MATH 34001 | Fundamental Concepts of Algebra | Spring 2015
MWF 11:00 | | Dr. Kracht

Name: KEY Score: ____/100

. *(104 pts available)*

Exam 1: Friday, February 27, 2015

To receive full credit, show all work necessary to justify answers and all steps of proofs clearly, in logical sequence, using notation developed in class. Reference properties and theorems explicitly where needed. Write proofs in complete sentences (with proper capitalization, punctuation, subject, verb, etc.). Introduce notation with "Let" statements and quantify all variables. Partial credit will be given only for significant progress toward a solution.

No calculators or other electronic devices are permitted. No books or notes are permitted.

1. (12 points) Complete each of the following definitions.

   (a) an integer is said to be **odd** if $n = 2k+1$ for some integer k.

   (b) A set S is said to be **closed** under the binary operation $\circ$ if

   $a \circ b \in S$ for all $a, b \in S$.

   (c) The number $\underline{1}$ is said to be the **multiplicative identity element** of $\mathbb{Z}$ since

   $a \cdot 1 = a$, for all $a \in \mathbb{Z}$.

   (d) For $a \in \mathbb{Z}$, the number $\underline{-a}$ is said to be the **additive inverse** of a since

   $a + (-a) = 0$.

   (e) *(Complete the following using set notation.)*

   $\mathbb{Q} = \left\{ \frac{a}{b} \mid a, b \in \mathbb{Z} \text{ and } b \neq 0 \right\}$

   (f) If $\frac{a}{b}, \frac{c}{d} \in \mathbb{Q}$, where $a, b, c, d \in \mathbb{Z}$, then we say $\frac{a}{b} = \frac{c}{d}$ if and only if

   $ad = bc$.

2. (4 points) Order the sets $\mathbb{N}, \mathbb{Q}, \mathbb{R}, \mathbb{W}$, and $\mathbb{Z}$ from "smallest' to "largest" by containment.

   $\mathbb{N} \subseteq \mathbb{W} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$

3. (2 points) TRUE or FALSE: The number 0 is neither odd nor even. 0 is even

4. (6 points) State the properties of operations in $\mathbb{N}$ illustrated by each of the following. *Be specific.*

   (a) $(ab)c = a(bc)$ associativity of multiplication

   (b) $(a+b)\cdot c = (b+a)\cdot c$ commutativity of addition

   (c) $(a+b)\cdot(c+d) = a\cdot(c+d) + b\cdot(c+d)$ right distributive law of multiplication over addition

5. (10 points) Perform the long division by hand to find the decimal expansion, terminating or repeating, of the following rational number.

$\frac{13}{11} = 1.\overline{18}$

$$
\begin{array}{r}
1.1818 \\
11\,\overline{)\,13.0000} \\
\underline{11} \\
\to 20 \\
\underline{11} \\
90 \\
\underline{88} \\
\to 20 \\
\underline{11} \\
90 \\
\underline{88} \\
\to 2
\end{array}
$$

6. (10 points) Convert the following repeating decimal expansion to a fraction in lowest terms.

$0.3\overline{26} = 0.3262626\ldots$

Let $x = 0.3\overline{26} = 0.3262626\cdots$

Then $100x = 32.6\overline{26} = 32.6262626\cdots$

Then

$100x - x = 32.3$

$99x = 32.3$

$x = \frac{32.3}{99}$

$x = \frac{323}{990}$

7. (15 points) Let $x, y \in \mathbb{Z}$. Prove that if x is even and y is odd, then $x+y$ is odd.

Proof. Let $x, y \in \mathbb{Z}$ with x even and y odd. Then there exist integers k and l such that

$$x = 2k \quad \text{and} \quad y = 2l+1.$$

Then we have

$$\begin{aligned} x+y &= 2k + (2l+1) \\ &= (2k+2l)+1 \quad &&\text{, by associativity of addition} \\ &= 2(k+l)+1 \quad &&\text{, by the distributive law.} \end{aligned}$$

Since $\mathbb{Z}$ is closed under addition, $k+l \in \mathbb{Z}$. Thus, $x+y$ is odd, by definition of odd. ■

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8. (15 points) Prove that if $a \in \mathbb{Z}$, then $(-1) \cdot a$ is equal to $-a$, the additive inverse of a.

Proof. Let $a \in \mathbb{Z}$. We compute

$$\begin{aligned} a + (-1)\cdot a &= 1\cdot a + (-1)\cdot a \quad &&\text{, multiplicative identity prop.} \\ &= (1+(-1))\cdot a \quad &&\text{, distributive law} \\ &= 0 \cdot a \quad &&\text{, additive inverses} \\ &= 0 \quad &&\text{, proposition from class.} \end{aligned}$$

Since the additive inverse of an element is unique, we conclude that $(-1)\cdot a = -a$. ■

9. (15 points) Using the definition of $\mathbb{Q}$ (including the definitions of addition and multiplication in $\mathbb{Q}$) and the algebraic properties of the ring $\mathbb{Z}$ of integers, prove that multiplication in $\mathbb{Q}$ is commutative. That is, if $q, r \in \mathbb{Q}$, then $q \cdot r = r \cdot q$.

Proof. Let $q, r \in \mathbb{Q}$, say $q = \frac{a}{b}$ and $r = \frac{c}{d}$, where $a, b, c, d \in \mathbb{Z}$.

Then
$$q \cdot r = \frac{a}{b} \cdot \frac{c}{d}$$
$= \frac{ac}{bd}$, by definition of mult. in $\mathbb{Q}$;

$= \frac{ca}{db}$, by the commutativity of multiplication in $\mathbb{Z}$;

$= \frac{c}{d} \cdot \frac{a}{b}$, by defn of mult in $\mathbb{Q}$

$= r \cdot q$, as desired. ■

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10. (15 points) Using the definition of $\mathbb{Q}$ (including the definitions of addition and multiplication in $\mathbb{Q}$) and the algebraic properties of the ring $\mathbb{Z}$ of integers, prove that $\mathbb{Q}$ is closed under addition.

Proof. Let $q, r \in \mathbb{Q}$, say $q = \frac{a}{b}$ and $r = \frac{c}{d}$, where $a, b, c, d \in \mathbb{Z}$.

Then, by definition,
$$\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}.$$

Now $\mathbb{Z}$ is closed under multiplication, so $ad, bc, bd \in \mathbb{Z}$. Also, $\mathbb{Z}$ is closed under addition, so $ad+bc \in \mathbb{Z}$. Therefore, $\frac{ad+bc}{bd}$ is a ratio of integers. Since $b \neq 0$ and $d \neq 0$, $bd \neq 0$.

This means $\frac{ad+bc}{bd} = \frac{a}{b} + \frac{c}{d} \in \mathbb{Q}$, as desired. ■