Chapter 4 Sample Exam Solutions

1. (a) $f' (x)$ is always $-1$ or $1$, so no such $c$ exists.
   (b) There is no contradiction, because $f$ is not differentiable in the relevant interval.

2. $f'' (x) = 0$ at $a$, $f'' (x)$ does not exist at $b$ (vertical tangent), $f'' (x)$ does not exist at $c$ (cusp)

3. $F (x) = \begin{cases} \cos x & \text{if } -\pi \leq x \leq 0 \\ -\cos x + 2 & \text{if } 0 < x \leq \pi \end{cases}$

4. (a) Vertical: $x = -3, x = 1$. Horizontal: $y = 0$
   (b) Vertical: $x = 3, x = -2$. Horizontal: $y = -4$
   (c) The numerator factors as $(x - 1) (x + 2) (x + 1)$. Vertical: $x = 2$. Slant: $y = x + 2$
   (d) Vertical: $x = 1$. Horizontal: $y = 1$
   (e) Vertical: $x = \pm \sqrt{2}$. Slant: $y = x$ (as $x \to \infty$) and $y = -x$ (as $x \to -\infty$)

5. Let $m$ be the slope of the line. Then the equation of the line is $y = mx - 3m + 2$ and the area of the triangle is $\frac{1}{2}bh = \frac{1}{2} \left( 3 - \frac{2}{m} \right) (-3m + 2) = -\frac{1}{2} \frac{(3m - 2)^2}{m}$. The minimum area occurs when $m = -\frac{2}{3}$, giving an area of 12. There is no maximum area, because the area increases without bound as $m$ gets close to zero.

6. (a) Always true, by the Mean Value Theorem
   (b) Never true, since $f''$ is never zero or undefined.
   (c) Sometimes true. For example, it is true for $f (x) = -(x - 5)^2$, but false for $f (x) = -(x - 2)^2$.
   (d) Always true, because $f'$ is always decreasing.

7. (a) With an initial approximation of $x = -0.5$, the successive approximations $x_n$ will approach the root at $x = -1$.
   (b) The horizontal tangent near $x = 0.5$ will result in successive approximations $x_n$ which oscillate about the minimum at $x \approx 1.2$.

8. (a) $f$ must be continuous on $[1, 3]$ and differentiable on $(1, 3)$.
   (b) (i) $\lim_{x \to 0^+} \frac{f(x + h) - f(x)}{h} = 0$ and $\lim_{x \to 0^-} \frac{f(x + h) - f(x)}{h} = 0$. Because the left- and right-hand derivatives exist and are equal, the derivative exists.
   (ii) The Mean Value Theorem states that there exists $c \in (-10, 1)$ for which
       $$f'(c) = \frac{f(1) - f(-10)}{1 - (-10)} = \frac{1}{11} - \frac{0}{11} = \frac{1}{11}.$$ Now for $x > 0$, $f'(x) = 2x$. So we set $2c = \frac{1}{11} \iff c = \frac{1}{22}$
9. The first can’t be the derivative, because it is zero for \([-1, 0]\), whereas \(g'(0) > g'(-1)\). The second can’t be the derivative, because the MVT guarantees a \(c\) in \((-1, 0)\) such that \(g'(c) = 1\). (This answer works for the first graph too.)

10. \(f(x) = x^3 + ax^2 + bx + c \Rightarrow f'(x) = 3x^2 + 2ax + b \Rightarrow f''(x) = 6x + 2a\)

(a) Concave up on \((-\frac{1}{3}a, \infty)\); concave down on \((-\infty, -\frac{1}{3}a)\)

(b) \(x = -\frac{1}{3}a\) is the only candidate, and the concavity does change there.

(c) We have \(0 = -\frac{1}{3}a\), giving that \(a = 0\). We have \(-2 = c\) from the first equation. \(f'(x) = 3x^2 + b\). Since \(b > 0\), there is no critical point.

11. (a) Answers will vary. Look for a graph that is increasing and concave up until \(x = \frac{1}{2}\), and then increasing and concave down.

(b) Answers will vary. Look for a graph with a local minimum at \(x = 0\) and concave up until \(x = \frac{1}{2}\), and then concave down.

(c) Answers will vary. Look for a graph that is decreasing and concave up until \(x = \frac{1}{2}\), and then decreasing and concave down.

12. (a) \(\frac{dL}{dt} = \frac{2}{3\sqrt{h}} \cdot \frac{dh}{dt}\). If \(t\) is in years, \(\frac{dh}{dt} = 5\), so \(\frac{dL}{dt} = \frac{10}{3\sqrt{h}}\) in yr. If \(t\) is in days, \(\frac{dh}{dt} = \frac{5}{365} = \frac{1}{73}\) in/yr,

\(\frac{dL}{dt} = \frac{2}{219\sqrt{h}}\) in/day.

(b) \(\frac{dL}{dt}\) decreases with time, so the fastest rate of growth occurs when \(t = 0\), and that rate is \(\frac{5}{12}\) in/year or \(\frac{1}{876}\) in/day.

13. \(F'(x) = \frac{1}{1 + x^4} + A\), so to find critical points we need to solve the equation \(\frac{1}{1 + x^4} + A = 0\).

(a) If \(A = 2\), then the equation \(\frac{1}{1 + x^4} + A = 0\) has no real solution.

(b) If \(A = -1\), then the equation \(\frac{1}{1 + x^4} + A = 0\) has one real solution, namely \(x = 0\).

(c) If \(A\) is any number between \(-1\) and \(0\), then the equation \(\frac{1}{1 + x^4} + A = 0\) has the two real solutions \(x = \pm \sqrt[4]{-1 - \frac{1}{A}}\).
14. (a) \( x_{n+1} = x_n - \frac{x_n^3 - a}{3x_n^2} \)

(b) \( x_{n+1} = x_n - \frac{x_n^3 - a}{3x_n^2} = \frac{3x_n^2}{3x_n^2} - \frac{x_n^3 - a}{3x_n^2} = \frac{2x_n^3 + a}{3x_n^2} - \frac{1}{3} (2x_n^2 + a) \)

(c) \( x_0 = 3 \Rightarrow x_1 \approx 3.074074074 \Rightarrow x_2 \approx 3.072317830 \Rightarrow x_3 \approx 3.072316826 \)

(d) \( 3.072316826 - \sqrt{29} \approx 3.141527067 \times 10^{-10} \) (answers may vary)

15. The volume of the conical part is \( \frac{1}{3} \pi 10^2 (20) - \frac{1}{3} \pi 5^2 (10) = \frac{1750}{3} \pi \). The volume of liquid when the height is \( h > 10 \) is \( \frac{1750}{3} \pi + 25 \pi (h - 10) \).

(a) \( 2 = 25 \pi \frac{dh}{dt} \Rightarrow \frac{dh}{dt} = \frac{2}{25 \pi} \) cm/s

(b) \( \frac{1750}{3} \pi \) cm\(^3\) \cdot \frac{1}{3} \) s/cm\(^3\) \cdot \frac{875}{3} \) s

(c) For \( h < 10 \), the volume is \( V = \frac{1}{3} \pi 10^2 (20) - \frac{1}{3} \pi (10 - \frac{1}{2}h)^2 (20 - h) = 100 \pi h - 5 \pi h^2 + \frac{1}{12} \pi h^3 \).

Taking derivatives with respect to time, we get \( 2 = \frac{dV}{dt} = 100 \pi \frac{dh}{dt} - 10 \pi h \frac{dh}{dt} + \frac{1}{4} \pi h^2 \frac{dh}{dt} \). When \( h = 5 \), \( \frac{dh}{dt} (100 \pi - 50 \pi + \frac{25}{4} \pi) = 2 \Rightarrow \frac{dh}{dt} = \frac{2}{100 \pi - 50 \pi + \frac{25}{4} \pi} \approx 1.132 \times 10^{-2} \).

16. (a) Answers will vary. Look for a smooth graph going through the indicated points.

(b) Apply the Mean Value Theorem to the interval \((0, 5)\) to show the existence of \( a \), and then again to \((5, 10)\) to show the existence of \( b \).

(c) Apply the Intermediate Value Theorem to \( f' \) using \( a \) and \( b \) to show the existence of \( c \). For another solution, use the Intermediate Value Theorem to show that \( f (\ell) = 3 \) for some \( \ell \) with \( 5 < \ell < 10 \), and then \( f' (c) = 0 \) using the Mean Value Theorem on the interval \([0, \ell]\).

17. (a) Minima at \( x = -5 \), 0, and 5; maxima at \( x = -4 \) and 4

(b) \( g \) is concave up when its derivative is increasing; that is, on \((-6, 4.5)\), \((2, 2)\), and \((4.5, 6)\).

(c) \( g \) is larger than \( g \) (2), because the function is increasing on \((2, 4)\).

18. (a) The numerator of \( f \) doesn’t contain a factor of \( x - 2 \), so there is a vertical asymptote at \( x = 2 \). There is a slant asymptote at \( y = 2x - 1 \).

(b) \( 2x^2 - 5x + 5 = 0 \) has no real solution, so \( f \) has no root.

(c) Solve \( 2x^2 - 8x + 5 = 0 \) to find critical points at \( x = 2 + \frac{1}{2} \sqrt{6} \) and at \( x = 2 - \frac{1}{2} \sqrt{6} \). The Second Derivative Test gives that the first is a minimum, the second a maximum.

(d) There is no inflection point, but the concavity changes at \( x = 2 \), where the function is undefined.)

(e) [Diagram]
19. No, it does not. There is a second vertical asymptote at $x = 100$.

20. (a) Yes. $f'$ can be negative, and $f(0) = 0$, so $f$ can decrease for $x \geq 0$.

(b) Yes. $f'$ can be positive, and $f(0) = 0$, so $f$ can be negative at $x = -3$.

(c) $f(3)$ will have its largest possible value if $f'(x) = 5$ for $0 < x < 3$. So $f(3) \leq 15$. $f(-3)$ will have its largest possible value if $f'(x) = -1$ for $0 < x < 3$. So $f(-3) \leq 3$.

(d) No, it need not have a critical point. For example, let $f(x) = x$.

21. (a) $g'(x) = 2 \sin \frac{x}{2} \cos \frac{x}{2}$

(b) $g'(x) = \sin x$

(c) $g(x) = -\cos x$

22. Answers will vary; the following are samples only.

(a) $f(x) = x^2$, $g(x) = x$

(b) $f(x) = 6x$, $g(x) = x$

(c) $f(x) = x$, $g(x) = x^2$

(d) This is not possible. For $\lim_{x \to \infty} \frac{f(x)}{g(x)} = -1$, either $f$ or $g$ would have to be negative for large $x$. This contradicts the assumption that $\lim_{x \to \infty} f(x) = \lim_{x \to \infty} g(x) = \infty$. 

256