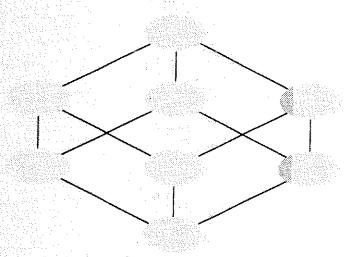
AN INTRODUCTION TO ABSTRACT MATHEMATICS



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MATHEMATICAL REASONING

Introduction: Early Mathematics

Mathematics in one form or another has probably existed in every civilization. Many cultures flourished in the area of the Tigris and Euphrates rivers known as Mesopotamia in what is now Iraq. The Babylonian civilization of that area has left us records of mathematical activity as far back as the years 1800–1600 B.C. Writing on clay tablets, they recorded solutions of algebraic problems and compiled tables of squares, cubes, square roots, and cube roots, even some logarithms. They also listed Pythagorean triples, numbers such as 3, 4, 5 and 5, 12, 13, which make up the sides of a right triangle (using of course their own symbols and number system). So the Babylonians knew the Pythagorean Theorem more than a thousand years before the time of Pythagoras! Even earlier than the Babylonians, the so-called Middle Kingdom of Egypt (2000–1800 B.C.) produced some sophisticated calculations of areas and volumes. One of their great achievements was the calculation of the volume of a truncated pyramid with square base and square top. (See Exercise D4 of Section 1.3.)

Beginning in about the sixth century B.C. in Greece, an extraordinary new chapter in the history of mathematics began. For the first time (at least as far as we know) the methods of reasoning and logical deduction, as opposed to trial and error calculations, were employed to arrive at new mathematical truths. Prior to the Greeks, in Egypt and Babylonia for example, geometric results and algebraic formulas were discovered by empirical methods; that is, by trying special cases and then extrapolating to the general case. This empirical method, called *inductive* reasoning, has been used by mathematicians throughout history. It is not merely a perfectly acceptable method of making new discoveries but is really an *indispensable* way to arrive at new results. But the Greeks insisted that any new results had to be *proved* and that meant using the rules of logic. This method is called *deductive* reasoning. In this chapter we introduce some of the important ideas of logic that are frequently used in mathematics.

1.1 STATEMENTS

The Notion of Proof

Contrary to popular opinion, mathematics is not just computations and equations. It might be better described as an attempt to determine which statements are true and which are not. The subject matter may vary from numbers to geometric figures to just about anything, but the form is always the same.

The process of discovery in mathematics is twofold. First comes the formulation of a mathematical statement or conjecture. This formulation often comes after much hard work that usually includes a trial-and-error process, many false starts, and sometimes extensive calculation. The second part of the process is the verification or proof that the statement that we have formulated is true or false. This part too can involve much trial and error and long, hard work. It is this part of the process that we will study in this text.

To begin to prove a mathematical statement it is necessary to begin with certain statements that we accept as given, called **axioms**, and try to logically deduce other statements from them. These deductions are called **propositions**, or, if they're particularly important, **theorems**. The arguments, the logic we use to make the deductions, are called **proofs**.

The beauty of mathematics often comes from the fact that propositions are not always obvious, and can in fact be surprising. Moreover, proofs may not be easy to construct, and may require insight and cleverness. They certainly require a little experience, especially some familiarity with the most commonly used logical methods. This "sophistication" won't come overnight; it's one of the major goals of this book.

Let's begin with an example.

EXAMPLE 1 Suppose that the following question were posed to you: Is the square of an even integer itself an even integer? You might start by trying some calculations:

$$2^2 = 4, \qquad 4^2 = 16, \qquad 6^2 = 36, \qquad 8^2 = 64, \qquad 10^2 = 100.$$

Each one of these examples gives a positive answer to the question. Do these answers constitute a proof?

If the problem asked us to show that the squares of *some* even integers are even, we would be done. But that is not the question. There is an implied universality about the question. Is the square of *every* even integer even? Clearly, more must be done than trying a few examples. In fact just working out examples will not suffice, because no matter how many we do, there will always be an infinite number of cases that we haven't done.

So how do we proceed? The first step might be to reword the question as a statement.

"The square of every even integer is even."

Now let's reword the statement using symbols:

"If n is an integer and n is even, then n^2 is even."

In order to begin a proof, we have to ask ourselves: What does it mean for an integer to be even? Well, we know that an integer is even if it is two times an integer. That is the *definition* of even integer.

Now we need to translate this definition into symbols:

"n is an even integer if $n = 2m$ for some integer m."

Now let's think about what we're trying to prove. We want to show that the square of the even integer n is even. So we start by letting n be an even integer and try to show that n^2 is twice an integer. We do the obvious algebra step; we write $n = 2m$ and square both sides.

We get $n^2 = 4m^2 = 2(2m^2)$. Since $2m^2$ is an integer, this shows that n^2 is twice an integer and therefore n^2 is even.

The proof is now complete.

This is an example, admittedly not a complicated one, of the proof of a mathematical statement.

Note that in this last example, we made some assumptions about multiplication of integers; namely, that multiplication satisfies a commutative law and an associative law and that the product of integers is still an integer. These laws make up some of the axioms of the integers and will be discussed in Chapter 5. For the purpose of examples in this and later chapters, we will assume the well-known arithmetic properties of the integers and real numbers. (See Section 5.1 for properties of the integers and Section 7.1 for properties of the real numbers.)

The notion of multiples of an integer is used in some examples. An integer x is called a **multiple** of an integer n if $x = kn$ for some integer k. So the even integers are the multiples of 2. The multiples of 3 consist of the integers 0, ± 3, ± 6, ± 9, and so on.

Statements

In the course of this chapter, we will discuss many of the rules of logic and inference needed in the previous example and others like it in mathematics.

In the previous example, we used the word "statement" several times. In this text, the word will have a specific meaning.

Definition 1.1.1 A **statement** is any declarative sentence that is either true or false.

A statement then will have a truth value. It is either true or false. It cannot be *neither* true nor false and it cannot be *both* true and false.

Some examples of statements are given next:

EXAMPLE 2 John Fitzgerald Kennedy was the 35th president of the United States.

EXAMPLE 3 Marie Curie did not win the Nobel Prize.

EXAMPLE 4 $3 + 1 = 4$.

The fourth example is a mathematical statement, which of course is true. We commonly write mathematical statements with symbols for convenience, although you should think of them not as "formulas," but as full-fledged sentences, with a subject, a verb, and possibly other parts of speech.

In the rest of this chapter, we examine what mathematical statements can look like and some methods that can be used to prove them. For convenience, we often use letters, most often P or Q, to denote statements.

EXAMPLE 5 Some sentences, even some mathematical ones, are not statements. For example, consider a typical sentence from algebra: $x + 1 = 2$. Here, x is a **variable**; it's a symbol that stands for an undetermined number. The sentence is a statement if we specify what number x stands for. It's a true statement if x stands for 1, and it's false for any other x. We could label this sentence $P(x)$ because it depends on the variable x. So $P(1)$ is a true statement and $P(x)$ is a false statement if $x \neq 1$.

Note, however, that if x is not specified, then $P(x)$ is *not* a statement.

We will call any sentence like the one from the previous example an *open sentence*.

Definition 1.1.2 An **open sentence** is any declarative sentence containing one or more variables that is not a statement but becomes a statement when the variables are assigned values.

The values that can be assigned to the variables of an open sentence will depend on the context. They may come from the real numbers as in the example $x + 1 = 2$ or from the complex numbers or even just the positive integers. The values do not even have to be mathematical. For example, the sentence "He was the 16th president of the United States" is an open sentence containing the variable "he" and is therefore a true statement when "he" is assigned the value "Abraham Lincoln" and is false otherwise.

An open sentence is usually written $P(x)$, $P(x, y)$, $P(x, y, z)$, and so on, depending on the number of variables used.

Quantifiers

An open sentence like $x + 1 = 2$ can, as we have seen, be made into a statement by substituting a value for the variable or, in the case of an open sentence with more than one variable, by substituting a value for each of the variables.

Another way an open sentence can be made into a statement is by introducing **quantifiers**. For example, for the open sentence $x + 1 = 2$, we could say: For every real number x, $x + 1 = 2$. This sentence is now a mathematical statement that happens to be false. The quantifier introduced here is the phrase "for every real number x" and is called a **universal** quantifier. Another way to modify $P(x)$ is to write: there is a real number x such that $x + 1 = 2$. Note that this statement is true. The quantifier in this example, "there is a real number x," is called **existential**.

Once a quantifier is applied to a variable, then the variable is called a **bound** variable. In the example "For every real number x, $x + 1 = 2$" of the previous paragraph, then, the variable x is a bound variable. A variable that is not bound is called a **free** variable.

If $P(x)$ is an open sentence, then the statement: "For all x, $P(x)$" means that for *every* assigned value a of the variable x, the statement $P(a)$ is true.

The statement "For some x, $P(x)$" means that for *some* assigned value of the variable x, say $x = a$, the statement $P(a)$ is true. This statement may also be worded: "There exists a value of x such that $P(x)$."

Sometimes, in a statement containing universal quantifiers, the words "for all" or "for every" are not actually in the sentence but are implied by the meaning of the words. Here are some examples.

EXAMPLE 6 If n is an even integer, then n^2 is even.

On the surface, this sentence might seem to be an open sentence rather than a statement since it contains the variable n. However, implicit in the wording is the meaning: for every integer n, if n is even, then n^2 is even. So the variable n has been modified by a universal quantifier and is now a bound variable, making the sentence a statement. As we saw in Example 1, it is actually a true statement.

EXAMPLE 7 A triangle has three sides.

This statement contains a universal quantifier since it is really asserting that every triangle has three sides. Another way to word this statement is "For every plane figure T, if T is a triangle, then T has three sides," or more simply "If T is a triangle, then T has three sides."

EXAMPLE 8 The square of a real number is nonnegative.

Again this statement has a universal quantifier since it is saying that the square of every real number is nonnegative. It could also be written as:

"If x is a real number, then $x^2 \geq 0$."

EXAMPLE 9 All triangles are isosceles.

This statement has a universal quantifier as well. It just happens to be false.

We will sometimes use the symbol $\forall$ to mean "for all" or "for every." Example 8 can be rewritten: "$\forall\, x$, if x is real, then $x^2 \geq 0$."

The following examples give some of the forms that a statement with an existential quantifier can take.

EXAMPLE 10 Some even numbers are multiples of 3.

First note that even though this statement is written in plural form ("some even numbers"), it may be phrased: "There exists an even integer that is a multiple of 3." To prove this statement, one need only find *one* even number that is a multiple of 3. Since 6 is such a number, the proof is complete.

EXAMPLE 11 Some real numbers are irrational.

This statement asserts something about some, but not all, real numbers. It may be reworded as: "There exists a real number x such that x is irrational." It is a true statement provided that there is at least one real number that is not rational. In Section 1.4, we will prove this statement by showing that $\sqrt{2}$ is irrational.

EXAMPLE 12 There is a real number whose square is negative.

This statement also makes an assertion about some real numbers. Note that it is a false statement.

The symbol $\exists$ is used to mean "there exists" or "there is." The symbol $\ni$ is read "such that." Example 11 then can be expressed as: $\exists\, x$, x real, $\ni x$ is irrational.

A statement of course may have more than one quantifier.

EXAMPLE 13 For every real number x, there is an integer n such that $n > x$.

This statement contains both a universal and an existential quantifier.

EXAMPLE 14 The following statement, which is a definition from calculus, also has both a universal and an existential quantifier: A real-valued function $f(x)$ is **bounded** on the closed interval $[a, b]$ if $f(x)$ is defined on $[a, b]$ and there exists a positive real number M such that $|f(x)| \leq M$ for all $x \in [a, b]$.

For example, the function $f(x) = x^2 + 1$ is bounded on $[0, 2]$ because $|f(x)| \leq 5$ for all $x \in [0, 2]$.

The order in which quantifiers appear in a statement is important. If $P(x, y)$ is an open sentence in the variables x and y, then the statement

$$\forall x, \exists y \ni P(x, y)$$

does not always mean the same as the statement

$$\exists y \ni \forall x, P(x, y).$$

To see this, consider the following example.

EXAMPLE 15 The statement "Every real number has a cube root" can be written in the form:

$$\forall x \in \mathbf{R}, \exists y \in \mathbf{R} \ni y^3 = x.$$

This statement is true and is a consequence of the *Intermediate Value Theorem* of Calculus. However, the statement

$$\exists y \in \mathbf{R} \ni \forall x \in \mathbf{R}, y^3 = x$$

means that every real number x is the cube of a single number y and is clearly false.

Negations

Two of the statements just given, "All triangles are isosceles" and "There is a real number whose square is negative," were noted to be false. To *prove* that they are false it is necessary to prove that the negations of these statements are true.

Definition 1.1.3 If P is a statement, the **negation** of P, written $\neg P$ (and read "not P"), is the statement "P is false."

There are several alternative ways to express $\neg P$. For example, "P is not true" and "It is not true that P" are the same as our definition. In addition,

EXAMPLE 26 P: There is a real number whose square is negative.

Statement P says that we can find a real number whose square is negative. The negation of P means that we cannot find such a number. In other words, the negation of P is

The square of every real number is not negative.

To negate P by saying "There is a real number whose square is nonnegative" would not be correct because this statement and P could, theoretically at least, both be true.

EXAMPLE 27 P: Some real-valued functions are not integrable.
$\neg P$: Every real-valued function is integrable.

As Examples 22–27 show, there are thus two basic rules about negating statements with quantifiers.

> *Rule 1:* The negation of the statement "For all x, $P(x)$" is the statement "For some x, $\neg P(x)$."
>
> *Rule 2:* The negation of the statement "For some x, $P(x)$" is the statement "For all x, $\neg P(x)$."

Of course, if a statement contains both universal and existential quantifiers, then in order to negate the statement, it is necessary to apply both of these rules.

If a statement S has the form "For all x, $\exists y \ni P(x, y)$" then the negation of S is "For some x, $\neg(\exists y \ni P(x, y))$" by Rule 1. Since Rule 2 tells us that the negation of "$\exists y \ni P(x, y)$" is "For all y, $\neg(P(x, y))$," then the negation of S becomes "For some x, for all y, $\neg(P(x, y))$" or "$\exists x \ni$ for all y, $\neg(P(x, y))$."

Similarly, the negation of "$\exists x \ni \forall y$, $P(x, y)$" is "$\forall x$, $\exists y \ni \neg(P(x, y))$."

EXAMPLE 28 P: For every real number x, there is an integer n such that $n > x$.
$\neg P$: There is a real number x such that for every integer n, $n \leq x$.

The statement in this last example is known as the Archimedean Principle. See the Historical Comments at the end of Section 1.2.

EXAMPLE 29 P: There is a continuous real-valued function $f(x)$ such that $f(x)$ is not differentiable at any real number c.
$\neg P$: For every continuous real-valued function $f(x)$, there is a real number c such that $f(x)$ is differentiable at c.

Surprisingly, statement P of Example 29 is true. There are continuous functions that are not differentiable at any point! An example, due to K. Weierstrass, is

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{2^n} \cos((15)^n \pi x).$$

This is a function that is continuous everywhere but differentiable nowhere.

Writing Proofs

As you start to read and write proofs, you will see that they require a certain writing style to ensure clarity and readability. Take, for example, the proof in Example 1, that if n is an even integer, then n^2 is even. Suppose that a proof were written as follows:

n = even = $2t$.
$n^2 = 4t^2$ = even.

This rather brief proof has the correct mathematical steps, but is lacking in explanation and hence in clarity and also suffers from poor notation. One should begin by clearly stating the assumption: "Let n be an integer and suppose that n is even." Then it can be noted that this means that "$n = 2t$ for some integer t." But writing "n = even" is sloppy notation. The word "even" is an adjective and should precede a noun; in this case, the word "number." But even the expression "n = even number" is not appropriate. The phrase "even number" does not belong in an equation. Equations should only contain numbers and symbols.

After writing "$n = 2t$ for some integer t" an explanation should be given for the next step: "Squaring both sides, we get $n^2 = 4t^2 = 2(2t^2)$." Then finally, one should note that "since $2t^2$ is an integer, it follows that n^2 is even."

HISTORICAL COMMENTS: EARLY GREEK MATHEMATICS

A history of early Greek mathematics was written by **Eudemus** in the fourth century B.C. Although this book is now lost, a summary of it was written by **Proclus** in the fifth century A.D. According to Proclus's work, the earliest known mathematician to use the deductive method was **Thales** of Miletus. Thales founded the earliest Greek school of mathematics and philosophy. Among the results attributed to Thales are: the base angles of an isosceles triangle are congruent; triangles with corresponding angles equal have proportional sides; and an angle inscribed in a semicircle is a right angle. Each of these results was proven by deductive methods.

About sixty to eighty years after Thales, an important school of mathematics and philosophy was founded in southern Italy by **Pythagoras** (ca. 585–497 B.C.). One of the great contributions of the Pythagoreans, as the