MATH 4/50055 Fall 2014

Homework 1 Solutions

1. $$F_x(t) = \begin{cases} \frac{t}{100-x} &, t \in [0, 100-x) \\ 1 &, t \geq 100-x \end{cases}$$

Then $${}_tp_x = S_x(t) = 1 - F_x(t) = \begin{cases} 1 - \frac{t}{100-x} &, t \in [0, 100-x) \\ 0 &, t \geq 100-x \end{cases}$$

(a) $$\mathring{e}_x = \int_0^\infty {}_tp_x \, dt$$

$$= \int_0^{100-x} \left(1 - \frac{t}{100-x}\right) dt$$

$$= \left(t - \frac{t^2}{2(100-x)}\right) \Bigg|_0^{100-x}$$

Note: x is fixed, so $100-x$ is constant

$$= (100-x) - \frac{(100-x)^2}{2(100-x)} - 0$$

$$= (100-x) - \frac{100-x}{2}$$

$$= \frac{2(100-x) - (100-x)}{2}$$

$$= \frac{100-x}{2}.$$

(b) $$E[T_x^2] = 2 \int_0^\infty t \, {}_tp_x \, dt$$

$$= 2 \int_0^{100-x} t \left(1 - \frac{t}{100-x}\right) dt$$

$$= 2 \int_0^{100-x} \left(t - \frac{t^2}{100-x}\right) dt$$

1(b) cont'd

$$E[T_x^2] = 2\left[\frac{1}{2}t^2 - \frac{t^3}{3(100-x)}\right]_0^{100-x}$$

$$= 2\left[\frac{(100-x)^2}{2} - \frac{(100-x)^3}{3(100-x)}\right] - 2\cdot 0$$

$$= 2\left[\frac{3(100-x)^2 - 2(100-x)^3}{6}\right]$$

$$= \frac{(100-x)^2}{3}.$$

So $V[T_x] = E[T_x^2] - \mathring{e}_x^{\,2}$

$$= \frac{(100-x)^2}{3} - \left[\frac{100-x}{2}\right]^2$$

$$= \frac{(100-x)^2}{3} - \frac{(100-x)^2}{4}$$

$$= \frac{4(100-x)^2 - 3(100-x)^2}{12}$$

$$= \frac{(100-x)^2}{12}.$$

(c) Let $m = \text{median}[T_x]$. Then

$$F_x(m) = \frac{1}{2}$$

$$\frac{m}{100-x} = \frac{1}{2}$$

$$m = \frac{100-x}{2}.$$

2. $f_x(t) = ce^{-ct}$, for $t \geq 0$, where $c > 0$ is constant.

One approach is to find ${}_tp_x = S_x(t) = 1 - F_x(t)$ first:

$$F_x(t) = \int_0^t f_x(s)\,ds$$

$$= \int_0^t ce^{-cs}\,ds$$

$$= -\frac{c}{c}e^{-cs}\Big|_0^t = -e^{-ct} + 1 = 1 - e^{-ct}.$$

So ${}_tp_x = e^{-ct}$.

(a) $$\mathring{e}_x = \int_0^\infty {}_tp_x\,dt$$

$$= \int_0^\infty e^{-ct}\,dt$$

$$= -\frac{1}{c}e^{-ct}\Big|_0^\infty = 0 - \left(-\frac{1}{c}e^0\right) = \frac{1}{c}.$$

(b) $$E[T_x^2] = 2\int_0^\infty t\,{}_tp_x\,dt$$

$$= 2\int_0^\infty te^{-ct}\,dt$$

$$= 2\left[-\frac{te^{-ct}}{c}\Big|_0^\infty - \int_0^\infty -\frac{e^{-ct}}{c}\,dt\right]$$

$$= 2\left[-\frac{1}{c}\left[\lim_{t\to\infty}\frac{t}{e^{ct}} - 0\cdot e^0\right] + \frac{e^{-ct}}{-c\cdot c}\Big|_0^\infty\right]$$

Int by Parts:

Let $u = t$ $\quad dv = e^{-ct}dt$

$du = dt$ $\quad v = -\frac{e^{-ct}}{c}$

$\int u\,dv = uv - \int v\,du$

2(b) cont'd

$$\text{L'Hôpital's Rule}$$

$$E[T_x^2] = 2\left[-\frac{1}{c}\left(\lim_{t\to\infty}\frac{1}{ce^{ct}}\right) - \left(0 - \frac{1}{c^2}\right)\right]$$

$$= 2\left[0 + \frac{1}{c^2}\right]$$

$$= \frac{2}{c^2}.$$

So $V[T_x] = E[T_x^2] - (\mathring{e}_x)^2$

$$= \frac{2}{c^2} - \left(\frac{1}{c}\right)^2$$

$$= \frac{1}{c^2}.$$

(c) Let $m = \text{median}[T_x]$. Then

$$S_x(m) = \frac{1}{2}$$

$$_mp_x = \frac{1}{2}$$

$$e^{-cm} = \frac{1}{2}$$

$$-cm = \log\left(\frac{1}{2}\right)$$

$$-cm = -\log 2$$

$$m = \frac{\log 2}{c}.$$

3. (a). Can $\mu_x = ax^b$, where $a, b > 0$?

If so, then $S_x(t) = \exp\left\{-\int_x^{x+t} \mu_r \, dr\right\}$

$$= \exp\left\{-\int_x^{x+t} a r^b \, dr\right\}$$

$$= \exp\left\{-a\left(\frac{r^{b+1}}{b+1}\right)\Big|_x^{x+t}\right\}$$

$$= \exp\left\{-a\left(\frac{(x+t)^{b+1} - x^{b+1}}{b+1}\right)\right\}.$$

Check three conditions on $S_x(t)$:

(i) $S_x(0) = \exp\left\{-a\left(\frac{x^{b+1} - x^{b+1}}{b+1}\right)\right\} = e^0 = 1$ ✓

(ii) $\lim_{t\to\infty} S_x(t) = \lim_{t\to\infty} \exp\left\{-a\left(\frac{(x+t)^{b+1} - x^{b+1}}{b+1}\right)\right\}$

$$= \lim_{y\to\infty} \exp\{-y\} = 0 \quad ✓$$

Chain Rule

(iii) $\frac{d}{dt} S_x(t) = \exp\left\{-a\left(\frac{(x+t)^{b+1}}{b+1} - \frac{x^{b+1}}{b+1}\right)\right\} \frac{d}{dt}\left\{-a\left(\frac{(x+t)^{b+1}}{b+1} - \frac{x^{b+1}}{b+1}\right)\right\}$

$$= \exp\left\{-a\left(\frac{(x+t)^{b+1} - x^{b+1}}{b+1}\right)\right\}\left[-a\left[\left(\frac{b+1}{b+1}\right)(x+t)^b(1) - 0\right]\right]$$

$$= \underbrace{\exp\left\{-a\left(\frac{(x+t)^{b+1} - x^{b+1}}{b+1}\right)\right\}}_{\text{pos}} \underbrace{(-a)}_{\text{neg}} \underbrace{(x+t)^b}_{\text{pos}}$$

< 0 ✓

Thus, $S_x(t)$ can be a survival function and so $\mu_x = ax^b$ can be a force of mortality function if $a, b > 0$. (This is called the Weibull Model.)

3(b) Can $\mu_x = \frac{1}{(1+x)^3}$, $x \geq 0$, be a force of mortality function. If so, then the survival function is

$$S_x(t) = \exp\left\{- \int_x^{x+t} \mu_r \, dr\right\}$$

$$= \exp\left\{- \int_x^{x+t} (1+r)^{-3} \, dr\right\}$$

$$= \exp\left\{- \frac{(1+r)^{-2}}{-2} \Big|_x^{x+t}\right\}$$

$$= \exp\left\{\frac{1}{2}\left[(1+x+t)^{-2} - (1+x)^{-2}\right]\right\}$$

$$= \exp\left\{\frac{1}{2(1+x+t)^2} - \frac{1}{2(1+x)^2}\right\}.$$

Now $\lim_{t \to \infty} S_x(t) = \lim_{t \to \infty} \exp\left\{\frac{1}{2(1+x+t)^2} - \frac{1}{2(1+x)^2}\right\}$

$$= \exp\left\{0 - \frac{1}{2(1+x)^2}\right\}$$

$$= \exp\left\{- \frac{1}{2(1+x)^2}\right\}$$

$$\neq 0.$$

Therefore, $S_x(t)$ cannot be a survival function and so $\frac{1}{(1+x)^3}$ cannot be a force of mortality function. ∎

4. $e_{x:\overline{n}|} = \mathbb{E}[\min(K_x, n)]$, where $n \in \mathbb{Z}^+$.

$$= \sum_{k=0}^{\infty} \min(k,n) \Pr[K_x = k]$$

$$= \sum_{k=0}^{n-1} k \Pr[K_x = k] + \sum_{k=n}^{\infty} n \Pr[K_x = k]$$

$$= \sum_{k=0}^{n-1} k \left[{}_kp_x - {}_{k+1}p_x \right] + n \sum_{k=n}^{\infty} \Pr[K_x = k]$$

$$= \sum_{k=0}^{n-1} k \left[{}_kp_x - {}_{k+1}p_x \right] + n \Pr[K_x \geq n]$$

$$= \sum_{k=0}^{n-1} k \left[{}_kp_x - {}_{k+1}p_x \right] + n \cdot {}_np_x$$

$$= 0 + 1({}_1p_x - {}_2p_x) + 2({}_2p_x - {}_3p_x) + 3({}_3p_x - {}_4p_x) + \cdots + (n-1)({}_{n-1}p_x - {}_np_x) + n \cdot {}_np_x$$

$$= {}_1p_x + {}_2p_x + {}_3p_x + \cdots + {}_np_x$$

$$= \sum_{k=1}^{n} {}_kp_x . \quad \blacksquare$$

note that we start at 1.

x	l_x	d_x	p_x	q_x
20	97741	118	0.9987927277	0.0012072723
21	97623	124	0.9987298075	0.0012701925
22	97499	129	0.9986769095	0.0013230905
23	97370	130	0.9986648865	0.0013351135
24	97240	130	0.9986631016	0.0013368984
25	97110	128	0.9986819071	0.0013180929
26	96982	126	0.9987007898	0.0012992102
27	96856	126	0.9986990997	0.0013009003
28	96730	126	0.9986974051	0.0013025949
29	96604			

table made in Excel

5a) $d_{28} = 126$

b) $p_{28} = 0.9986974051$

c) $q_{28} = 0.0013025949$

d) $_4p_{23} = \frac{l_{27}}{l_{23}} = \frac{96856}{97370} = 0.994721166 7$

e) $_5q_{20} = \frac{l_{20} - l_{25}}{l_{20}} \approx 0.0064558374$

f) $_{2|3}q_{20} = \frac{l_{22} - l_{25}}{l_{20}} = 0.0039799061$

6. Using the UDD assumption:

(a)
$$
\begin{aligned}
{}_{0.3}p_{27} &= 1 - {}_{0.3}q_{27} \\
&\overset{UDD}{=} 1 - 0.3\,q_{27} \\
&= 0.999609729 \ldots
\end{aligned}
$$

(b)
$$
\begin{aligned}
{}_{0.5}q_{23.2} &= 1 - {}_{0.5}p_{23.2} \\
&= 1 - \frac{l_{23.7}}{l_{23.2}} \\
&\overset{UDD}{=} 1 - \frac{l_{23} - 0.7\,d_{23}}{l_{23} - 0.2\,d_{23}} \\
&= 0.00066773504 27 \ldots
\end{aligned}
$$

(c)
$$
\begin{aligned}
{}_{0.8}q_{25.4} &= 1 - {}_{0.8}p_{25.4} \\
&= 1 - \frac{l_{26.2}}{l_{25.4}} \\
&\overset{UDD}{=} 1 - \frac{l_{26} - 0.2\,d_{26}}{l_{25} - 0.4\,d_{25}} \\
&= 0.00105091 \ldots
\end{aligned}
$$

(d)
$$
\begin{aligned}
{}_{3.2}p_{21.1} &= \frac{l_{24.3}}{l_{21.1}} \overset{UDD}{=} \frac{l_{24} - 0.3\,d_{24}}{l_{21} - 0.1\,d_{21}} \\
&= 0.99580373 \ldots
\end{aligned}
$$

7. Using the CFM assumption

(a) $${}_{0.3}p_{27} \overset{CFM}{=} (p_{27})^{0.3} = 0.9996095521\cdots$$

(b) $${}_{0.5}q_{23.2} = 1 - {}_{0.5}p_{23.2}$$

$$\overset{CFM}{=} 1 - (p_{23})^{0.5} \qquad \text{since } 0.5 + 0.2 < 1$$

$$= 0.00066777797072$$

(c) $${}_{0.8}q_{25.4} = {}_{0.6}q_{25.4} + {}_{0.6}p_{25.4} \cdot {}_{0.2}q_{26}$$

$$= (1 - {}_{0.6}p_{25.4}) + {}_{0.6}p_{25.4}(1 - {}_{0.2}p_{26})$$

$$\overset{CFM}{=} (1 - (p_{25})^{0.6}) + (p_{25})^{0.6}(1 - (p_{26})^{0.2})$$

$$= 0.00105125\cdots$$

(d) $${}_{3.2}p_{21.1} = {}_{0.9}p_{21.1} \cdot {}_{2}p_{22} \cdot {}_{0.3}p_{24}$$

$$\overset{CFM}{=} (p_{21})^{0.9} \cdot \frac{l_{24}}{l_{22}} \cdot (p_{24})^{0.3}$$

$$= 0.9958036 2\cdots$$