EXAM 2—Tuesday, February 19, 2008

Do the easier questions first. Read the directions carefully. Show your reasoning. GOOD LUCK!

1. (15 points) Find the derivative $f'(x)$ using the definition of derivative where $f(x) = \frac{2}{3x}$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{2}{3(x+h)} - \frac{2}{3x}}{h}$$

$$= \lim_{h \to 0} \frac{3x(x+h) \left[ \frac{2}{3(x+h)} - \frac{2}{3x} \right]}{3x(x+h)h}$$

$$= \lim_{h \to 0} \frac{2x - 2(x+h)}{3hx(x+h)}$$

$$= \lim_{h \to 0} \frac{2x - 2x - 2h}{3hx(x+h)}$$

$$= \lim_{h \to 0} \frac{-2h}{3hx(x+h)}$$

$$= \lim_{h \to 0} \frac{-2}{3x(x+h)}$$

$$= \frac{-2}{3x(x+0)}$$

$$= \frac{-2}{3x^2}$$
2. (2 points) Circle True or False for each statement.

(a) **True** or **False**: If a function $f$ is differentiable at $a$, then $f$ is continuous at $a$. 
- **e.g.** $f(x) = |x|$ at $x = 0$

(b) **True** or **False**: If a function $f$ is continuous at $a$, then $f$ is differentiable at $a$.

3. (10 points) Suppose a particle moves in a straight line so its position after $t$ seconds is $s(t) = t^3 - 5t^2$ meters for $t \in [0, 20]$.

(a) What is the particle's position after 10 seconds? **Include units with your answer.**

$$s(10) = 10^3 - 5(10)^2$$
$$= 1000 - 500$$
$$= 500 \text{ meters}$$

(b) Find the velocity $v(t)$ of the particle after $t$ seconds. **Include units with your answer.**

$$v(t) = s'(t)$$
$$= 3t^2 - 10t \frac{m}{s}$$

(c) Find the velocity $v(10)$ of the particle after 10 seconds. **Include units with your answer.**

$$v(10) = 3(10)^2 - 10(10)$$
$$= 300 - 100$$
$$= 200 \text{ m/s}$$

(d) Find the acceleration $a(t)$ of the particle after $t$ seconds. **Include units with your answer.**

$$a(t) = v'(t)$$
$$= 6t - 10 \frac{m}{s^2}$$

(e) Find the acceleration $a(10)$ of the particle after 10 seconds. **Include units with your answer.**

$$a(10) = 6(10) - 10$$
$$= 60 - 10$$
$$= 50 \text{ m/s}^2$$

4. (8 points) Find $\frac{d^4 y}{dx^4}$ where $y = \sqrt[3]{x^4}$. **Use Leibniz notation. Simplify your answer.**

$$y = \sqrt[3]{x^4} = x^{4/3}$$
$$\frac{dy}{dx} = \frac{4}{3} x^{1/3}$$
$$\frac{d^2 y}{dx^2} = \frac{4}{3} \left( \frac{1}{3} \right) x^{-2/3}$$
$$= \frac{4}{9} x^{-2/3}$$

$$\frac{d^3 y}{dx^3} = \frac{4}{9} \left( \frac{-2}{3} \right) x^{-5/3}$$
$$= \frac{-8}{27} x^{-5/3}$$

$$\frac{d^4 y}{dx^4} = \frac{-8}{27} \left( \frac{-5}{3} \right) x^{-8/3}$$
$$= \frac{40}{81} x^{-8/3}$$

**USE THE LAWS OF EXPONENTS!**
5. (10 points) Let \( f(x) = \frac{2}{x} \). Use proper mathematical notation to indicate what you are computing for each of the following.

(a) Find the \( y \)-coordinate of the point on the curve \( y = f(x) \) where \( x = 7 \).

\[ f(7) = \frac{2}{7} \]

(b) Find \( f'(x) \).

\[ f'(x) = 2(-1)x^{-2} \]

\[ = \frac{-2}{x^2} \]

(c) Find the slope \( m \) of the line tangent to the curve \( y = f(x) \) at the point where \( x = 7 \).

\[ m = f'(7) = \frac{-2}{7^2} = \frac{-2}{49} \]

(d) Find an equation of the line tangent to the curve \( y = f(x) \) at the point where \( x = 7 \). Write it in the form \( y = mx + b \) with exact coefficients, simplified.

\[
\begin{align*}
  y - \frac{2}{7} &= m(x - 7) \\
  y - \frac{2}{7} &= \frac{-2}{49}(x - 7) \\
  y - \frac{2}{7} &= \frac{-2}{49}x + \frac{2}{7}
\end{align*}
\]

\[ y = \frac{-2}{49}x + \frac{4}{7} \]

6. (10 points) Use the definition of derivative to prove the Difference Rule for Derivatives. That is, prove that if \( f \) and \( g \) are differentiable functions, then \( (f - g)'(x) = f'(x) - g'(x) \).

\[
(f - g)'(x) = \lim_{h \to 0} \frac{(f - g)(x+h) - (f - g)(x)}{h} \quad \text{defn of derivative}
\]

\[
= \lim_{h \to 0} \frac{f(x+h) - g(x+h) - [f(x)-g(x)]}{h} \quad \text{defn of } f-g
\]

\[
= \lim_{h \to 0} \frac{f(x+h) - g(x+h) - f(x) + g(x)}{h} \quad \text{algebra}
\]

\[
= \lim_{h \to 0} \frac{f(x+h) - f(x) - [g(x+h) - g(x)]}{h} \quad \text{algebra}
\]

\[
= \lim_{h \to 0} \left[ \frac{f(x+h) - f(x)}{h} - \frac{g(x+h) - g(x)}{h} \right] \quad \text{algebra}
\]

\[
= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} - \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} \quad \text{limit laws, dfct that } f, g \text{ are diff}
\]

\[ = f'(x) - g'(x) \quad \text{defn of derivative} \]
7. (10 points) Find \( \frac{dy}{dx} \) by implicit differentiation where \( y^3 = 1 + x^2y \). Simplify.

\[
\frac{dy}{dx} = \frac{d}{dx} (1 + x^2y) \\
3y^2 \frac{dy}{dx} = 0 + 2xy + x^2 \frac{dy}{dx}
\]

\[
3y^2 \frac{dy}{dx} - x^2 \frac{dy}{dx} = 2xy
\]

\[
(3y^2 - x^2) \frac{dy}{dx} = 2xy
\]

\[
\frac{dy}{dx} = \frac{2xy}{3y^2 - x^2}
\]

8. (10 points) For what values of \( x \) does the graph of \( f(x) = \sin^2 x + \sqrt{3} \cos x \) have a horizontal tangent line? Give exact values of all solutions, in radians.

\[
f'(x) = 2\sin x \cos x + \sqrt{3} (-\sin x)
\]

\[
= \sin x (2\cos x - \sqrt{3})
\]

Horizontal tangent line:

\[
f'(x) = 0
\]

\[
\sin x (2\cos x - \sqrt{3}) = 0
\]

\[
\sin x = 0 \quad \text{or} \quad 2\cos x - \sqrt{3} = 0
\]

\[
\sin x = 0 \quad \text{or} \quad 2\cos x = \sqrt{3}
\]

\[
\cos x = \frac{\sqrt{3}}{2}
\]

\[
\begin{align*}
\text{or} \quad x &= \frac{\pi}{6} + 2n\pi \\
\text{or} \quad x &= \frac{\pi}{3} + 2n\pi
\end{align*}
\]
9. (10 points)

(a) \( \frac{d}{dx} \sin x = \cos x \); \hspace{1cm} \frac{d}{dx} \cos x = -\sin x \);

(b) Use the above facts, basic trigonometric identities, and the product, quotient, and/or chain rules to prove that \( \frac{d}{dx} \sec x = \sec x \tan x \). Supply a reason for each step.

**Solution 1**

\[
\frac{d}{dx} \sec x = \frac{d}{dx} \left( \frac{1}{\cos x} \right)
= -\left( \frac{1}{\cos^2 x} \right) \frac{d}{dx} \cos x
= \frac{\sin x}{\cos^2 x}
= \sec x \tan x
\]

**Solution 2**

\[
\frac{d}{dx} \sec x = \frac{d}{dx} \left( \frac{1}{\cos x} \right)
= \frac{0 - (-\sin x)}{\cos^2 x}
= \frac{\sin x}{\cos^2 x}
= \sec x \tan x
\]

10. (15 points) A streetlight sits at the top of an 8-foot pole. A man 6 feet tall walks at the rate of 4.5 ft/sec away from the pole. How fast is the length of his shadow changing at the moment when he is 20 feet from the pole?

Introduce your variables (and their units) with “Let” statements. Sketch and label a diagram. Show your reasoning explicitly. Give both an exact answer and a decimal approximation, rounded to the nearest tenth. Include units with your answer.

Let \( x \) be the distance from man to pole, in ft.

Let \( s \) be length of shadow, in ft.

Let \( t \) be time, in sec.

Given: \( \frac{dx}{dt} = 4.5 \text{ ft/sec} \). To find: \( \frac{ds}{dt} \) when \( x = 20 \text{ ft} \).

By similar triangles,

\[
\frac{x + s}{8} = \frac{s}{6}
24(x + s) = 24 \left( \frac{5}{6} \right)
3x + 3s = 45
3x = 15
3x = s
\]

So

\[
\frac{ds}{dt} = \frac{d}{dt} \left( \frac{3x}{4} \right)
= \frac{3}{4} \frac{dx}{dt}
= \frac{3}{4} \times 4.5 \text{ ft/sec}
= 13.5 \text{ ft/sec}
\]

His shadow is lengthening at a rate of 13.5 ft/sec.
BONUS. (10 points)

(a) \( \lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1 \); \( \lim_{\theta \to 0} \frac{\cos \theta - 1}{\theta} = 0 \).

(b) \( \sin(\alpha + \beta) = \sin \alpha \cos \beta + \sin \beta \cos \alpha \).

(c) Use the above facts and the definition of derivative to prove that \( \frac{d}{dx} \sin x = \cos x \).

\[
\frac{d}{dx} \sin x = \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h} = \\
= \lim_{h \to 0} \frac{\sin x \cos h + \sin h \cos x - \sin x}{h} = \\
= \lim_{h \to 0} \frac{\sin x \cos h - \sin x + \cos x \sin h}{h} = \\
= \lim_{h \to 0} \left\{ \sin x \left[ \frac{\cosh h - 1}{h} \right] + \cos x \left[ \frac{\sin h}{h} \right] \right\} = \\
= \left[ \lim_{h \to 0} \sin x \right] \left[ \lim_{h \to 0} \frac{\cosh h - 1}{h} \right] + \left[ \lim_{h \to 0} \cos x \right] \left[ \lim_{h \to 0} \frac{\sin h}{h} \right] = \\
= \left[ \sin x \right] [0] + \left[ \cos x \right] [1] = \\
= \cos x. \quad \blacksquare