

Prop 88. If $z \in \mathbb{C}$, then $|z| = \sqrt{z\bar{z}}$.

Proof. Let $z \in \mathbb{C}$, say $z = a+bi$, where $a, b \in \mathbb{R}$. Then

$$\text{by Prop. 75, } \sqrt{z\bar{z}} = \sqrt{a^2+b^2} = |z|. \blacksquare$$

Prop 90. If $z \in \mathbb{C}$, then $|z|^2 = (\operatorname{Re} z)^2 + (\operatorname{Im} z)^2$.

Proof. Let $z \in \mathbb{C}$. Then by definition, $|z| = \sqrt{(\operatorname{Re} z)^2 + (\operatorname{Im} z)^2}$,
so $|z|^2 = (\operatorname{Re} z)^2 + (\operatorname{Im} z)^2$. \blacksquare

Prop 91. If $z \in \mathbb{C}$, then $|z|^2 \geq (\operatorname{Re} z)^2$ and $|z|^2 \geq (\operatorname{Im} z)^2$.

Proof. Let $z \in \mathbb{C}$. By Prop. 90, $|z|^2 = (\operatorname{Re} z)^2 + (\operatorname{Im} z)^2$.

Now $\operatorname{Im} z$ is a real number, so $(\operatorname{Im} z)^2 \geq 0$. Therefore,

$$|z|^2 = (\operatorname{Re} z)^2 + (\operatorname{Im} z)^2 \geq (\operatorname{Re} z)^2 + 0 = (\operatorname{Re} z)^2.$$

Similarly, $\operatorname{Re} z$ is a real number, so $(\operatorname{Re} z)^2 \geq 0$. Hence,

$$|z|^2 = (\operatorname{Re} z)^2 + (\operatorname{Im} z)^2 \geq 0 + (\operatorname{Im} z)^2 = (\operatorname{Im} z)^2. \blacksquare$$

Prop 92. If $z \in \mathbb{C}$, then $|z| \geq \operatorname{Re} z$ and $|z| \geq \operatorname{Im} z$.

Proof. Let $z \in \mathbb{C}$, say $z = a+bi$, where $a, b \in \mathbb{R}$. Then $|z| = \sqrt{a^2+b^2}$.

Since $a, b \in \mathbb{R}$, $a^2, b^2 \geq 0$ and so $a^2+b^2 \geq 0$. Hence $|z|$ is a nonnegative real number. By Prop. 91, $|z|^2 \geq a^2$.

First, suppose $a \geq 0$. Then $\sqrt{a^2} = a$. Since the square root function is increasing on $[0, \infty)$, it follows that $|z| \geq a$.

Next, suppose $a < 0$. Then $|z| > a$, so $|z| \geq a$. Thus, $|z| \geq \operatorname{Re} z$ for all z . A similar argument shows $|z| \geq \operatorname{Im} z$. \blacksquare

Conjectures (They are, in fact, true.)

105 Let $z, w \in \mathbb{C}$. Then $|zw| = |z||w|$.

106. Let $z, w \in \mathbb{C}$, Then $\text{Arg}(zw)$ is coterminal to $\text{Arg}(z) + \text{Arg}(w)$.

107. Let $z \in \mathbb{C}$ with $z \neq 0$. Then $|\frac{1}{z}| = \frac{1}{|z|}$.

108. Let $z \in \mathbb{C}$ with $z \neq 0$. Then $\text{Arg}(\frac{1}{z}) = -\text{Arg}(z)$.