EXAM 3—Wednesday, March 19, 2003

** As always, you will be graded on the quality of your exposition not just the correctness of your answers. **
Read each question carefully and do the easier ones first. GOOD LUCK!

1. (15 points) Let \( f(x) = 2^x \).

(a) Carefully sketch the graph of \( f \). Sketch asymptotes with dotted lines and label with their equations. Plot and label (with their ordered pairs) at least five points on the graph.

(b) Give the formula for \( f^{-1} \), the inverse of \( f \).

\[
 f^{-1}(x) = \log_2 x
\]

(c) Carefully sketch the graph of \( f^{-1} \) on the same coordinate system. Sketch asymptotes with dotted lines and label with their equations. Plot and label (with their ordered pairs) at least five points on the graph.

2. (10 points) Let \( y = \ln x \). Show that \( \frac{dy}{dx} = \frac{1}{x} \) by using the fact that \( \frac{d}{dx} e^x = e^x \).

\[
\begin{align*}
\text{If} & \quad y = \ln x, \text{ then} \\
& \quad e^y = x. \quad \text{So}
\end{align*}
\]

\[
\frac{d}{dx} e^y = \frac{d}{dx} x
\]

\[
\frac{d}{dx} e^y = 1
\]

\[
\frac{dy}{dx} = \frac{1}{e^y}
\]

\[
\frac{dy}{dx} = \frac{1}{x}. \quad \text{Since} \quad e^y = x.
\]
3. (20 points) Evaluate each limit. Show your reasoning.

(a) \[ \lim_{x \to \infty} \frac{\ln x}{\sqrt[3]{x}} = \lim_{x \to \infty} \frac{\frac{1}{x}}{\frac{1}{3} x^{-\frac{2}{3}}} = \lim_{x \to \infty} \frac{3}{x^{\frac{1}{3}}} = 0. \]

(b) \[ \lim_{x \to 0} \frac{\sec x - 1}{x^2} = \lim_{x \to 0} \frac{\frac{d}{dx} (\sec x - 1)}{\frac{d}{dx} x^2} = \lim_{x \to 0} \frac{\sec x \tan x}{2x}. \]

\[ = \lim_{x \to 0} \frac{\sec x \tan x + \sec x \tan x}{2} = \frac{\sec x \tan x}{2} \]

\[ = \frac{\sec 0 \tan 0}{2} = \frac{1}{2}. \]

4. (10 points)

(a) Complete the statement of Rolle’s Theorem.

Suppose \( f \) is a function satisfying the following hypotheses.

i. \( f \) is continuous on the closed interval \([a, b]\);

ii. \( f \) is differentiable on the open interval \((a, b)\);

iii. \( f(a) = f(b) \).

Then there exists some \( c \in (a, b) \) such that \( f'(c) = 0 \).

(b) Sketch an example to illustrate that the conclusion of Rolle’s Theorem need not be true if hypotheses (ii) and (iii) hold, but (i) does not.

![Graph showing an example](image-url)
5. (15 points) Use calculus to find the absolute maximum and absolute minimum values of the function \( f(x) = x^3 - x^2 - x + 1 \) on the interval \([-1, 3]\). Give exact values. Show your reasoning clearly.

\[
\begin{align*}
\text{Locate critical points:} & \\
f'(x) &= 3x^2 - 2x - 1 \\
&= (3x+1)(x-1) \\
\text{Domain of } f' & = \mathbb{R} \\
f'(x) &= 0 \\
(3x+1)(x-1) &= 0 \\
x &= -\frac{1}{3}, \quad x = 1 \\
\text{Both are in } [-1, 3].
\end{align*}
\]

\[
\begin{align*}
\text{Test at endpoints:} & \\
f(-1) &= (-1)^3 - (-1)^2 - (-1) + 1 = 0 \\
f(3) &= 3^3 - 3^2 - 3 + 1 \\
&= 27 - 9 - 3 + 1 = 16 \\
\end{align*}
\]

\[
\begin{align*}
\text{Critical points:} & \\
f(-\frac{1}{3}) &= (-\frac{1}{3})^3 - (-\frac{1}{3})^2 - (-\frac{1}{3}) + 1 \\
&= -\frac{1}{27} - \frac{1}{9} + \frac{1}{3} + 1 \\
&= -\frac{1}{27} - \frac{9}{27} + \frac{9}{27} = \frac{32}{27} \\
f(1) &= 1^3 - 1^2 - 1 + 1 = 0 \\
\end{align*}
\]

The absolute maximum value of \( f \) on \([-1, 3]\) is \( 16 \) and it occurs at \( x = 3 \).

The absolute minimum value of \( f \) on \([-1, 3]\) is \( 0 \) and it occurs at \( x = -\frac{1}{3} \).

6. (15 points) Let \( f(x) = \frac{5x^2 - 9x - 2}{4x^2 - 1} \).

(a) Give the ordered pairs for all intercepts of \( f \). (Write "none" if there aren't any.) Show your reasoning clearly.

\[
\begin{align*}
\text{y-intercept(s) of } f & : (0, 2) \\
\text{x-intercept(s) of } f & : \left(-\frac{1}{5}, 0\right), (2, 0)
\end{align*}
\]

(b) Give the equation of the line for all asymptotes of \( f \). (Write "none" if there aren't any.) Show the limit(s) to verify that your horizontal asymptote(s) is(are) correct.

\[
\begin{align*}
\text{Vertical asymptote(s): } x = \frac{1}{2}; \quad x = -\frac{1}{2} \\
\text{Horizontal asymptote(s): } y = \frac{5}{4}
\end{align*}
\]
7. (15 points) Suppose the function $h$ has the following properties.

$h(-3) = -2$  \hspace{1cm} h'(-3) = 0  \hspace{1cm} h'(2) = 0  \hspace{1cm} h'(-1) = 0  \hspace{1cm} h''(2) = 0$

(a) Fill in the charts with the phrases “increasing”, “decreasing”, “concave up(3,4),(997,994)”, or “concave down”.

<table>
<thead>
<tr>
<th>interval</th>
<th>$(-\infty, -3)$</th>
<th>$(-3, 2)$</th>
<th>$(2, \infty)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$h'(x)$</td>
<td>$-$</td>
<td>$+$</td>
<td>$+$</td>
</tr>
<tr>
<td>$h$ is</td>
<td>decreasing</td>
<td>increasing</td>
<td>increasing</td>
</tr>
</tbody>
</table>

<table>
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<tr>
<td>$h''(x)$</td>
<td>$+$</td>
<td>$-$</td>
<td>$+$</td>
</tr>
<tr>
<td>$h$ is</td>
<td>concave up</td>
<td>concave down</td>
<td>concave up</td>
</tr>
</tbody>
</table>

(b) Give the $x$-coordinates for all of the following. (Write “none” if there aren’t any.)

critical points of $f$: $x = -3, 2$  \hspace{1cm} local maximum points of $f$: none

inflection points of $f$: $x = -1, 2$  \hspace{1cm} local minimum points of $f$: $x = -3$

c) Sketch a possible graph of the function $h$. Indicate all critical points and inflection points as well as other important behavior very clearly.