A FIRST COURSE IN ABSTRACT ALGEBRA

FIFTH EDITION

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CHAPTER ZERO

A FEW PRELIMINARIES

It is our experience that spending two or three weeks on background material at the start of an abstract algebra course may destroy interest in the subject. Accordingly, we introduce mathematical tools as they are needed, provided that their presentation can be kept short so that the flow of the text is not interrupted. Since they need longer discussion, we present equivalence relations and partitions of sets in Section 0.2 and proof by mathematical induction in Section 0.3. Section 0.1, which well might be left for students to read on their own, attempts to prepare students for this axiomatic, definition-theorem-proof treatment of algebra. Section 0.4 summarizes some of the algebra of complex numbers and matrices for students who may not be familiar with it.

0.1 Mathematics and Proofs

You have probably never had a laboratory course in mathematics. Mathematics is not considered to be an experimental science, whereas physics, chemistry, and biology are. Research for a chemist can consist of a laboratory experiment designed to validate a conjecture or simply to see what happens. There is little comparable activity in mathematics.

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The main business of mathematics is proving theorems.

Just examine any research journal. Few meaningful theorems can be proved by experimentation. In mathematics, experimentation might lead to a *conjecture* which may or may not be correct. If the conjecture is later proved, then it is elevated to the status of a *theorem*. Exercise 10 illustrates experimentation leading to conjecture.

In theory, all of mathematics is an *axiomatic* study, consisting of chains of valid conclusions (theorems) deduced by valid reasoning (proofs) from the axioms of set theory. This is the current view of mathematical logicians who wrestle valiantly with the foundations of mathematics. Probably most research mathematicians would be unable to write down the axioms of set theory or describe all the rules of valid reasoning in a way that would satisfy logicians. In spite of this, logicians would agree that the results in the great majority of mathematical research papers are valid. In case an assertion is not valid, the author of the paper would probably admit to a mistake when the difficulty is pointed out. With perhaps no formal training in mathematical logic, the research mathematician learned as a student the rules of the game and can contribute successfully to the subject.

Keeping in mind the preceding paragraphs, we can try to put one feature of abstract algebra in perspective. Abstract algebra is the most axiomatic study undertaken by the typical mathematics major. It gives a lot of exposure to the rules of the game, mathematics. However, it would be absurd to pretend that this text is a totally axiomatic study. For example, we shall feel free to use familiar properties of the real numbers without any axiomatic verification. An abstract algebra course does represent a big step from the typical freshman-sophomore calculus course toward the modern mathematical method.

Courses in linear algebra vary widely in axiomatic approach. If you used a text that gave axioms for a vector space in the first chapter and then developed the subject from them, your study of linear algebra was similar to the study of abstract algebra in this text. On the other hand, if vector-space axioms did not appear at all or were relegated to an appendix of your linear algebra text, the orientation of the course was probably close to that of the typical first course in calculus.

An axiomatic approach is not used merely to expose students to proofs, although it does serve that function quite well. It is the most efficient way we have found to present algebra. Once a body of theorems has been deduced from axioms, we know that the theorems hold for *every* structure that satisfies the axioms. For example, we will start our study by examining structures called groups, which satisfy three axioms. If we were to prove a theorem in terms of one particular group, perhaps involving addition of real numbers, it might not be clear whether the theorem holds for all groups. We would have to reexamine the proof, doing the same work all over again if we change the group. But if we prove a theorem just in terms of the axioms of a group, without using any other properties, then this single proof allows us to use the theorem freely for any group. This is a virtue of the axiomatic approach, and our study of abstract algebra will illustrate this technique. The adjective *abstract* indicates that algebra is being studied by properties that have been *abstracted* from the subject.

Abstract algebra is often considered an ideal subject for drill in proofs since the lists of axioms used are quite short. However, devising a proof in algebra often amounts to finding just the right method of attack, perhaps considering just the right algebraic expression. If the right method is found, a proof may fall out easily. Otherwise we may struggle a long time without finding a proof. Geometric pictures are usually no help in finding a proof in algebra. For this reason point-set topology might be a better course for training students in proofs, for pictures can often be used in topology as an aid in understanding why a theorem must be true. Emphasis can then be placed on writing a correct proof.

It is not possible for us to give any meaningful outline on how to prove theorems; experience is the best guide. For the remainder of this section we make a few general observations. We will start by pointing out that it is essential to know what we are talking about, that is, to understand *definitions* of the terms we are using.

Definitions

Many students do not realize the great importance of definitions to mathematics. This importance stems partly from the need for mathematicians to communicate with each other about their work. If two people are trying to communicate about some subject, they must have the same understanding of its technical terms.

A very important ingredient of mathematical creativity is the ability to formulate useful definitions, ones that will lead to interesting results. A mathematics student commencing graduate study may find that he or she spends a great deal of time discussing definitions with other graduate students. When I was in graduate school, a physics graduate student once complained to me that at the evening meal the mathematics students always sat together and argued, and that the subject of their argument was always a definition. Graduate students are usually asked to give several definitions on oral examinations. If they cannot explain the meaning of a term, they probably cannot give sensible answers to questions involving that concept.

Every definition is understood to be an *if and only if* type of statement, even though it is customary to suppress the *only if*. Thus we may define an isosceles triangle as follows: "A triangle is **isosceles** if it has two sides of equal length," when we really mean that a triangle is isosceles if and only if it has two sides of equal length.

Do not feel that you have to memorize a definition word for word. The important thing is to *understand* the concept, so that you can define precisely the same concept in your own words. Thus the definition "An **isosceles** triangle is one having two equal sides" is perfectly correct.

Throughout the text, a term that appears in boldface type is being *defined* at that point. Specifically labeled definitions are used for the main algebraic concepts with which we are concerned. Many other terms are defined, using the boldface convention, outside a labeled definition. You will find ideas defined in this fashion in text paragraphs, theorems, and exercises.

Observations on Proofs

Observation 1 If some concept has just been defined and we are asked to prove something concerning the concept, we *must* use the definition as an integral part of the proof.

> Immediately after a concept is defined, the definition is the only information one has available regarding the concept.

EXAMPLE 1 An integer n is defined to be **even** if $n = 2m$ for some integer m. It is a theorem that the sum of two even integers is even. The definition of an even integer must be used to prove this theorem. We leave the proof to Exercise 5. ▲

Observation 2 The statement of a theorem consists of two parts: the *hypotheses* and the *conclusion*. If all the hypotheses are needed to prove the theorem, that is, if no hypothesis is redundant, then each hypothesis must be cited somewhere in the proof.

EXAMPLE 2 It is a theorem that the sum of an even integer r and an odd integer s is an odd integer. In proving this theorem, which we leave to Exercise 7, it is essential to use both hypotheses, namely, that r is even and that s is odd. ▲

Observation 3 If even one example can be found for which a statement is not true, then the statement is not a theorem. In fact, the standard way to show that a statement is not a theorem is to provide such a *counterexample.*

EXAMPLE 3 Is the statement "The square of every real number is positive" a theorem? The answer is no, since $0^2 = 0$ and 0 is a real number but is not positive. This is the only counterexample that can be given, but one such example is all that is needed to show that a statement is not a theorem. ▲

Observation 4 Never tacitly assume any hypothesis that is not explicitly stated. Never take for granted any quantifying words or phrases such as *only, for all, for every,* or *for some* that do not actually appear.

EXAMPLE 4 The statement "There are four real numbers whose squares are less than 2" is true (a theorem). As a proof, we need only observe that $(-1)^2$, 0^2, 1^2, and $(\frac{1}{2})^2$ are all less than 2. The statement "There are only four real numbers whose squares are less than 2" is false (not a theorem). We need only observe that $(-1/2)^2 < 2$ also. The word *only* makes all the difference. ▲

Observation 5 A theorem of the form

If *hypotheses* then *conclusion*

cannot be proved by giving a specific example where the hypotheses and conclusion all are true. We must show that *for all* examples where the hypotheses are true, the conclusion is true also.

EXAMPLE 5 Consider the statement "If $f(x)$ is continuous, then $f(x)$ is differentiable." Now $f(x) = x^2$ is continuous and is also differentiable, for $f'(x) = 2x$ at every point in the domain of $f(x)$. However, the statement is not a theorem. A counterexample is given by $f(x) = |x|$, which is continuous but not differentiable since $f'(0)$ does not exist. Of course, in classifying this statement as a theorem or not a theorem, we had to know the *definitions* of a continuous function and of a differentiable function. ▲

There are a few types of theorems for which the method of attack for a proof is fairly standard. As our final observation, we mention one type that appears a few times in the text.

Observation 6 Suppose we wish to show that an element having some property exists and is *unique,* that is, that there is one and only one such element. First, show that there is such an element. To show uniqueness, assume that there are two such elements, say r and s, and try to show that r and s must be equal (the same).

EXAMPLE 6 Show that there is a unique real number r such that $rx = r$ for all real numbers x.

Solution We know that $0x = 0$ for all real numbers x, so that 0 has the property described for the number r. Suppose that a number s also has this property, so that $sx = s$ for all real numbers s. We use the fact that $ab = ba$ for all real numbers a and b and proceed to use an algebraic trick, namely, we consider $0s$. Since both 0 and s have the required property, we see that

$$0s = 0 \qquad \text{and also} \qquad 0s = s0 = s.$$

Thus $0 = s$ since each is equal to $0s$. ▲

The exercises that follow are designed to illustrate the preceding observations further, with special emphasis on the use of the quantifying words and phrases *only, there exists, for all, for every, for each,* and *for some.*