Real Analysis Problem Sheet 1: The version with typed solutions.

1. Suppose that \((X,d)\) is a metric space. Prove that if there is a function \(f : X \rightarrow \mathbb{R}\), and a sequence \((f_n)_n \in C(X)\) such that
\[
\|f_n - f\|_\infty := \sup_{x \in X} |f_n(x) - f(x)| \rightarrow 0 \text{ as } n \rightarrow \infty,
\]
then \(f \in C(X)\).

**Solution.** Let \(x \in X\) and \(\varepsilon > 0\). First fix \(n \in \mathbb{N}\) with \(\|f_n - f\|_\infty < \frac{\varepsilon}{3}\). Since \(f_n \in C(X)\), we have that \(\|f_n\|_\infty \leq M\) for some \(M \in (0, \infty)\), and that there exists \(\delta > 0\) such that if \(x' \in X\) satisfies \(d(x,x') < \delta\), then \(|f_n(x) - f_n(x')| \leq \frac{\varepsilon}{3}\).

Thus, if \(x' \in X\) satisfies \(d(x,x') < \delta\), then
\[
|f(x) - f(x')| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(x')| + |f_n(x') - f(x')| < \varepsilon,
\]
and so \(f\) is continuous at \(x\).

On the other hand, \(|f(x)| \leq |f_n(x) - f(x)| + |f_n(x)| \leq \frac{\varepsilon}{3} + M\), which shows that \(f\) is bounded.

2. Suppose that \((X,d)\) is a complete metric space, and that \(\rho : X \times X \rightarrow [0, \infty)\) is a metric on \(X\) such that there are positive constants \(c_0, C_1\) such that
\[
c_0 d(x,x') \leq \rho(x,x') \leq C_1 d(x,x')
\]
for every \(x,x' \in X\). Prove that \((X,\rho)\) is a complete metric space.

**Solution.** Consider a Cauchy sequence in \((X,\rho)\). Since \(d(x,x') \leq \frac{\rho(x,x')}{c_0}\) for any \(x,x' \in X\), this Cauchy sequence is also Cauchy in the complete metric space \((X,d)\). Consequently, it converges in \((X,d)\). However, as \(\rho(x,x') \leq C_1 d(x,x')\) for any \(x,x' \in X\), this sequence also converges to the same limit in \((X,d)\).

3. Let \(X\) be a compact metric space. Prove that if a set of functions \(\mathcal{F} \subset C(X)\) is relatively compact, then it is bounded and uniformly equicontinuous.

**Solution.** Since \(\mathcal{F}\) is relatively compact, it is totally bounded. In any metric space, a totally bounded set is bounded (you should try and prove this). Let’s demonstrate equicontinuity. Let \(\varepsilon > 0\). Choose a finite \(\frac{\varepsilon}{3}\) net \(f_1, \ldots, f_N\) in \(\mathcal{F}\). Since each \(f_j\) is uniformly continuous (\(X\) is compact), and there are only a finite number of them, there
exists $\delta > 0$ such that whenever $x, x' \in X$ satisfy $d(x, x') < \delta$, then $|f_j(x) - f_j(x')| < \frac{\varepsilon}{3}$ for every $j \in \{1, \ldots, N\}$.

Now, fix $f \in F$, and choose $j$ such that $\|f - f_j\|_\infty < \frac{\varepsilon}{3}$. But then if $x, x' \in X$ satisfy $d(x, x') < \delta$, then

$$|f(x) - f(x')| \leq |f(x) - f_j(x)| + |f_j(x) - f_j(x')| + |f_j(x') - f(x')| < \varepsilon.$$

We’re done here.

4. Let

$$F = \left\{ f \in C([0, 1]) \cap C^1((0, 1)) : \|f\|_\infty \leq 1, \text{ and } \int_0^1 |f'(x)|^2 \, dx \leq 1 \right\}.$$

Prove that $F$ is relatively compact in $C([0, 1])$.

**Solution.** Since $[0, 1]$ is a compact set, it suffices to verify the hypotheses of the Arzela-Ascoli theorem, namely that $F$ is bounded and uniformly equicontinuous. We are told that $F$ is bounded (every member $f$ of the set satisfies $\|f\|_\infty \leq 1$). For the equicontinuity, we shall first write, for $0 \leq x < y \leq 1$,

$$|f(x) - f(y)| = \left| \int_x^y f'(t) \, dm_1(t) \right| \leq \int_x^y |f'(t)| \, dm_1(t).$$

But now we shall appeal to the Hölder (or Cauchy-Schwarz) inequality with exponent $p = 2$,

$$\int_x^y |f'(t)| \, dm_1(t) = \int_x^y |f'(t)| \cdot 1 \, dm_1(t) \leq \left( \int_x^y |f'(t)|^2 \, dm_1(t) \right)^{1/2} \left( \int_x^y 1^2 \, dm_1(t) \right)^{1/2}.$$

The right hand side here equals $(\int_x^y |f'(t)|^2 \, dm_1(t))^{1/2} \sqrt{|y - x|}$ which for $f \in F$ is bounded by $\sqrt{|y - x|}$.

We have found a uniform modulus of continuity: For every $f \in F$, $|f(x) - f(y)| \leq \sqrt{|x - y|}$ for every $x, y \in [0, 1]$. This proves the equicontinuity.

5. (The convolution). Let $\phi : \mathbb{R}^n \to \mathbb{R}$ be a mollifier, that is, a function satisfying

- $\phi \in C^\infty(\mathbb{R}^n)$,
- $\phi \geq 0$ in $\mathbb{R}^n$,
- $\phi(x) = 0$ whenever $|x| \geq 1$, and
- $\int_{\mathbb{R}^n} \phi \, dm_n = 1$. 

(a). For $\varepsilon > 0$, set $\phi_\varepsilon(x) = \varepsilon^{-n} \phi\left(\frac{x}{\varepsilon}\right)$, $x \in \mathbb{R}$. Prove that

$$\int_{\mathbb{R}^n} \phi_\varepsilon(x) \, dm_n(x) = 1.$$  

**Solution.** Consider the change of variable $y = \frac{x}{\varepsilon}$. The Jacobian matrix of this change of variable has determinant $\varepsilon^{-n}$.

Now, for $\varepsilon > 0$, form the convolution $(\phi_\varepsilon * f)(x) = \int_{\mathbb{R}^n} \phi_\varepsilon(x - y) f(y) \, dm_n(y)$.

(b). Prove that if $f$ is continuous on $\mathbb{R}^n$, then $\phi_\varepsilon * f$ converges to $f$ uniformly on compact subsets of $\mathbb{R}^n$ as $\varepsilon \to 0^+$.

**Solution.** Let $K$ be a compact subset of $\mathbb{R}^n$. Since the closure of $\{x \in \mathbb{R}^n : \text{dist}(x, K) \leq 1\}$ is a compact set, $f$ is uniformly continuous on it: for each $\varkappa > 0$ there exists $\delta \in (0, 1)$ such that $|f(x) - f(x')| < \varkappa$ whenever $x \in K$ and $|x - x'| < \delta$. But now, if $\varepsilon < \delta$ and $x \in K$

$$|\phi_\varepsilon * f(x) - f(x)| = \left| \int_{\mathbb{R}^n} \phi_\varepsilon(x - y) [f(y) - f(x)] \, dm_n(y) \right|$$

$$\leq \int_{\mathbb{R}^n} \phi_\varepsilon(x - y) |f(y) - f(x)| \, dm_n(y).$$

where, in the first equality, the fact that $\int_{\mathbb{R}^n} \phi_\varepsilon(x - y) \, dm_n(y) = 1$ was used, while the inequality following it relies on the fact that $\phi$ is nonnegative.

Next, note that the function $y \mapsto \phi_\varepsilon(x - y)$ vanishes outside of the ball $B(x, \varepsilon)$, so we may write

$$|\phi_\varepsilon * f(x) - f(x)| \leq \int_{B(x, \varepsilon)} \phi_\varepsilon(x - y) |f(y) - f(x)| \, dm_n(y).$$

Since $\varepsilon < \delta$ and $x \in K$, $|f(y) - f(x)| < \varkappa$ whenever $y \in B(x, \varepsilon)$. Appealing again to the fact that $\int_{\mathbb{R}^n} \phi_\varepsilon(x - y) \, dm_n(y) = 1$ yields $|\phi_\varepsilon * f(x) - f(x)| < \varkappa$.

(c). Now suppose that $f \in C(\mathbb{R}^n)$ with $\|f\|_\infty = \sup_{x \in \mathbb{R}^n} |f(x)| \leq M$. Prove that, for every $\varepsilon > 0$, $\phi_\varepsilon * f \in C^1(\mathbb{R}^n)$, and moreover:

(i) $\|\phi_\varepsilon * f\|_\infty \leq M$, and

(ii) For each $i \in \{1, \ldots, n\}$,

$$\left\| \frac{\partial}{\partial x_i} (\phi_\varepsilon * f) \right\|_\infty \leq \frac{1}{\varepsilon} \left\| \frac{\partial}{\partial x_i} \phi \right\|_\infty m_n(B(0, 1)) M.$$
(The derivative of the convolution was done in detail in class (even with $f \in L^1_{\text{loc}}(\mathbb{R}^n)$) on Monday 9th February.)