Real Analysis Problem Sheet 1: The version with typed solutions.

1. Suppose that (X, d) is a metric space. Prove that if there is a function $f : X \to \mathbb{R}$, and a sequence $(f_n)_n \in C(X)$ such that

$$\|f_n - f\|_\infty := \sup_{x \in X} |f_n(x) - f(x)| \to 0 \text{ as } n \to \infty,$$

then $f \in C(X)$.

Solution. Let $x \in X$ and $\varepsilon > 0$. First fix $n \in \mathbb{N}$ with $\|f_n - f\|_\infty < \frac{\varepsilon}{3}$. Since $f_n \in C(X)$, we have that $\|f_n\|_\infty \leq M$ for some $M \in (0, \infty)$, and that there exists $\delta > 0$ such that if $x' \in X$ satisfies $d(x, x') < \delta$, then $|f_n(x) - f_n(x')| \leq \frac{\varepsilon}{3}$.

Thus, if $x' \in X$ satisfies $d(x, x') < \delta$, then

$$|f(x) - f(x')| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(x')| + |f_n(x') - f(x')| < \varepsilon,$$

and so f is continuous at x.

On the other hand, $|f(x)| \leq |f_n(x) - f(x)| + |f_n(x)| \leq \frac{\varepsilon}{3} + M$, which shows that f is bounded.

2. Suppose that (X, d) is a complete metric space, and that $\rho : X \times X \mapsto [0, \infty)$ is a metric on X such that there are positive constants c_0, C_1 such that

$$c_0 d(x, x') \leq \rho(x, x') \leq C_1 d(x, x')$$

for every $x, x' \in X$. Prove that (X, ρ) is a complete metric space.

Solution. Consider a Cauchy sequence in (X, ρ). Since $d(x, x') \leq \frac{\rho(x,x')}{c_0}$ for any $x, x' \in X$, this Cauchy sequence is also Cauchy in the complete metric space (X, d). Consequently, it converges in (X, d). However, as $\rho(x, x') \leq C_1 d(x, x')$ for any $x, x' \in X$, this sequence also converges to the same limit in (X, d).

3. Let X be a compact metric space. Prove that if a set of functions $\mathcal{F} \subset C(X)$ is relatively compact, then it is bounded and uniformly equicontinuous.

Solution. Since $\mathcal{F}$ is relatively compact, it is totally bounded. In any metric space, a totally bounded set is bounded (you should try and prove this). Let's demonstrate equicontinuity. Let $\varepsilon > 0$. Choose a finite $\frac{\varepsilon}{3}$ net $f_1, \ldots, f_N$ in $\mathcal{F}$. Since each f_j is uniformly continuous (X is compact), and there are only a finite number of them, there

exists $\delta > 0$ such that whenever $x, x' \in X$ satisfy $d(x, x') < \delta$, then $|f_j(x) - f_j(x')| < \frac{\varepsilon}{3}$ for every $j \in \{1, \ldots, N\}$.

Now, fix $f \in \mathcal{F}$, and choose j such that $\|f - f_j\|_\infty < \frac{\varepsilon}{3}$. But then if $x, x' \in X$ satisfy $d(x, x') < \delta$, then

$$|f(x) - f(x')| \leq |f(x) - f_j(x)| + |f_j(x) - f_j(x')| + |f_j(x') - f(x')| < \varepsilon.$$

We're done here.

4. Let

$$\mathcal{F} = \Big\{ f \in C([0,1]) \cap C^1((0,1)) : \|f\|_\infty \leq 1, \text{ and } \int_0^1 |f'(x)|^2 dx \leq 1 \Big\}.$$

Prove that $\mathcal{F}$ is relatively compact in $C([0,1])$.

Solution. Since $[0,1]$ is a compact set, it suffices to verify the hypotheses of the Arzela-Ascoli theorem, namely that $\mathcal{F}$ is bounded and uniformly equicontinuous. We are told that $\mathcal{F}$ is bounded (every member f of the set satisfies $\|f\|_\infty \leq 1$). For the equicontinuity, we shall first write, for $0 \leq x < y \leq 1$,

$$|f(x) - f(y)| = \Big| \int_x^y f'(t) dm_1(t) \Big| \leq \int_x^y |f'(t)| dm_1(t).$$

But now we shall appeal to the Hölder (or Cauchy-Schwarz) inequality with exponent $p = 2$,

$$\begin{aligned} \int_x^y |f'(t)| dm_1(t) &= \int_x^y |f'(t)| \cdot 1 dm_1(t) \\ &\leq \Big(\int_x^y |f'(t)|^2 dm_1(t) \Big)^{1/2} \Big(\int_x^y 1^2 dm_1(t) \Big)^{1/2}. \end{aligned}$$

The right hand side here equals $\big(\int_x^y |f'(t)|^2 dm_1(t) \big)^{1/2} \sqrt{|y - x|}$ which for $f \in \mathcal{F}$ is bounded by $\sqrt{|y - x|}$.

We have found a uniform modulus of continuity: For every $f \in \mathcal{F}$, $|f(x) - f(y)| \leq \sqrt{|x - y|}$ for every $x, y \in [0,1]$. This proves the equicontinuity.

5. (The convolution). Let $\phi : \mathbb{R}^n \to \mathbb{R}$ be a mollifier, that is, a function satisfying

- $\phi \in C^\infty(\mathbb{R}^n)$,
- $\phi \geq 0$ in $\mathbb{R}^n$,
- $\phi(x) = 0$ whenever $|x| \geq 1$, and
- $\int_{\mathbb{R}^n} \phi \, dm_n = 1$.

(a). For $\varepsilon > 0$, set $\phi_\varepsilon(x) = \varepsilon^{-n}\phi\big(\frac{x}{\varepsilon}\big)$, $x \in \mathbb{R}$. Prove that

$$\int_{\mathbb{R}^n} \phi_\varepsilon(x) dm_n(x) = 1.$$

Solution. Consider the change of variable $y = \frac{x}{\varepsilon}$. The Jacobian matrix of this change of variable has determinant ε^{-n}.

Now, for $\varepsilon > 0$, form the convolution $(\phi_\varepsilon * f)(x) = \int_{\mathbb{R}^n} \phi_\varepsilon(x - y) f(y) dm_n(y)$.

(b). Prove that if f is continuous on $\mathbb{R}^n$, then $\phi_\varepsilon * f$ converges to f uniformly on compact subsets of $\mathbb{R}^n$ as $\varepsilon \to 0^+$.

Solution. Let K be a compact subset of $\mathbb{R}^n$. Since the closure of $\{x \in \mathbb{R}^n : \operatorname{dist}(x, K) \leq 1\}$ is a compact set, f is uniformly continuous on it: for each $\varkappa > 0$ there exists $\delta \in (0, 1)$ such that $|f(x) - f(x')| < \varkappa$ whenever $x \in K$ and $|x - x'| < \delta$. But now, if $\varepsilon < \delta$ and $x \in K$

$$\begin{aligned} |\phi_\varepsilon * f(x) - f(x)| &= \Big| \int_{\mathbb{R}^n} \phi_\varepsilon(x - y)[f(y) - f(x)] dm_n(y) \Big| \\ &\leq \int_{\mathbb{R}^n} \phi_\varepsilon(x - y)|f(y) - f(x)| dm_m(y). \end{aligned}$$

where, in the first equality, the fact that $\int_{\mathbb{R}^n} \phi_\varepsilon(x - y) dm_n(y) = 1$ was used, while the inequality following it relies on the fact that ϕ is nonnegative.

Next, note that the function $y \mapsto \phi_\varepsilon(x - y)$ vanishes outside of the ball $B(x, \varepsilon)$, so we may write

$$|\phi_\varepsilon * f(x) - f(x)| \leq \int_{B(x,\varepsilon)} \phi_\varepsilon(x - y)|f(y) - f(x)| dm_n(y).$$

Since $\varepsilon < \delta$ and $x \in K$, $|f(y) - f(x)| < \varkappa$ whenever $y \in B(x, \varepsilon)$. Appealing again to the fact that $\int_{\mathbb{R}^n} \phi_\varepsilon(x - y) dm_n(y) = 1$ yields $|\phi_\varepsilon * f(x) - f(x)| < \varkappa$.

(c). Now suppose that $f \in C(\mathbb{R}^n)$ with $\|f\|_\infty = \sup_{x \in \mathbb{R}^n} |f(x)| \leq M$. Prove that, for every $\varepsilon > 0$, $\phi_\varepsilon * f \in C^1(\mathbb{R}^n)$, and moreover:

(i) $\|\phi_\varepsilon * f\|_\infty \leq M$, and

(ii) For each $i \in \{1, \ldots, n\}$,

$$\Big\| \frac{\partial}{\partial x_i}(\phi_\varepsilon * f) \Big\|_\infty \leq \frac{1}{\varepsilon} \Big\| \frac{\partial}{\partial x_i} \phi \Big\|_\infty m_n(B(0, 1))M.$$

(The derivative of the convolution was done in detail in class (even with $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$) on Monday 9th February.)