

66(i). The set \mathbb{C} is closed under addition.

Proof. Let $z, w \in \mathbb{C}$. Then $z = a+bi$ and $w = c+di$, for some $a, b, c, d \in \mathbb{R}$. By the definition of addition in \mathbb{C} ,

$$\begin{aligned} z+w &= (a+bi) + (c+di) \\ &= (a+c) + (b+d)i. \end{aligned}$$

Now $a+c$ and $b+d$ are both real numbers, because \mathbb{R} is closed under addition. Therefore, $z+w$ is a complex number. This shows that \mathbb{C} is closed under addition. \blacksquare

66(v). If $z = a+bi$, where $a, b \in \mathbb{R}$, then the additive inverse of z in \mathbb{C} is $(-a)+(-b)i$.

Proof. Consider the complex number $a+bi$, where $a, b \in \mathbb{R}$. We compute the sum

$$\begin{aligned} (a+bi) + [(-a)+(-b)i] &= (a-a) + (b-b)i \\ &= 0 + 0i \\ &= 0. \end{aligned}$$

Since additive inverses are unique, this shows that $(-a)+(-b)i$ is the additive inverse of $a+bi$. \blacksquare

(optional)

69 (i) If $r \in \mathbb{R}$, then $\bar{r} = r$.

(ii) Let $z \in \mathbb{C}$. If $\bar{z} = z$, then $z \in \mathbb{R}$.

Proof. (i) Suppose $r \in \mathbb{R}$. Then $r = r + 0i \in \mathbb{C}$.

Thus, $\bar{r} = \overline{r+0i} = r - 0i = r$, as desired. \blacksquare

(ii) Let $z \in \mathbb{C}$, say $z = a+bi$, where $a, b \in \mathbb{R}$.

Suppose that $\bar{z} = z$. This means $a-bi = a+bi$.

By the definition of equivalence of complex numbers,
it follows that $-b = b$. But the only real number
equal to its additive inverse is 0. Thus, $b = 0$.

Therefore, $z = a+0i = a$, a real number, as
desired. \blacksquare

71 If $z, w \in \mathbb{C}$, then $\overline{zw} = \overline{z} \cdot \overline{w}$.

Proof. Let $z, w \in \mathbb{C}$, say $z = a+bi$ and $w = c+di$, where $a, b, c, d \in \mathbb{R}$. Then

$$\begin{aligned}\overline{z} \cdot \overline{w} &= \overline{(a+bi)} \cdot \overline{(c+di)} \\&= (a-bi)(c-di) \quad \text{definition of conjugation in } \mathbb{C} \\&= (ac-bd) + (-ad-bc)i \quad \text{defn of mult. in } \mathbb{C} \\&= (ac-bd) + [-(ad-bc)]i \quad \text{dist. law in } \mathbb{R} \\&= (ac-bd) - (ad+bc)i \\&= \overline{(ac-bd) + (ad+bc)i} \quad \text{defn of conjugation} \\&= \overline{(a+bi)(c+di)} \quad \text{defn of mult in } \mathbb{C} \\&= \overline{zw},\end{aligned}$$

as desired. ■

75 If $z \in \mathbb{C}$, then $z\overline{z} = (\operatorname{Re} z)^2 + (\operatorname{Im} z)^2$.

Proof. Let $z \in \mathbb{C}$, say $z = a+bi$, where $a, b \in \mathbb{R}$.

Then

$$\begin{aligned}z\overline{z} &= (a+bi)\overline{(a+bi)} \\&= (a+bi)(a-bi) \quad \text{defn of conjugation in } \mathbb{C} \\&= (a^2 - (-b)^2) + (ab - ab)i \quad \text{defn of mult. in } \mathbb{C} \\&= a^2 + b^2 \\&= (\operatorname{Re} z)^2 + (\operatorname{Im} z)^2, \quad \text{as desired. ■}\end{aligned}$$