Real Analysis Problem Sheet 4: Building Banach Spaces out of Banach Spaces.

Due Monday 6th April

Suppose that \((X, \mathcal{A}, \mu)\) is a \(\sigma\)-finite measure space. Set \(L^p = L^p(X, \mathcal{A}, \mu)\).

Throughout this sheet, set \(1 < p_0 < p_1 < \infty\). Also \(p' = \frac{p}{p-1}\).

1. (Direct Sum) Let \(L^{p_0} \oplus L^{p_1} = \{ (f_0, f_1) : f_j \in L^{p_j}, j \in \{0, 1\} \}\). Set \(\|(f_0, f_1)\| = \|f_0\|_{p_0} + \|f_1\|_{p_1}\).

Prove that \(\|\cdot\|\) defines a norm on \(L^{p_0} \oplus L^{p_1}\), and that this normed space is complete.

2. (A random question). A sequence \((x_n)\) in a normed space \((X, \|\cdot\|)\) is absolutely convergent if \(\sum_{j=1}^{\infty} \|x_j\| < \infty\). Prove that a normed space \((X, \|\cdot\|)\) is a Banach space if and only if every absolutely convergent sequence converges.

3. (Sum) \(L^{p_0} + L^{p_1} = \{ f \text{ measurable} : f = f_0 + f_1 \text{ for } f_j \in L^{p_j} \}\). Define, for \(f \in L^{p_0} + L^{p_1}\)

\[\|f\| = \inf\{\|f_0\|_{p_0} + \|f_1\|_{p_1} : f = f_0 + f_1\}\]

where the infimum is taken over all decompositions \(f = f_0 + f_1\). Prove that \(\|\cdot\|\) is a norm and it turns \(L^{p_0} + L^{p_1}\) into a Banach space.

4. (Intersection) Consider the vector space \(L^{p_0} \cap L^{p_1}\). For \(f \in L^{p_0} \cap L^{p_1}\) set

\[\|f\| = \|f\|_{p_0} + \|f\|_{p_1}\]

Prove that this is a norm and \(L^{p_0} \cap L^{p_1}\) is a Banach space when equipped with this norm.

5. Prove that \((L^{p_0} + L^{p_1})^* = L^{p_0'} \cap L^{p_1'}\) up to an equivalence of norms.

6. Prove that \((L^{p_0} \oplus L^{p_1})^* = L^{p_0'} \oplus L^{p_1'}\) up to an equivalence of norms.

7. Prove that \((L^{p_0} \cap L^{p_1})^* = L^{p_0'} + L^{p_1'}\) up to an equivalence of norms.

(This is the hardest problem, and I’ll be happy to give a hint.)