1. We wish to find $_{3|2.5}q_{90} = {}_3p_{90} \cdot {}_{2.5}q_{93}$.

First fill in the table:

x	l_x	d_x	p_x	q_x
95	$\boxed{600}$	$\boxed{240}$	$\boxed{0.60}$	0.40
96	$\boxed{360}$	$\boxed{288}$	0.20	$\boxed{0.80}$
97	$\boxed{72}$	72	$\boxed{0}$	1.00

Since $p_x + q_x = 1$ for all x, and $l_{96} = \frac{72}{0.20} = 360$

and $l_{95} = \frac{360}{0.60} = 600$.

Now $_3p_{90} = \frac{l_{93}}{l_{90}} = \frac{825}{1000} = 0.825$; and

$$
\begin{aligned}
{}_{2.5}q_{93} &= 1 - {}_{2.5}p_{93} \\
&= 1 - {}_2p_{93} \; {}_{0.5}p_{95} \\
&= 1 - {}_2p_{93} \left(1 - {}_{0.5}q_{95}\right) \\
&\overset{UDD}{=} 1 - {}_2p_{93} \left(1 - 0.5 \cdot q_{95}\right) \\
&= 1 - \frac{l_{95}}{l_{93}} \left[1 - 0.5(0.40)\right] \\
&= 1 - \frac{600}{825}(1 - 0.2) \\
&= 1 - \frac{600}{825}(0.8)
\end{aligned}
$$

So $_{3|2.5}q_{90} = {}_3p_{90} \; {}_{2.5}q_{93}$

$$= (0.825)\left(1 - \frac{480}{825}\right) = \underline{\underline{0.345}} \qquad ■$$

2. Timeline:

	0	1	2	3
	\$1	\$2	\$3	
	1	v	v^2	
	1	p_{30}	${}_2p_{30}$	

Now $p_{30} = 1 - q_{30} = 1 - 0.01 = 0.99$

$${}_2p_{30} = p_{30}\, p_{31} = 0.99(1 - q_{31})$$
$$= 0.99\,(0.985)$$

So the actuarial present value of this annuity is

$$(I\ddot{a})_{30:\overline{3}|} = EPV = 1 + 2(1.04)^{-1}(0.99) + 3(1.04)^{-2}(0.99)(0.985)$$

$$= \$\,\underline{\underline{5.61}} \quad \blacksquare$$

3. We wish to find $\bar{a}_{\overline{x:\overline{20}|}} = \bar{a}_{\overline{20}|} + \bar{a}_x \overset{\text{minus}}{\cancel{+}} \bar{a}_{x:\overline{20}|}$

So we compute:

$$\bar{a}_{\overline{20}|} = \frac{1-v^{20}}{\delta} = \frac{1-1.05^{-20}}{\ln(1.05)} = 12.77123\ldots$$

$$\bar{a}_x = \frac{1-\bar{A}_x}{\delta} = \frac{1-0.3}{\ln(1.05)} = \frac{0.7}{\ln(1.05)}$$

$$\bar{a}_{x:\overline{20}|} = \frac{1-\bar{A}_{x:\overline{20}|}}{\delta} = \frac{1-0.4}{\ln(1.05)} = \frac{0.6}{\ln(1.05)}$$

So $\bar{a}_x - \bar{a}_{x:\overline{20}|} = \dfrac{0.7-0.6}{\ln(1.05)} = \dfrac{0.1}{\ln(1.05)} = 2.04959\ldots$

Hence, $\bar{a}_{\overline{x:\overline{20}|}} = \bar{a}_{\overline{20}|} + \bar{a}_x - \bar{a}_{x:\overline{20}|}$

$$= 12.77123\ldots + 2.04959\ldots$$
$$= \underline{\underline{\$14.82}}$$

$\blacksquare$

4. In general, $\bar{a}_{x:\overline{n|}} = \int_0^n v^t \, {}_tp_x \, dt$,
but there are complications with the change in interest rate.

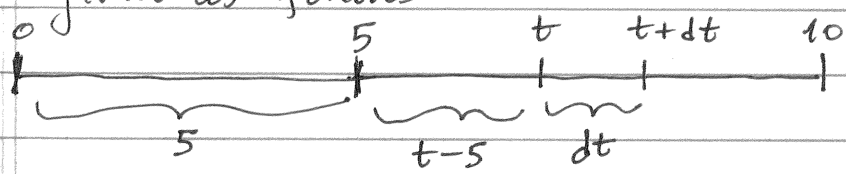
First, find ${}_tp_x = e^{-\int_0^t \mu_{x+s} \, ds}$

$$= e^{-\int_0^t 0.01 \, ds}$$

$$= e^{-[0.01s|_0^t]}$$

$$= e^{-0.01t}.$$

We will break the integral up into two parts. The discount factor for the second integral is given as follows:



$$e^{-0.03(t-5)} e^{-0.04(5)} = e^{-0.03t + 0.15 - 0.20}$$

$$= e^{-0.03t - 0.05}$$

$$= e^{-0.03t} e^{-0.05}$$

So we get

$$\bar{a}_{x:\overline{10|}} = \int_0^5 e^{-0.04t} e^{-0.01t} \, dt + e^{-0.05} \int_5^{10} e^{-0.03t} e^{-0.01t} \, dt$$

$$= \int_0^5 e^{-0.05t} \, dt + e^{-0.05} \int_5^{10} e^{-0.04t} \, dt$$

4 (cont'd)

$$\bar{a}_{x:\overline{10}|} = -\frac{1}{0.05} e^{-0.05t}\Big|_0^5 + e^{-0.05}\left[-\frac{1}{0.04} e^{-0.04t}\right]\Big|_5^{10}$$

$$= 20\left(1-e^{-0.25}\right) + 25e^{-0.05}\left(e^{-0.20} - e^{-0.40}\right)$$

$$= 20\left(1-e^{-0.25}\right) + 25\left(e^{-0.25} - e^{-0.45}\right)$$

$$= \underline{\underline{\$7.95}}$$

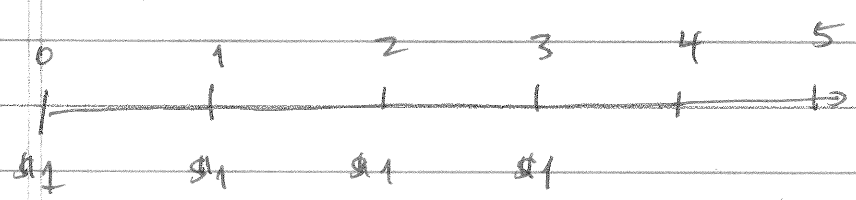
5. Sum over all possible dates of death.

$$\ddot{a}_{x:\overline{4}|} = \overset{\text{dies during 1st year}}{\ddot{a}_{\overline{1}|}\, q_x} + \overset{\text{dies during 2nd year}}{\ddot{a}_{\overline{2}|}\, {}_{1|}q_x} + \overset{\text{dies during 3rd year}}{\ddot{a}_{\overline{3}|}\, {}_{2|}q_x} + \overset{\text{dies after 3 years}}{\ddot{a}_{\overline{4}|}\, {}_{3}p_x}$$

$$= (1.00)(0.33) + (1.93)(0.24) + (2.80)(0.16) + (3.62)\left[1 - 0.33 - 0.24 - 0.16\right]$$

$$= \$\,2.22$$

Note: even if (x) dies during the 4th year (doesn't live a full 4 years), s/he still receives all 4 payments, so the last term involves ${}_3p_x$ not ${}_4p_x$.



probability of receiving this 4th payment is ${}_3p_x$.

probability that this is the last payment received is ${}_{2|}q_x$

6. $\bar{a}_{x:\overline{1|}} = \int_0^1 e^{-\delta t} \, {}_tp_x \, dt$

$= \int_0^1 e^{-\delta t} e^{-\mu t} \, dt$

$= \int_0^1 e^{-(\delta+\mu)t} \, dt$

$= \left. \dfrac{e^{-(\delta+\mu)t}}{-(\delta+\mu)} \right|_0^1$

$= \dfrac{e^{-(\delta+\mu)} - 1}{-(\delta+\mu)}$

$= \dfrac{1 - e^{-(\delta+\mu)}}{\delta+\mu}$. ∎

${}_tp_x = e^{-\int_0^t \mu \, ds}$

$= e^{-\mu s|_0^t}$

$= e^{-\mu t}$

We want to find $\bar{a}_x$.

7. We know that

$$Var[\bar{a}_{\overline{T_x}|}] = \frac{{}^2\bar{A}_x - (\bar{A}_x)^2}{\delta^2}$$

So $$4 = \frac{0.08 - (\bar{A}_x)^2}{(0.10)^2}$$

$$0.04 = 0.08 - (\bar{A}_x)^2$$

$$(\bar{A}_x)^2 = 0.04$$

$$\Rightarrow \underline{\bar{A}_x = 0.2}$$

Then $$\bar{a}_x = \frac{1 - \bar{A}_x}{\delta}$$

$$= \frac{1 - 0.2}{0.1}$$

$$= \frac{0.8}{0.1}$$

$$= \$8$$ ■

8. Using UDD, we get

$$ {}_{5|}\ddot{a}^{(12)}_{x:\overline{10|}} \overset{UDD}{=} \alpha(12)\,{}_{5|}\ddot{a}_{x:\overline{10|}} - \beta(12)\left[v^5\,{}_5p_x - v^{15}\,{}_{15}p_x\right] $$

$$ = \alpha(12)\,{}_{5|}\ddot{a}_{x:\overline{10|}} - \beta(12)\left[{}_5E_x - {}_{15}E_x\right] $$

store all these in memory of calculator

Now $i^{(12)} = 12\left(1.1^{1/12} - 1\right) = 0.0956897\ldots$

$d^{(12)} = 12\left[1 - \left(\frac{1}{1.1}\right)^{1/12}\right] = 0.0949327\ldots$

So $\alpha(12) = \dfrac{id}{i^{(12)}d^{(12)}} = \dfrac{(0.1)\left(\frac{0.1}{1.1}\right)}{(0.095\ldots)(0.0949\ldots)} = 1.000752$

and $\beta(12) = \dfrac{i - i^{(12)}}{i^{(12)}d^{(12)}} = \dfrac{0.1 - 0.0956897\ldots}{(0.09568\ldots)(0.0949\ldots)} = 0.474491$

Thus,

$$ {}_{5|}\ddot{a}^{(12)}_{x:\overline{10|}} = (1.000752)(5.3) - (0.474491)(0.60 - 0.20) $$

$$ = \$5.12 \quad \blacksquare $$