Extra Credit: Review for Final Exam

Due by 5:00 p.m., Friday, May 5, 2006

- Give to me or else leave in my mailbox in 233 MSB (open 8-5 weekdays).
- No extra credit assignments will be accepted after 5:00 p.m., Friday, May 5, 2006.
- Solutions will be posted on the web page shortly after the due date.
- Assignments will not be handed back before the final exam, so you might want to make a copy to keep.
- You will earn 1 point for each completed problem. One problem will be randomly chosen to be graded for an additional 10 points.

1. Consider the function \( f \) depicted below.

\[
\begin{array}{c}
\text{\( f(-2) = \)} \\
\text{\( f(1) = \)} \\
\text{\( f(2) = \)} \\
\text{\( f(4) = \)}
\end{array}
\]

\[
\begin{array}{c}
\text{\( \lim_{x \to -2^-} f(x) = \)} \\
\text{\( \lim_{x \to -1^-} f(x) = \)} \\
\text{\( \lim_{x \to -2^+} f(x) = \)} \\
\text{\( \lim_{x \to -4^+} f(x) = \)}
\end{array}
\]

(a) Find each of the following (or state “does not exist”).

(b) Answer “Yes” or “No.” (This refers to the function depicted above.)

   i. Is \( f \) continuous at \( x = -2 \)?

   ii. Is \( f \) continuous at \( x = 1 \)?

   iii. Is \( f \) continuous at \( x = 2 \)?

   iv. Is \( f \) continuous at \( x = 4 \)?

(c) True or False: For a function \( f \), the value of \( \lim_{x \to a} f(x) \) depends upon the value of \( f(a) \).
2. (a) State the formal definition of derivative.

\[ f'(x) = \]

(b) Use the definition of the derivative to find the derivative of the function \( f(x) = \frac{3}{1-x} \). Show all steps.

\[ f'(x) = \]
3. Let \( P(x) \) be the profit, in dollars, obtained from manufacturing \( x \) widgets. Fill in the table with a mathematical expression and appropriate units corresponding to each description.

<table>
<thead>
<tr>
<th>Description</th>
<th>Mathematical Expression</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>profit from manufacturing 5000 widgets</td>
<td></td>
<td></td>
</tr>
<tr>
<td>profit from manufacturing the 5001\textsuperscript{st} widget</td>
<td></td>
<td></td>
</tr>
<tr>
<td>average rate of change of profit from a production level of 5000 widgets</td>
<td></td>
<td></td>
</tr>
<tr>
<td>instantaneous rate of change of profit at a production level of 5000widgets</td>
<td></td>
<td></td>
</tr>
<tr>
<td>instantaneous rate of change of profit at a production level of 5000widgets (another expression)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

4. Let \( P(t) \) be population of black squirrels on campus at time \( t \). For each scenario, fill in the blanks with one of the symbols \(<\), \(>\), \(=\), or \(?\) (if the sign cannot be determined).

(a) The population of squirrels has remained steady all year.

\[ P'(t) \quad 0 \quad P''(t) \quad 0 \]

(b) An epidemic of the deadly squirrel flu has been causing the population to fall more and more rapidly.

\[ P'(t) \quad 0 \quad P''(t) \quad 0 \]

(c) The population has been growing at a slow but steady rate.

\[ P'(t) \quad 0 \quad P''(t) \quad 0 \]

(d) Since students started feeding the squirrels, the population has been growing more and more rapidly.

\[ P'(t) \quad 0 \quad P''(t) \quad 0 \]

(e) The population has been growing, but not as rapidly as in the early spring.

\[ P'(t) \quad 0 \quad P''(t) \quad 0 \]
5. (a) Write the mathematical expression representing the exponent to which 5 must be raised to obtain 41.7.

(b) Write the mathematical expression representing the exponent to which $e$ must be raised to obtain $\frac{2}{3}$.

(c) Evaluate each of the following.

i. $\log_5 \frac{1}{25}$

ii. $\log_8 1$

iii. $\log_{27} 3$

6. A rocket is traveling with velocity $v(t) = t^2 + 2t + 100$ feet per second at time $t$ seconds after take-off. Show your reasoning. Include appropriate units with your answers.

   (a) Find the acceleration $a(t)$ of the rocket after $t$ seconds.

   (b) Find the acceleration of the rocket after 3 seconds.

   (c) If the rocket took off from a platform 10 feet above the ground, find the height $h(t)$ of the rocket above the ground at $t$ seconds.

   (d) Find the height of the rocket after 3 seconds.
7. Consider the function \( f(x) = x^4 - 8x^3 + 256 \).

(a) Find the \( y \)-intercept of the graph of \( f \). (Express it as an ordered pair.)

(b) Find \( f'(x) \) and factor it completely.

(c) Make a sign chart for \( f' \) and indicate on which intervals the original function \( f \) is increasing and on which intervals \( f \) is decreasing. *Show your reasoning.*

(d) Find all critical points of \( f \). (Express them as ordered pairs.)

(e) Give the ordered pairs for all

i. relative minimum points

ii. relative maximum points

(f) Find \( f''(x) \) and factor it completely.

(g) Make a sign chart for \( f'' \) and indicate on which intervals the original function \( f \) is concave up and on which intervals \( f \) is concave down.

(h) Find all inflection points of \( f \). (Express them as ordered pairs.)

*continued on next page*
(i) Carefully sketch the graph $y = f(x)$. Plot and label all points found above. Indicate all important aspects of the graph clearly.
8. Find the derivatives of the following functions and simplify your answer. It might also help to simplify the original function before differentiating.

(a) \( f(x) = \sqrt[3]{x^2} + \frac{1}{\sqrt[3]{x^2}} \)

(b) \( f(x) = 25x + \frac{x}{25} + \frac{25}{x} + \frac{1}{25x} \)

(c) \( f(x) = \frac{2 + x^3}{2 - x^3} \)
(d) $f(x) = \frac{1}{\sqrt{5x - x^2}}$

(e) $f(x) = x \ln x - x$

(f) $f(x) = e^{x^2-2x+1}$
9. An open-top box is to be made from a square sheet of cardboard that measures 18 inches by 18 inches by removing a square from each corner and folding up the sides. What are the dimensions and volume of the largest box that can be made in this way?

Show your reasoning below. Be sure to:

- Introduce all variables with “Let” statements. Include the units. Draw and label diagrams, if appropriate.
- Verify that you have indeed found the maximum or minimum point (on the appropriate domain)
- Answer the question posed in the problem in a complete sentence, using appropriate units.
10. An airline finds that if it prices a cross-country ticket at $200, it will sell 300 tickets per day. It estimates that each $10 price reduction will result in 30 more tickets sold per day. Find the ticket price (and the number of tickets sold) that will maximize the airline’s revenue.

Show your reasoning below. Be sure to:

- Introduce all variables with “Let” statements. Include the units. Draw and label diagrams, if appropriate.
- Verify that you have indeed found the maximum or minimum point (on the appropriate domain)
- Answer the question posed in the problem in a complete sentence, using appropriate units.
11. A company’s cost function is $C(x) = 20 + 3x + \frac{54}{\sqrt{x}}$ dollars.

(a) Find the marginal cost function.

(b) Find the marginal cost at $x = 9$ and interpret your answer. (Explain what it really means in plain English.)

12. At the birth of a child, the parents invest $500 in an account earning 3.75% annual interest. Suppose interest is compounded continuously.

(a) Find a formula for $FV(t)$, the future value of the account after $t$ years.

(b) Find (to the nearest penny) how much the child will have in the account on her 21st birthday.

(c) How long will it take for the balance in the account to reach $800? Set up and solve an equation. Round your answer to one decimal place.
13. Find each indefinite integral (general antiderivative). Show all steps for full credit.

(a) \( \int \left( 25x + \frac{x}{25} + \frac{25}{x} + \frac{1}{25x} \right) \, dx \)

(b) \( \int \left( \frac{1}{\sqrt{x^5}} + e^{5x} \right) \, dx \)

(c) \( \int \left( x^9 - x \right) \sqrt{x^{10} - 5x^2} \, dx \)

(d) \( \int \frac{x^2 - 25}{x + 5} \, dx \)
14. Find each definite integral. Give exact answers, simplified. Show all steps for full credit.

(a) \[ \int_{-1}^{0} \frac{1}{2 - 3x} \, dx \]

(b) \[ \int_{e^2}^{e^5} \frac{1}{x (\ln x)^2} \, dx \]
15. Let $f(x) = 9 - x^2$ and $g(x) = 3 - x$. Compute the area $A$ of the region bounded by the curves $y = f(x)$ and $y = g(x)$ by following the steps below.

(a) Use algebra to find all points of intersection of $f$ and $g$. (Set up and solve an equation.)

(b) Determine algebraically whether $f$ or $g$ is the “top” function on the interval determined in the first part.

(c) Sketch the graphs of $f$ and $g$. Plot and label the points of intersection. Shade the region whose area we wish to compute.

(d) Set up and evaluate the integral to find $A$. Express your answer as a fraction, reduced to lowest terms.
16. (a) If \( h'(t) \) is the rate of growth of height of a child in inches per year, what does \( \int_3^5 h'(t) \, dt \) represent?

(b) If a bacteria culture starts with 50 individuals and increases at a rate \( n'(t) \) individuals per hour, what does 
\[ 50 + \int_0^3 n'(t) \, dt \]
represent?

17. The marginal cost of manufacturing \( x \) feet of a certain silver chain is \( 2 + 0.04x - 0.0003x^2 \) (in dollars per foot). Find the increase in cost if the production level is raised from 50 feet to 100 feet. Introduce your function(s) with a “Let” statement.
18. (a) The demand function for “I Love Intuitive Calculus” T-shirts is \( d(x) = 128 - 0.3x^2 \). Find the consumers’ surplus, \( CS \), at the demand level \( x = 20 \) T-shirts. (Set up and evaluate an integral.)

(b) The supply function for “I Love Intuitive Calculus” T-shirts is \( s(x) = 0.4x \). Find the producers’ surplus, \( PS \), at the demand level \( x = 20 \) T-shirts. (Set up and evaluate an integral.)
19. The manager of TeleStar Cable Service estimates that the total number of subscribers to cable service in a certain city \( t \) yr from now will be

\[
N(t) = \frac{-40,000}{\sqrt{1 + 0.2t}} + 50,000
\]

Find the average number of cable television subscribers over the next five years if this prediction holds true.

20. The Lorenz curve for income distribution in the United States in 1996 was approximately \( L(x) = x^{2.7} \).

(a) Find the Gini Index, \( GI \), for income distribution in the United States in 1996. (Set up and evaluate an integral. Round your answer to two decimal places.)

(b) In 1993, the Gini Index for income distribution in the United States was approximately 0.39. Between 1993 and 1996 did income distribution become more or less equal?