MATH 11008 | Explorations in Modern Mathematics | Fall 2014

**Circle one: 9:55 / 12:05** | Dr. Kracht

**Print Name:** KEY | Exam Score: /100 *(105 pts available)*

## Exam 2: Version A

$F_n$ refers to the $n^{th}$ Fibonacci number and $\Phi = \frac{1+\sqrt{5}}{2} \approx 1.618$ is the golden ratio.

### Part I: Long Answer.

*No credit for answers without sufficient justification. Use standard mathematical notation correctly.*

1. (7 pts) Write out the first fifteen terms of the Fibonacci Sequence ($F_1$, $F_2$, $F_3$, ..., $F_{15}$).

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610

---

2. (14 pts) Compute the value of each of the following. *HINT: You should be able to use your answer to the previous question for all of these.*

(a) $F_1 + F_2 + F_3$ = 1+1+2 = 4

(b) $F_{1+2+3}$ = $F_6$ = 8

(c) $F_3 \times F_5$ = (2)(5) = 10

(d) $F_{3\times5}$ = $F_{15}$ = 610

(e) $F_{F_4}$ = $F_3$ = 2

(f) $2\,F_{12} - 5$ = 2(144) − 5 = 288 − 5 = 283

(g) $2\,F_{12-5}$ = $2F_7$ = 2(13) = 26

---

3. (4 pts) Given that $F_{32} = 2{,}178{,}309$ and $F_{33} = 3{,}524{,}578$, find $F_{31}$.

$F_{33} = F_{31} + F_{32}$

So $F_{31} = F_{33} - F_{32}$

$= 3{,}524{,}578 - 2{,}178{,}309$

$= 1{,}346{,}269$

3 5 2 4 5 7 8
2 1 7 8 3 0 9
1 3 4 6 2 6 9



25

Version A

4. (15 pts) Recall that $\Phi$ satisfies the Golden Property, $\Phi^2 = \Phi + 1$. It is also true that $\Phi^5 = 5\Phi + 3$. Use only these facts and algebra to express $\Phi^6$ in terms of $\Phi$. *Show your reasoning clearly.*

Given: $\Phi^5 = 5\Phi + 3$

$\Phi \cdot \Phi^5 = \Phi(5\Phi + 3)$

$\Phi^6 = 5\Phi^2 + 3\Phi$

$\Phi^6 = 5(\Phi + 1) + 3\Phi$ (Golden property: $\Phi^2 = \Phi + 1$)

$\Phi^6 = 5\Phi + 5 + 3\Phi$

$\boxed{\Phi^6 = 8\Phi + 5}$

---

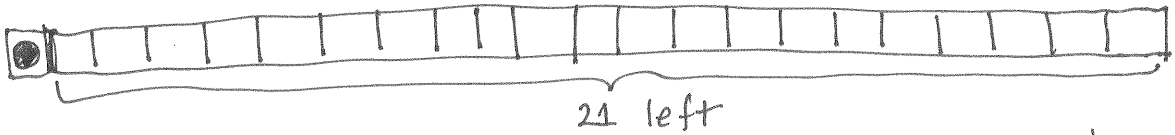
5. (15 points) Recall that $T_n$ represents the number of tilings by pennies and paperclips of an $n$-board, where each penny occupies one square and each paperclip occupies two adjacent squares.

(a) Find each of the following.

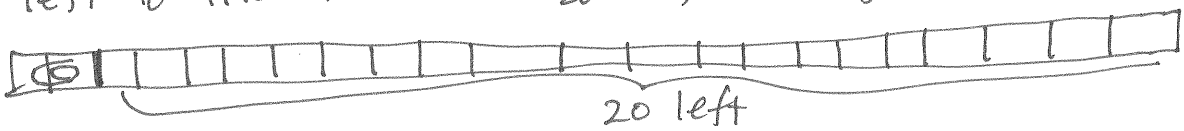
$T_1 = 1$ $T_2 = 2$ $T_3 = 2+1 = 3$ $T_4 = 3+2 = 5$

(b) Given that $T_{20} = 10{,}946$ and $T_{21} = 17{,}711$, find $T_{22}$, explaining your answer in terms of pennies, paperclips, and 22-boards.

Every tiling of a 22-board starts with either a penny or a paperclip. If it starts with a penny, then a 21-board is left to tile. There are $T_{21} = 17{,}711$ ways to do that.




If it starts with a paperclip, then there is a 20-board left to tile. There are $T_{20} = 10{,}946$ ways to do that.




Hence, there are a total of $T_{21} + T_{20} = 17{,}711 + 10{,}946 = 28{,}657$ ways to tile a 22-board. ∎

Version A

## Part II: Multiple Choice (5 points each)

*Circle the letter of the best answer.*

D

6. Kent State started with one male-female pair of baby immortal black squirrels. Immortal black squirrels begin to breed their second month. Each month, each adult pair gives birth to another male-female pair. If there were 2584 pairs in month $n$ and 4181 pairs in month $n+1$, how many pairs of immortal black squirrels were there at month $n+2$?

(a) 1597
(b) 8341
(c) 7333
(d) 6765
(e) None of the above

2584
4181
6765

---

D

7. The golden ratio $\Phi$ is the positive solution of which of the following equations?

(a) $x^2 = \frac{1}{x}$
(b) $x = 1 + x^2$
(c) $x^2 = 1 - x$
(d) $x^2 = 1 + x$
(e) None of the above

---

A

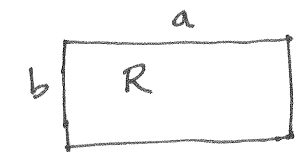
8. For large values of $n$, the ratio $\frac{F_n}{F_{n-1}}$ is approximately equal to what?

(a) $\Phi$
(b) $\Phi^2$
(c) $\pi$
(d) $F_{n+1}$
(e) None of the above

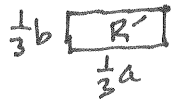
C

9. Suppose that $R$ and $R'$ are similar rectangles. The longest side of $R$ has length $a$ ft and the longest side of $R'$ has length $\frac{1}{3}a$ ft. If the perimeter of $R$ is 30 ft, find the perimeter of $R'$.

(a) 15 ft
(b) 90 ft
(c) 10 ft
(d) 30 ft
(e) None of the above




$P = 2a + 2b$
$= 30$ ft




$P' = 2(\frac{1}{3}a) + 2(\frac{1}{3}b)$
$= \frac{1}{3}(2a+2b)$
$= \frac{1}{3}P = \frac{1}{3} \cdot 30 = 10$ ft

---

A

10. Suppose that $R$ and $R'$ are similar rectangles. The longest side of $R$ has length $a$ ft and the longest side of $R'$ has length $\frac{1}{3}a$ ft. If the area of $R$ is 36 ft$^2$, find the area of $R'$.

(a) 4 ft$^2$
(b) 12 ft$^2$
(c) 108 ft$^2$
(d) 324 ft$^2$
(e) None of the above

See figure above:

$A = ab = 36$ ft$^2$

$A' = (\frac{1}{3}a)(\frac{1}{3}b)$
$= \frac{1}{9}(ab)$
$= \frac{1}{9}(36)$
$= 4$ ft$^2$

Version A

11. The Lucas Numbers are defined recursively by

$L_1 = 1,\ L_2 = 3,\ L_N = L_{N-1} + L_{N-2}.$

What is $L_7$?

C

(a) 18
(b) 7
(c) 29
(d) 31
(e) None of the above

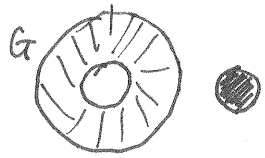
$L_1 = 1$
$L_2 = 3$
$L_3 = 4$
$L_4 = 7$
$L_5 = 11$
$L_6 = 18$
$L_7 = 29$

---

12. The circular ring $G$ has an inner radius of 10 and an outer radius of 16. Figure $G$ is a gnomon to which of the following?

C

(a) A circular ring with inner radius 16 and outer radius 20.
(b) A circular ring with inner radius 10 and outer radius 32.
(c) A circular disk of radius 10.
(d) A circular disk of radius 16.
(e) None of the above

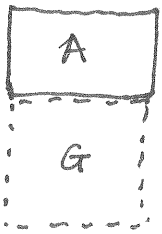



---

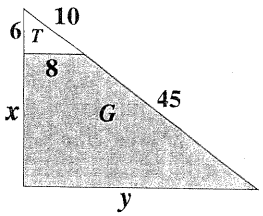
13. If $A$ is a **golden rectangle**, then which of the following is a gnomon to $A$?

D

(a) A square of sides equal to the shorter side of $A$.
(b) Another golden rectangle whose longer side equals the shorter side of $A$.
(c) Another golden rectangle whose shorter side equals the longer side of $A$.
(d) A square of sides equal to the longer side of $A$.
(e) None of the above




14. Find the value of $y$ so that the shaded figure $G$ is a gnomon to the white triangle $T$.




base: hypotenuse

$\frac{y}{8} = \frac{10+45}{10}$

C

(a) $\frac{225}{4}$
(b) $\frac{45}{8}$
(c) 44
(d) 55
(e) None of the above

$\frac{y}{8} = \frac{55}{10}$

$10y = 8 \cdot 55$

$y = \frac{8 \cdot 55}{10}$

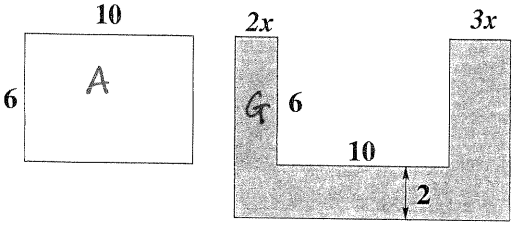
$y = \frac{2 \cdot 4 \cdot 5 \cdot 11}{2 \cdot 5}$

$= 4 \cdot 11$

$= 44$

---

15. Find the value of $x$ so that the shaded U-shaped region is a gnomon to the white rectangle.

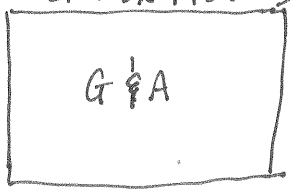



A

(a) $\frac{2}{3}$
(b) $\frac{7}{3}$
(c) $\frac{3}{5}$
(d) $\frac{5}{9}$
(e) None of the above

$2x + 3x + 10 = 5x + 10$

$6 + 2 = 8$




long : short

$\frac{5x+10}{8} = \frac{10}{6}$

$24\left(\frac{5x+10}{8}\right) = \frac{5}{3} \cdot 24$

$3(5x+10) = 5 \cdot 8$

$15x + 30 = 40$

$15x = 10$

$x = \frac{10}{15} = \frac{2}{3}$.