Real Analysis Problem Sheet 4: Building Banach Spaces out of Banach Spaces.

Due Monday 6th April

Suppose that $(X, \mathcal{A}, \mu)$ is a σ-finite measure space. Set $L^p = L^p(X, \mathcal{A}, \mu)$.

Throughout this sheet, set $1 < p_0 < p_1 < \infty$. Also $p' = \frac{p}{p-1}$.

1. (Direct Sum) Let $L^{p_0} \oplus L^{p_1} = \{(f_0, f_1) : f_j \in L^{p_j}, j \in \{0, 1\}\}$. Set

$$\|(f_0, f_1)\| = \|f_0\|_{p_0} + \|f_1\|_{p_1}.$$

Prove that $\|\cdot\|$ defines a norm on $L^{p_0} \oplus L^{p_1}$, and that this normed space is complete.

2. (A random question). A sequence (x_n) in a normed space $(X, \|\cdot\|)$ is absolutely convergent if $\sum_{j=1}^{\infty} \|x_j\| < \infty$. Prove that a normed space $(X, \|\cdot\|)$ is a Banach space if and only if every absolutely convergent sequence converges.

3. (Sum) $L^{p_0} + L^{p_1} = \{f \text{ measurable} : f = f_0 + f_1 \text{ for } f_j \in L^{p_j}\}$. Define, for $f \in L^{p_0} + L^{p_1}$

$$\|f\| = \inf\{\|f_0\|_{p_0} + \|f_1\|_{p_1} : f = f_0 + f_1\}$$

where the infimum is taken over all decompositions $f = f_0 + f_1$. Prove that $\|\cdot\|$ is a norm and it turns $L^{p_0} + L^{p_1}$ into a Banach space.

4. (Intersection) Consider the vector space $L^{p_0} \cap L^{p_1}$. For $f \in L^{p_0} \cap L^{p_1}$ set

$$\|f\| = \|f\|_{p_0} + \|f\|_{p_1}.$$

Prove that this is a norm and $L^{p_0} \cap L^{p_1}$ is a Banach space when equipped with this norm.

5. Prove that $(L^{p_0} + L^{p_1})^* = L^{p_0'} \cap L^{p_1'}$ up to an equivalence of norms.

6. Prove that $(L^{p_0} \oplus L^{p_1})^* = L^{p_0'} \oplus L^{p_1'}$ up to an equivalence of norms.

7. Prove that $(L^{p_0} \cap L^{p_1})^* = L^{p_0'} + L^{p_1'}$ up to an equivalence of norms. (This is the hardest problem, and I'll be happy to give a hint.)