EXAM 4—Friday, April 11, 2003

There are 6 questions at 20 points each. Do any 5 of the 6 questions for full credit. You may do the 6th for extra credit. Suggestion: Read each question carefully and do the easier ones first. GOOD LUCK!

1. (20 points) A farmer wishes to fence an area of 6,000,000 square feet in a rectangular field and then divide it in half with a fence parallel to one of the sides of the rectangle. How can he do this so as to minimize the amount of fencing used?

   Sketch and label several possible diagrams. Introduce your variables with "let" statements. Show your reasoning, especially the calculus! Be sure to prove that you really have found the minimum point on the appropriate interval.
2. (20 points) The cost function for producing Magic Glow-in-the-Dark Widgets is \( C(x) = 25,000 + 120x + 0.1x^2 \) at a production level of \( x \) Widgets.

(a) Find a formula for the average cost (use standard notation).

(b) Find a formula for the marginal cost (use standard notation).

(c) Find the cost, average cost, and marginal cost at a production level of 1000 Magic Glow-in-the-Dark Widgets.

(d) Find the production level that will minimize the average cost.

(e) Find the minimum average cost.
3. (20 points)

(a) Find the most general antiderivative $F$ of $f(x) = x^5 + x^{2/3} - \frac{1}{x} + \frac{7}{x^2} + 82$.

(b) Find $f$ where $f''(x) = x^2 + 2x$, $f(0) = 5$ and $f(2) = 1$.

(c) A particle is moving with velocity $v(t) = \sin t + \cos t$ meters/second and initial position $s(0) = 0$ meters. Find the position function $s(t)$ for the particle.
4. (20 points) Let $f(x) = \sqrt{x}$.

(a) Find a formula for the differential $dy$.

(b) Now also assume that $x = 1$ and $\Delta x = dx = 2$. The graph $y = f(x)$ and its tangent line at $x = 1$ are shown.

i. Sketch and label line segments with their lengths using the expressions $\Delta x$, $dy$, and $\Delta y$.

ii. Compute $dy$.

iii. Use differentials (your work in the previous parts) or local linearization to approximate $\sqrt{3}$.

iv. Explain how you can tell from the graph and work above alone whether your approximation is an over- or underestimate for $\sqrt{3}$. 
5. (20 points) Let $A$ be the area under the graph of $f(x) = \frac{1}{x}$ from $x = 1$ to $x = 2$.

(a) Estimate $A$ using four approximating rectangles and left endpoints. (That is, find $L_4$.)

(b) Sketch the graph and the rectangles.

(c) Is your estimate an under- or an overestimate? Explain how you know.

(d) Now suppose that an object is moving with velocity $v(t) = \frac{1}{t}$ meters/second for $t \geq 1$ second. Explain why your answer to part (a) is an estimate of the displacement of the object from $t = 1$ to $t = 2$ seconds.
6. (20 points) Each of these should be done **WITHOUT** computing limits. (Use geometry or properties of integrals where necessary.)

(a) (5 pts) Express as a definite integral. (Do not evaluate.) \[ \lim_{n \to \infty} \sum_{i=1}^{n} \sqrt{x_i^+} \Delta x \text{ on } [1, 9]. \]

(b) (5 pts) Evaluate by interpreting in terms of areas. \[ \int_{0}^{3} 2x \, dx \]

(c) (2 pts) Evaluate. \[ \int_{2}^{3} \sin^2 \theta \, d\theta \]

(d) (2 pts) Given that \[ \int_{1}^{9} \sqrt{x} \, dx = \frac{52}{3}, \] find \[ \int_{9}^{1} \sqrt{u} \, du. \]

(e) (2 pts) If \[ \int_{0}^{3} f(x) \, dx = 38 \] and \[ \int_{3}^{7} f(x) \, dx = 20, \] find \[ \int_{0}^{3} f(x) \, dx \]

(f) (2 pts) If \[ \int_{-2}^{1} f(x) \, dx = 3.1 \] and \[ \int_{-2}^{1} g(x) \, dx = 10.8, \] find \[ \int_{-2}^{1} 2f(x) - g(x) \, dx \]

(g) (2 pts) If \[ -2 \leq f(x) \leq 10 \] for all \( x \in [3, 9] \), find a lower bound and an upper bound for the value of \[ \int_{3}^{9} f(x) \, dx. \]