MATH 34001
MWF 11:00

Fundamental Concepts of Algebra

Spring 2015
Dr. Kracht

Name: KEY Score: ____/100

Exam 2: Friday, April 10, 2015

To receive full credit, show all work necessary to justify answers and all steps of proofs clearly, in logical sequence, using notation developed in class. Reference properties and theorems explicitly where needed. Write proofs in complete sentences (with proper capitalization, punctuation, subject, verb, etc.). Introduce notation with "Let" statements and quantify all variables. Partial credit will be given only for significant progress toward a solution.

No calculators or other electronic devices are permitted. No books or notes are permitted.

1. (3 pts) *(Complete the following using set notation.)*

$\mathbb{C} = \{a+bi \mid a, b \in \mathbb{R} \text{ and } i^2 = -1\}$

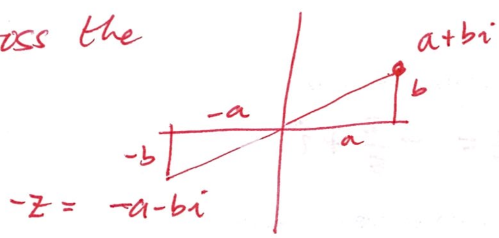
2. (4 points) Order the sets $\mathbb{C}, \mathbb{N}, \mathbb{Q}, \mathbb{R}, \mathbb{W}$, and $\mathbb{Z}$ from "smallest' to "largest" by containment.

$\mathbb{N} \subseteq \mathbb{W} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$

3. (6 pts) Let $z \in \mathbb{C}$.

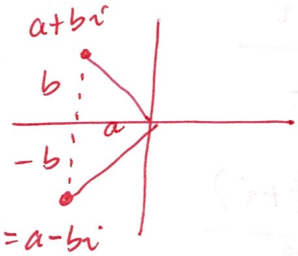
(a) How are z and $-z$ related geometrically?

They are reflections across the origin.



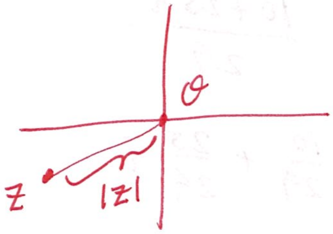
(b) How are z and $\bar{z}$ related geometrically?

They are reflections across the ~~x~~ real axis.



(c) What does $|z|$ represent geometrically?

It represents the distance from z to the origin.



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4. (13 points) Let $z = \frac{2}{5} - i$. Find each of the following. Write answers in standard form, simplified.

(a) Re(z) $= \frac{2}{5}$

(b) Im(z) $= -1$

(c) $\overline{z} = \frac{2}{5} + i$

(d) $|z| = \sqrt{\left(\frac{2}{5}\right)^2 + (-1)^2}$

$= \sqrt{\frac{4}{25} + \frac{25}{25}}$

$= \sqrt{\frac{29}{25}}$

$= \frac{\sqrt{29}}{5}$

(e) the additive inverse of z

$-z = -\frac{2}{5} + i$

(f) the multiplicative inverse of z

$\frac{1}{z} = \frac{\overline{z}}{|z|^2} = \frac{\frac{2}{5} + i}{\frac{29}{25}}$

$= \frac{25\left(\frac{2}{5} + i\right)}{29}$

$= \frac{10 + 25i}{29}$

$= \frac{10}{29} + \frac{25}{29}i$

5. (10 points) Perform the indicated operations, writing your answers in standard form.

(a) $(-7 + 2i) - (5 - 4i) + (1 - 8i)$

$= (-7 - 5 + 1) + (2 + 4 - 8)i$

$= -11 - 2i$

(b) $(6 - i)(5 + 2i)$

$= (30 + 2) + (12 - 5)i$

$= 32 + 7i$

(c) $i^{823} = i^{820} \cdot i^3 = (i^4)^{205} \cdot i^3$

$= 1^{205} \cdot (-i)$

$= -i.$

(d) $\frac{1 + 9i}{-2 + 3i} = \frac{(1 + 9i)(-2 - 3i)}{(-2 + 3i)(-2 - 3i)}$

$= \frac{(-2 + 27) + (-3 - 18)i}{(-2)^2 + (3)^2}$

$= \frac{25 - 21i}{4 + 9}$

$= \frac{25}{13} - \frac{21}{13}i$

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6. (7 points) Write $z = 7 - 7i$ in polar form, using the principal value of the argument.

$r = |z| = \sqrt{7^2 + (-7)^2}$

$= \sqrt{7^2(1+1)}$

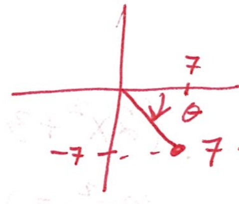
$= 7\sqrt{2}$

Let $\theta = \operatorname{Arg} z$. Then

$\tan\theta = \frac{(-7)}{7} = -1$

and $\theta \in (-\frac{\pi}{2}, 0)$

So $\theta = \frac{-\pi}{4}$



(Quad IV)

$\therefore \quad z = 7\sqrt{2}\left[\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right]$.

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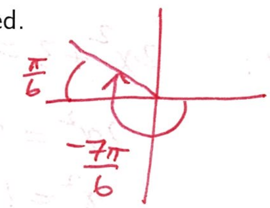
7. (7 points) Write $z = 4\sqrt{5}\left[\cos\left(-\frac{7\pi}{6}\right) + i\sin\left(-\frac{7\pi}{6}\right)\right]$ in standard form, simplified.



$z = 4\sqrt{5}\left[-\frac{\sqrt{3}}{2} + i\left(\frac{1}{2}\right)\right]$

$= -2\sqrt{15} + 2\sqrt{5}\, i$

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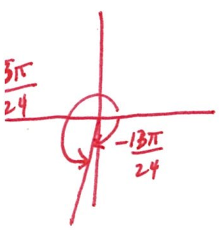
8. (10 points) Let $z = 6\left(\cos\frac{\pi}{8} + i\sin\frac{\pi}{8}\right)$ and $w = \frac{1}{2}\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right)$. Write in *exact* **polar** form, simplified.

(Any correct argument accepted.)

(a) $z \cdot w = \left(6 \cdot \frac{1}{2}\right)\left[\cos\left(\frac{\pi}{8} + \frac{4\pi}{3}\right) + i\sin\left(\frac{\pi}{8} + \frac{4\pi}{3}\right)\right]$



$= 3\left[\cos\left(\frac{3\pi + 32\pi}{24}\right) + i\sin\left(\frac{3\pi + 32\pi}{24}\right)\right]$

$= 3\left[\cos\left(\frac{35\pi}{24}\right) + i\sin\left(\frac{35\pi}{24}\right)\right] = 3\left[\cos\left(-\frac{13\pi}{24}\right) + i\sin\left(-\frac{13\pi}{24}\right)\right]$

(b) $\frac{1}{z} = \frac{1}{6}\left[\cos\left(-\frac{\pi}{8}\right) + i\sin\left(-\frac{\pi}{8}\right)\right]$

(c) $w^6 = \left(\frac{1}{2}\right)^6\left[\cos\left(6 \cdot \frac{4\pi}{3}\right) + i\sin\left(6 \cdot \frac{4\pi}{3}\right)\right]$

$= \frac{1}{64}\left[\cos(8\pi) + i\sin(8\pi)\right]$

$= \frac{1}{64}\left[\cos(0) + i\sin(0)\right]$

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9. (10 pts) Find all pairs of real numbers (x, y) satisfying:

$$(2x+i)(x-3i) = 35+2yi$$

$$(2x^2+3)+(-6x+x)i = 35+2yi$$

$$(2x^2+3)+(-5x)i = 35+2yi$$

equating real + imaginary parts, we get the system of equations : $\begin{cases} 2x^2+3 = 35 \\ -5x = 2y \end{cases}$

eqn 1: $2x^2+3=35$

$2x^2 = 32$

$x^2 = 16$

$x = \pm 4$

eqn 2: $-5x = 2y$

$y = \frac{-5}{2}x$

If $x=4$, $y = \frac{-5}{2}(4) = -10$

If $x=-4$, $y = \frac{-5}{2}(-4) = 10$.

Solutions: $(4,-10), (-4,10)$.

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10. (10 points) Show that if z is a complex number, then $\bar{\bar{z}} = z$.

Proof. Let $z \in \mathbb{C}$, say $z = a+bi$, where $a, b \in \mathbb{R}$. Then

$\bar{\bar{z}} = \overline{\overline{(a+bi)}}$

$= \overline{(a-bi)}$ (defn of complex conjugate)

$= a+bi$ (defn of complex conjugate)

$= z$, as desired. ■

11. (10 points) Using the definition of $\mathbb{C}$ (including the definitions of addition and multiplication in $\mathbb{C}$) and the algebraic properties of the field $\mathbb{R}$ of real numbers, prove that $\mathbb{C}$ is closed under addition.

Proof. Let $z, w \in \mathbb{C}$. Then $z = a+bi$ and $w = c+di$, for some $a,b,c,d \in \mathbb{R}$. We have

$$z+w = (a+bi)+(c+di)$$
$$= (a+c)+(b+d)i,$$

by the definition of addition in $\mathbb{C}$. Since $\mathbb{R}$ is closed under addition, $a+c$ and $b+d$ are real numbers. Therefore, $z+w \in \mathbb{C}$. Since z and w were arbitrary, this shows that $\mathbb{C}$ is closed under addition. ■

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12. (10 pts) Let $z \in \mathbb{C}$. Prove that if $(\bar{z})^2 = z^2$, then z is either real or pure imaginary.

Proof. Let $z \in \mathbb{C}$, say $z = a+bi$, where $a,b \in \mathbb{R}$. Suppose $(\bar{z})^2 = z^2$. That is,

$$(a-bi)^2 = (a+bi)^2, \text{ and so}$$

$$(a^2-b^2)+(-2ab)i = (a^2-b^2)+(2ab)i.$$

By the definition of equivalence in $\mathbb{C}$, this implies that $-2ab = 2ab$, which means $4ab = 0$. It follows that $a=0$ or $b=0$. If $b=0$, then $z=a$ is real. If $b \neq 0$, then $a=0$, and so $z = bi$ is pure imaginary. In conclusion, if $(\bar{z})^2 = z^2$, then z is either real or pure imaginary. ■