EXAM 2—Wednesday, February 22, 2006

Do the easier questions first. Read the directions carefully. GOOD LUCK!

1. (15 points) Find the derivative \( f'(x) \) using the formal (limit) definition of derivative where \( f(x) = \sqrt{2-3x} \).
Write out all steps completely. Simplify your answer.

\[
\frac{f'(x)}{h} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}
\]

\[
= \lim_{h \to 0} \frac{\sqrt{2-3(x+h)} - \sqrt{2-3x}}{h}
\]

\[
= 5 \lim_{h \to 0} \frac{\sqrt{2-3x-3h} - \sqrt{2-3x}}{h} \left[ \frac{\sqrt{2-3x-3h} + \sqrt{2-3x}}{\sqrt{2-3x-3h} + \sqrt{2-3x}} \right]
\]

\[
= \lim_{h \to 0} \frac{2-3x-3h - (2-3x)}{h} \left[ \frac{\sqrt{2-3x-3h} + \sqrt{2-3x}}{\sqrt{2-3x-3h} + \sqrt{2-3x}} \right]
\]

\[
= \lim_{h \to 0} \frac{-3h}{\sqrt{2-3x-3h} + \sqrt{2-3x}}
\]

\[
= \lim_{h \to 0} \frac{-3}{\sqrt{2-3x-3h} + \sqrt{2-3x}}
\]

\[
= -3 \frac{2-3x}{\sqrt{2-3x-3h} + \sqrt{2-3x}}
\]

\[
= -3 \frac{2-3}{2 \sqrt{2-3x}}
\]
2. (5 points) Circle TRUE or FALSE for each statement.

(a) TRUE or FALSE: If a function $f$ is continuous at $a$, then $f$ is differentiable at $a$.

(b) TRUE or FALSE: If $f'(a)$ exists, then $\lim_{x \to a} f(x) = f(a)$.

(c) TRUE or FALSE: $\frac{d}{dx} |x^2 - 4| = |2x|$.

(d) TRUE or FALSE: If $f$ and $g$ are differentiable and $g(x) \neq 0$, then $\frac{d}{dx} \left[ \frac{f(x)}{g(x)} \right] = \frac{f'(x)}{g'(x)}$.

(e) TRUE or FALSE: $\frac{d}{dx} (\tan^2 x) = \frac{d}{dx} (\sec^2 x)$. Actually compute each!

3. (10 points) Suppose a particle moves in a straight line so its position after $t$ seconds is $s(t) = t^3 - 8t + 5$ meters for $t > 0$.

(a) What is the particle's position after 2 seconds? Include units with your answer.

$$s(2) = 2^3 - 8(2) + 5 = 8 - 16 + 5 = -3 \text{ m}.$$ 

(b) Find the velocity $v(t)$ of the particle after $t$ seconds. Include units with your answer.

$$v(t) = s'(t) = 3t^2 - 8 \text{ m/s}.$$ 

(c) Find the velocity $v(2)$ of the particle after 2 seconds. Include units with your answer.

$$v(2) = 3(2)^2 - 8 = 12 - 8 = 4 \text{ m/s}.$$ 

(d) Find the acceleration $a(t)$ of the particle after $t$ seconds. Include units with your answer.

$$a(t) = v'(t) = 6t \text{ m/s}^2.$$ 

(e) Find the acceleration $a(2)$ of the particle after 2 seconds. Include units with your answer.

$$a(2) = 6(2) = 12 \text{ m/s}^2.$$
4. (10 points) Use the definition of derivative to prove the Subtraction Rule for Derivatives. That is, prove that if \( f \) and \( g \) are differentiable functions, then \( (f - g)'(x) = f'(x) - g'(x) \).

\[
(f - g)'(x) = \lim_{h \to 0} \frac{(f - g)(x + h) - (f - g)(x)}{h} \\
= \lim_{h \to 0} \frac{f(x + h) - g(x + h) - [f(x) - g(x)]}{h} \\
= \lim_{h \to 0} \frac{f(x + h) - g(x + h) - f(x) + g(x)}{h} \\
= \lim_{h \to 0} \frac{f(x + h) - f(x) - [g(x + h) - g(x)]}{h} \\
= \lim_{h \to 0} \left[ \frac{f(x + h) - f(x)}{h} - \frac{g(x + h) - g(x)}{h} \right] \\
= \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} - \lim_{h \to 0} \frac{g(x + h) - g(x)}{h} \\
= f'(x) - g'(x).
\]

5. (10 points) Let \( f(x) = \frac{1}{x^2} \). Use proper mathematical notation to indicate what you are computing for each of the following.

(a) Find the \( y \)-coordinate of the point on the curve \( y = f(x) \) where \( x = -3 \).
\[ f'(-3) = \frac{1}{(-3)^2} = \frac{1}{9} \]

(b) Find \( f'(x) \).
\[ f'(x) = -2x^{-3} = \frac{-2}{x^3} \]

(c) Find the slope \( m \) of the line tangent to the curve \( y = f(x) \) at the point where \( x = -3 \).
\[ m = f'(-3) = \frac{-2}{(-3)^3} = \frac{2}{27} \]

(d) Find an equation of the line tangent to the curve \( y = f(x) \) at the point where \( x = -3 \); write it in the form \( y = mx + b \) with exact coefficients, simplified.
\[
y - y_1 = m(x - x_1) \\
\frac{y}{4} - \frac{1}{4} = \frac{2}{27} (x - (-3)) \\
y - \frac{1}{4} = \frac{2}{27} x + \frac{2}{9} \\
y = \frac{2}{27} x + \frac{2}{9} + \frac{1}{9} \\
y = \frac{2}{27} x + \frac{1}{3}.
\]
6. (5 points) Find $D^{58} \sin(2x)$ by finding the first few derivatives and observing the pattern which occurs.

\[
\begin{align*}
D \sin(2x) &= 2 \cos(2x) \\
D^2 \sin(2x) &= -2^2 \sin(2x) \\
D^3 \sin(2x) &= -2^3 \cos(2x) \\
D^4 \sin(2x) &= 2^4 \sin(2x)
\end{align*}
\]

So, $D^{58} \sin(2x) = -2^{58} \sin(2x)$.

7. (10 points) Find $\frac{dy}{dx}$ by implicit differentiation where $x^2 - 2xy + 4y^2 = 6$. Simplify.

\[
\begin{align*}
\frac{d}{dx} (x^2 - 2xy + 4y^2) &= \frac{d}{dx} (6) \\
2x - [2y + 2x \frac{dy}{dx}] + 8y \frac{dy}{dx} &= 0 \\
2x - 2y - 2x \frac{dy}{dx} + 8y \frac{dy}{dx} &= 0 \\
-2x \frac{dy}{dx} + 8y \frac{dy}{dx} &= 2y - 2x \\
(x - 2) \frac{dy}{dx} &= 2y - 2x \\
2 [4y - x] \frac{dy}{dx} &= 2[y - x] \\
\frac{dy}{dx} &= \frac{y - x}{4y - x}
\end{align*}
\]
8. (10 points) For what values of x does the graph of \( f(x) = \sin^2 x + x \) have a horizontal tangent line? Give exact values of all solutions, in radians.

\[
\begin{align*}
2. \quad f'(x) &= 2\sin x \cos x + 1 \\
\frac{d}{dx} \sin x &= \cos x \\
\frac{d}{dx} \cos x &= -\sin x \\
2. \quad 2\sin x \cos x + 1 &= 0 \\
2\sin x \cos x &= -1 \\
2 \sin 2x &= -1 \\
\sin 2x &= -\frac{1}{2} \\
2x &= \frac{3\pi}{2} + 2\pi k, \quad \text{where } k \in \mathbb{Z}.
\end{align*}
\]

9. (10 points)
(a) \( \frac{d}{dx} \sin x = \cos x \); \( \frac{d}{dx} \cos x = -\sin x \).

(b) Use the above facts, basic trigonometric identities, and the product, quotient, and/or chain rules to prove that \( \frac{d}{dx} \csc x = -\csc x \cot x \). Write out all steps.

\[
\begin{align*}
\frac{d}{dx} \csc x &= \frac{d}{dx} \left( \sin x \right)^{-1} \\
&= -\left( \sin x \right)^{-2} \cos x \\
&= -\frac{\cos x}{\sin^2 x} \\
&= -\frac{1}{\sin^2 x} \cdot \cot x \\
&= -\csc x \cot x.
\end{align*}
\]
10. (15 points) A spotlight on the ground shines on a wall 25 m away. A man 2 m tall walks from the spotlight toward the building at a speed of 1.5 m/s. How fast is the length of his shadow on the building changing when he is 15 m from the building?

Introduce your variables (and their units) with “Let” statements. Sketch and label a diagram. Show your reasoning explicitly. Give both an exact answer and a decimal approximation, rounded to the nearest hundredth. Include units with your answer.

Let $x$ be distance from man to light in m.
but $s$ be length of shadow on bldg in m.
but $t$ be time in s.

Then given \( \frac{dx}{dt} = 1.5 = \frac{3}{2} \text{ m/s} \).

Want: \( \frac{ds}{dt} \) when $x = 25 - 15 = 10$.

By similar $\triangle$s, \[ \frac{s}{25} = \frac{2}{x} \]

So \[ \frac{d}{dt} \left( \frac{s}{25} \right) = \frac{d}{dt} \left( \frac{2}{x} \right) \]

\[ \frac{1}{25} \frac{ds}{dt} = -2x^{-2} \frac{dx}{dt} \]

\[ \frac{ds}{dt} = -\frac{50}{x^2} \frac{dx}{dt} \]

Thus, \[ \frac{ds}{dt} = -\frac{50}{x^2} \left( \frac{3}{2} \right) \]

\[ = -\frac{75}{x^2} \text{ m/s (true always)} \]

When he is 15 m from bldg, $x = 10$ and so

\[ \frac{ds}{dt} \bigg|_{x=10} = -\frac{75}{100} = -0.75 \text{ m/s}. \]
11. (15 points) Let \( f(x) = \sqrt{x} = x^{1/2} \)

(a) Find a formula for the differential \( dy \). Write out the steps.

\[
\frac{dy}{dx} = \frac{1}{2} x^{-1/2} = \frac{1}{2\sqrt{x}} \quad \text{so} \quad dy = \frac{1}{2\sqrt{x}} \, dx
\]

(b) Now also assume that \( x = 1 \) and \( \Delta x = dx = -0.7 \). Sketch the graph \( y = f(x) \) and its tangent line at \( x = 1 \).

(c) Sketch and label line segments with their lengths using the expressions \( \Delta x, \, dy, \, \text{and} \, \Delta y \).

(d) Evaluate \( dy \) for the given values of \( x \) and \( dx \).

\[ dy = \frac{1}{2\sqrt{x}} \, dx = \frac{-0.7}{2} = -0.35 \]

(e) Use differentials (your work in the previous parts) to approximate \( \sqrt{0.3} \). Write out the steps.

\[ \sqrt{0.3} \approx 1 + dy \]
\[ \approx 1 - 0.35 = 0.65 \]

(f) Is your approximation an over- or underestimate for \( \sqrt{0.3} \)? Explain how you can tell this from the graph and work above alone. \text{It is an overestimate since the tangent line lies above the curve. (so } \sqrt{0.3} < 0.65 \text{)} \]