The Golden Property

Question 1. What numbers have the property that the square of the number is one more than the number?

Let x be the number. Then we wish to find all x such that x^2 is equal to $x+1$. That is, we wish to solve the equation

$$x^2 = x + 1$$

for x. Rearranging, we obtain

$$x^2 - x - 1 = 0,$$

a quadratic equation. The Quadratic Formula tells us that the solutions of an equation of the form

$$ax^2 + bx + c = 0$$

are given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

For the equation $x^2 - x - 1 = 0$, we have

$$\begin{aligned} a &= 1 \\ b &= -1 \\ c &= -1. \end{aligned}$$

So the solutions are

$$\begin{aligned} x &= \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-1)}}{2(1)} \\ &= \frac{1 \pm \sqrt{1+4}}{2} \\ &= \frac{1 \pm \sqrt{5}}{2}. \end{aligned}$$

Since $\sqrt{5} > 1$, one of these solutions is positive and the other is negative. Denote the positive solution by Φ, the Greek letter phi. That is,

$$\Phi = \frac{1 + \sqrt{5}}{2} = 1.6108339875 \cdots .$$

This number is referred to as the Golden Ratio, the Golden Mean, the Golden Section, or the Divine Proportion. Since ancient times, people have believed that it is imbued with mystery and beauty.

Remember where Φ came from: it was a solution to the equation

$$x^2 = x + 1,$$

which means that

$$\boxed{\Phi^2 = \Phi + 1.}$$

This is the ***Golden Property***. Thus, we have the square of Φ expressed in terms of just plain Φ.

Question 2. Can we express Φ^3 in terms of just plain Φ?

Start with the Golden Property and then multiply both sides by Φ:

$$\begin{aligned}
\Phi^2 &= \Phi + 1 \\
\Phi \cdot \Phi^2 &= \Phi \cdot (\Phi + 1) \\
\Phi^3 &= \Phi^2 + \Phi
\end{aligned}$$

Now use the Golden Property again to replace Φ^2 with $\Phi + 1$:

$$\begin{aligned}
\Phi^3 &= \Phi^2 + \Phi \\
\Phi^3 &= (\Phi + 1) + \Phi \\
\Phi^3 &= 2\Phi + 1.
\end{aligned}$$

Great! We did it. Can we go a step further?

Question 3. Can we express Φ^4 in terms of just plain Φ?

You can do this, but I'll get you started. Start with the formula we just derived and then multiply both sides by Φ:

$$\begin{aligned}
\Phi^3 &= 2\Phi + 1 \\
\Phi \cdot \Phi^3 &= \Phi \cdot (2\Phi + 1)
\end{aligned}$$

Now use that formula to answer the following question.

Question 4. Can we express Φ^5 in terms of just plain Φ?

And keep going!

Question 5. Can we express Φ^6 in terms of just plain Φ?

Question 6. Can we express Φ^7 in terms of just plain Φ?