Real Analysis Problem Sheet 3.
Due Monday 16th March

1. Let $1 < p < \infty$. Suppose $f \in L^p(\mathbb{R}^n)$ and $g \in L^{p'}(\mathbb{R}^n)$ where $\frac{1}{p} + \frac{1}{p'} = 1$. Show that the convolution $f * g \in C(\mathbb{R}^n)$.

2. For $k \in \mathbb{N}$, define the trigonometric polynomial
   \[ P_k(t) = c_k \left( 1 + \cos \frac{t}{2} \right)^k, \quad t \in \mathbb{R}. \]
   where $c_k$ has been chosen to ensure that
   \[ \int_0^{2\pi} P_k(t) dm_1(t) = 1. \]
   Let $C(\mathbb{T})$ denote the space of $2\pi$-periodic continuous functions on $\mathbb{R}$.
   (a). Show that $c_k \leq k + 1$.
   (b). For $f \in C(\mathbb{T})$, show that the convolution $P_k * f$ converges uniformly to $f$ as $k \to \infty$.
   As $P_k * f$ is also a trigonometric polynomial, this shows that trigonometric polynomials are dense in $C(\mathbb{T})$.

3. Show that the Sobolev space $W^{1,p}(\mathbb{R}^n)$ is a Banach space. (See “More on $L^p$ spaces” for the definition)

4. Suppose that, for each $j \in \mathbb{N}$, $f_j : [0, 1] \to \mathbb{R}$ is a Lebesgue measurable function satisfying $0 \leq f_j \leq \frac{3}{2}$ and
   \[ \int_0^1 f_j dm_1 = 1. \]
   Prove that
   \[ m_1 \left( \{ x \in [0, 1] : \limsup_{j \to \infty} f_j(x) \geq \frac{1}{2} \} \right) \geq \frac{1}{2}. \]

5. Suppose that $f$ is a non-negative Lebesgue measurable function on $[0, 1]$ with $f(x) > 0$ for $m_1$-almost every $x \in [0, 1]$. Suppose that $E_k$ is a sequence of measurable subsets of $[0, 1]$ with
   \[ \lim_{k \to \infty} \int_{E_k} f dm_1 = 0. \]
   Prove that $\lim_{k \to \infty} m_1(E_k) = 0$. 
