The Golden Ratio, φ and $-\frac{1}{\varphi}$

Solve $x^2 = x+1$ for x:

$$x^2 - x - 1 = 0 \qquad a=1,\ b=-1,\ c=-1$$

By the quadratic formula, the solutions are

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$= \frac{-(-1) \pm \sqrt{(-1)^2-4(1)(-1)}}{2(1)}$$

$$= \frac{1 \pm \sqrt{5}}{2}$$

We define $\varphi = \frac{1+\sqrt{5}}{2}$.

Now consider $-\frac{1}{\varphi} = -\frac{2}{1+\sqrt{5}}$

$$= \frac{-2(1-\sqrt{5})}{(1+\sqrt{5})(1-\sqrt{5})}$$ rationalize the denom

$$= \frac{-2(1-\sqrt{5})}{1^2-\sqrt{5}^2}$$

$$= \frac{-2(1-\sqrt{5})}{1-5}$$

$$= \frac{-2(1-\sqrt{5})}{-4}$$

$$= \frac{1-\sqrt{5}}{2}$$

which is, lo and behold!, the second solution to the equation $x^2 = x+1$. So φ and $-\frac{1}{\varphi}$ are both solutions to $x^2 = x+1$. ∎