = 9$. $x = -2 \Rightarrow g(x) = 0$

Tangent line: $\frac{y-0}{x-(-2)} = 9$, or $y = 9x + 18$

(b). Find all points x at which the tangent to this curve is horizontal.

$g'(x) = 0 \iff 3x^2 - 3 = 0 \iff x^2 = 1 \iff$
$x = +1, \; x = -1.$

C. In each case, find the requested derivative:

(a). $h'(t)$ where $h(t) = te^{t^2}$.

$h'(t) = e^{t^2} + 2t^2 e^{t^2}$

(b). $h'(t)$ where $h(t) = (t^4 - 2t^3 + 6)^5$.

$h'(t) = 5(t^4 - 2t^3 + 6)^4 (4t^3 - 6t^2)$

(c). $f'(x)$ where $f(x) = \ln(x^2 + x)$.

$f'(x) = \frac{1}{x^2+x}(2x+1)$

(d). $g''(s)$ where $g(s) = e^{s^2}$.

$g'(s) = 2s e^{s^2}$. $g''(s) = 2\left[e^{s^2} + 2s^2 e^{s^2}\right]$

D. Calculate the following expressions:

(a). The limit, as $n \to \infty$, of

$$\frac{(4+\frac{1}{n})^2 - 4^2}{\frac{1}{n}}.$$

Let $f(x) = x^2$. $f'(4) = \lim_{h\to 0} \frac{f(4+h) - f(4)}{h}$

$= \lim_{h \to 0} \frac{(4+h)^2 - 4^2}{h} = 8$

(b). The limit, as $n \to \infty$, of

$$0^2(\frac{1}{n}) + (\frac{1}{n})^2(\frac{1}{n}) + (\frac{2}{n})^2(\frac{1}{n}) + ... + (\frac{n-1}{n})^2(\frac{1}{n}).$$

This is the formula for Riemann sum of $f(x) = x^2$, $0 \le x \le 1$.

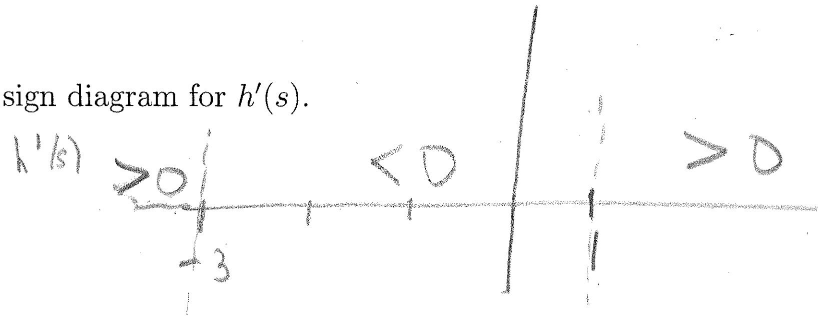
So, limit $= \int_0^1 x^2\,dx = \frac{x^3}{3}\Big|_0^1 = \frac{1}{3}$

E. Let $h(s) = s^3 + 3s^2 - 9s - 3$.

(a). Find the critical points of h.

$h'(s) = 3s^2 + 6s - 9 = 3(s^2 + 2s - 3) = 3(s+3)(s-1)$

(b). Draw the sign diagram for $h'(s)$.



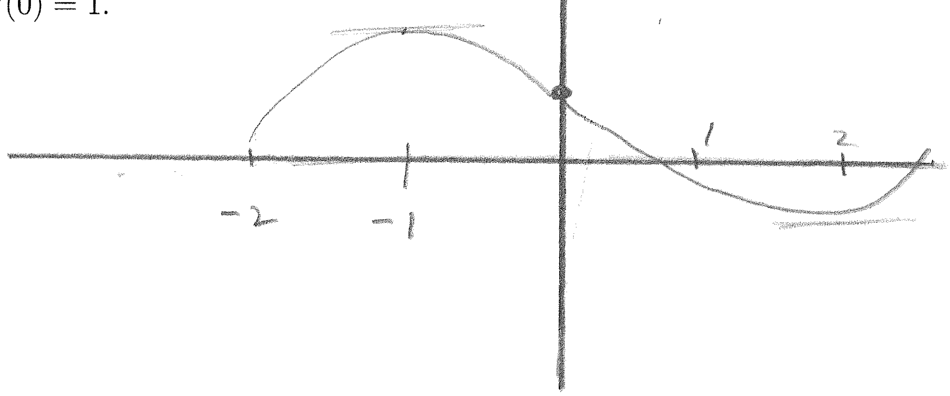
(c). Identify the relative minima and the relative maxima of h.

Rel. max at -3

Rel. min at 1.

F. Draw a very clear graph of a function $f(x)$, $-2 \le x \le 2$, having the following properties:

- f has critical points at -1 and 2.
- f has a point of inflection at 0.
- f is concave down on $(-2, 0)$.
- $f(0) = 1$.



G. Compute each of the following expressions involving exponentials and logarithms:

(a). $\ln(e^3) + \ln(\frac{2}{e^5})$.

$3 + \ln 2 - \ln e^5 = 3 + \ln 2 - 5$

$= -2 + \ln 2$

(b). $f'(t)$, where $f(t) = e^{\ln(3t^4+2t+2)}$.

$f(t) = 3t^4 + 2t + 2$, so $f'(t) = 12t^3 + 2$

(c). $g(x) = e^{x^4} + 5$. $g'(x) = e^{x^4} 4x^3$

(d). $\frac{e^2 e^3}{e^{-3} e}$. $= \frac{e^5}{e^{-2}} = e^7$

H. Let $f(x) = x^2 - 4x + 6$. Compute the extreme (i.e. the biggest and the smallest) values of f on the interval $[-1, 6]$.

$f'(x) = 2x - 4$. So, crit. point at $x = 2$. $f''(x) = 2$, so rel. min. at $x = 2$. $f(-1) = 11$, $f(2) = 2$, $f(6) = 18$

So, min. value is 2, at $x = 2$. Max. value is 18 at $x = 6$.

I. Compute each of the following integrals:

(a). $\int (3x^2 + x - 1)dx$.

$x^3 + \frac{x^2}{2} - x + C$

(b). $\int \frac{4t^5+2t^3-t^2-t}{t^2} dt.$ $= \int 4t^3 + 2t - 1 - \frac{1}{t} \, dt$

$= t^4 + t^2 - t - \ln t + C$

(c). $\int_{-1}^{1}(x^3+5)dx.$ $= \frac{x^4}{4}+5x\Big]_{-1}^{1} = \left(\frac{1}{4}+5\right)-\left(\frac{1}{4}-5\right)$

$= 10$

(d). $\int(x+2)(2x-1)dx.$ $= \int(2x^2+3x-2)\,dx$

$= \frac{2x^3}{3}+\frac{3x^2}{2}-2x+C$

J. Compute each of the following integrals:

(a). $\int(4x^3-3x^2+5)^6(2x^2-x)dx.$ Let $u = 4x^3-3x^2+5$, so

$du = (12x^2-6x)\,dx = 6(2x^2-x)\,dx.$ So, integral $=$

$\int u^6 \frac{1}{6}\,du = \frac{u^7}{6\cdot 7}+C = \frac{(4x^3-3x^2+5)^7}{42}+C$

(b). $\int_0^e \frac{x+1}{x^2+2x+1}dx.$ $= \int_0^e \frac{x+1}{(x+1)^2}\,dx =$

$\int_0^e \frac{1}{x+1}\,dx$. Let $u = x+1$, so $du = dx$, $x=0 \Rightarrow u=1$,

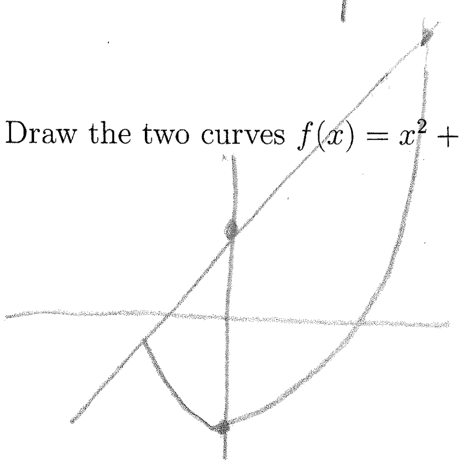
$x=e \Rightarrow u=e+1$. Get $\int_1^{e+1}\frac{1}{u}\,du = \ln u\Big|_1^{e+1} = \ln(e+1)-\ln 1$

$= \ln(e+1)$

K. What is the area under the curve $y=x^3$ and the $x-$axis, where x varies from 1 to 4.

$\int_1^4 x^3\,dx = \frac{x^4}{4}\Big|_1^4 = 64-\frac{1}{4} = 63\frac{3}{4}$

L.(a). Draw the two curves $f(x)=x^2+2x-5$ and $g(x)=2x+4$.



(b). Find the area between these two curves.

Curves meet at $x=-3$, $x=3$. Area $=$

$$\int_{-3}^{3}\left[(2x+4)-(x^2+2x-5)\right]dx = \int_{-3}^{3}(-x^2+9)\,dx = -\frac{x^3}{3}+9x\Big]_{-3}^{3}$$
$$= (-9+27)-(9-27) = 36$$

M.(a). A rocket can rise to a height of $h(t) = t^3 + 0.5t^2$ feet in t seconds. Find its velocity and acceleration 10 seconds after it is launched.

$h'(t) = 3t^2 + t$, $h''(t) = 6t+1$.

Velocity $= h'(10) = 310$

Acceleration $= 61$

Note misprint!

(b). If the height of a bullet that is shot straight into the air is given by $s(t) = -16t^2 + 4160t$ meters, how far above the ground does the bullet get?

$s'(t) = -32t + 4160$. $s'(t) = 0 \Rightarrow t = \frac{4160}{32} = 130$

$s(130) = -16(130)^2 + 4160(130) = 270,400$ meters.

N. A producer of audio tapes estimates the yearly demand for a tape to be 1,000,000. It costs \$800 to set up the machinery for the tape, plus \$10 for each tape produced. If it costs the company \$1 to store a tape for a year, how many should be produced at a time and how many productions runs will be needed to minimize costs?

Let x = # tapes to be produced at 1 time. So, there will be $\frac{1000000}{x}$ production runs.

Storage cost $= \frac{x}{2}\cdot 1 = \frac{x}{2}$

Production cost/run: $10x + 800$. Total production cost $= (10x+800)\left(\frac{1000000}{x}\right) = 10,000,000 + 800,000,000x^{-1} = C(x)$.

Total cost $= \frac{x}{2} + 10,000,000 + 800,000,000x^{-1}$. Setting $C'(x) = 0$, solving for x, we get

$C'(x) = \frac{1}{2} - 800000000x^{-2}$.

$x = 40,000$. There will be $\frac{1,000,000}{40,000} = 25$ production runs.

O. One bank is offering 20 year certificates of deposit paying 5 % per year, compounded quarterly. Another is offering 20 year certificates of deposit paying 4.5 % per year, compounded continuously. You have \$1,000 to invest. In which of the two banks should you deposit your money and why?

Bank 1: $\left(1+\frac{.05}{4}\right)^{80} = 2.70148\ldots$ So, your \$1000 would yield \$2701.48.

Bank 2: $e^{(.05)20} = 2.71828\ldots$. So, your \$1000 would yield \$2718.28.

So, Bank 2 should receive your deposit.

P. You buy a brand new Cadillac for \$ 50,000. The car depreciates at the rate of 20% per year. When is the car worth half the price you paid?

$\$50{,}000(1-0.20)^t = \$25{,}000$. That is,

$(0.80)^t = 1/2$, or $t \ln(.8) = \ln(1/2) = -\ln 2$, or

$t = \frac{-\ln 2}{\ln(.8)} = 3.11$ years.

Q. World consumption of lead is running at the rate of $6.1e^{0.01t}$ million metric tons per year, where t is measured in years, with $t = 0$ corresponding to 2008. Find a formula for the total amount of lead that will be consumed within t years of 2008.

$\int_0^t 6.1e^{0.01u}\,du$ (To avoid too many "t")

$= 6.1\int_0^t e^{.01u}\,du = 6.1\left.\frac{e^{.01u}}{.01}\right|_0^t = 610\left[e^{.01t}-1\right].$

R. A company's profit from producing x tons of a product is given by $P(x) = \sqrt{x^3 + 2x^2 + 4}$ thousand dollars (for $0 \leq x \leq 10$).

(a). Calculate the company's marginal profit, $MP(x)$.

$MP(x) = P'(x) = \left[(x^3+2x^2+4)^{1/2}\right]'$

$= \frac{1}{2}(x^3+2x^2+4)^{-1/2}\left[3x^2+4x\right]$

$= \frac{1}{2}(3x^2+4x)\frac{1}{\sqrt{x^3+2x^2+4}}$

(b). Calculate $P'(4)$ and interpret the result.

$MP(4) = P'(4) = \frac{1}{2}(64) \cdot \frac{1}{10} = 3.2$

Roughly, when 4 tons have been produced, the profit to produce 1 more ton is about \$3,200.

S. A restaurant manager knows that on a typical day, 100 cheeseburgers will be sold at a price of \$2.00 each. She also knows that if for each 20 cent reduction in price, the restaurant will sell 25 more cheeseburgers. Find the price that the restaurant should charge (and the number of cheeseburgers sold) that will maximize the restaurant's revenue for cheeseburgers.

Let x = # price reductions

So, price of a cheeseburger is $200-20x$. (in ¢).

Number of cheeseburgers sold is $100+25x$

Revenue $R(x) = (200-20x)(100+25x)$

$= 20000 - 2000x + 5000x - 500x^2$

$= 20000 + 3000x - 500x^2.$

To maximize, $R'(x) = 3000 - 1000x,$

or $3000 = 1000x$, so $x = 3$

(Note: $R''(x) = -1000$, so $x = 3$ is a min.)

To maximize revenue, she should charge

$(200-60)¢ = \$1.40$/cheeseburger.