EXAM 1: RED Version

Please make sure the color of your scantron sheet matches the version color.

NO CALCULATORS.

Some Formulas

Product Rule: If \( p(x) = f(x) \cdot g(x) \), then \( p'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x) \).

Quotient Rule: If \( q(x) = \frac{f(x)}{g(x)} \), then \( q'(x) = \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{[g(x)]^2} \).

Generalized Power Rule: If \( h(x) = [g(x)]^n \), then \( h'(x) = n[g(x)]^{n-1} \cdot g'(x) \).

Multiple Choice: 18 questions at 4 points each.

Fill in the letter of the best response on the scantron sheet.

For your own future reference, you may wish to record your answers on this sheet as well.

1. Evaluate \( \lim_{x \to 2} \frac{3x^2 + 5}{x^2} \).
   \( \frac{3 \cdot 2^2 + 5}{2^2} = \frac{3 \cdot 4 + 5}{4} = \frac{12 + 5}{4} = \frac{17}{4} \).
   (a) 0
   (b) 41
   (c) 17
   (d) 11
   (e) none of these

3. Let \( f(x) = \begin{cases} \frac{x}{2} & \text{if } x \leq 1; \\ 2x & \text{if } x > 1. \end{cases} \)
   \( f(1) = 1 \), so it is defined.
   (a) Yes
   (b) No, since \( \lim_{x \to 1^-} f(x) \) does not exist.
   (c) No, since \( f(1) \) is undefined.
   (d) No, since \( f(x) \) is a piecewise-defined function
   (e) none of these

2. Evaluate \( \lim_{x \to 6} \frac{x - 6}{x^2 - 36} \).
   \( \lim_{x \to 6} \frac{x - 6}{(x - 6)(x + 6)} = \lim_{x \to 6} \frac{1}{x + 6} = \frac{1}{12} \).
   (a) 0
   (b) 1
   (c) 12
   (d) does not exist
   (e) none of these

4. Let \( f(x) = x^3 + 2x + 1 \). What is the average rate of change in \( f(x) \) between the values \( x = 1 \) and \( x = 5 \)?
   (a) 132 because \( f(5) - f(1) = 132 \)
   (b) 33 because \( \frac{f(5) - f(1)}{4} = 33 \)
   (c) 41 because \( \frac{f'(3) + f'(5)}{2} = 41 \)
   (d) 29 because \( f'(3) = 29 \)
   (e) none of these
5. Find the instantaneous rate of change of the function \( f(x) = 10x^4 + 5 \) at \( x = -2 \).

(a) \(-320\)  
(b) \(320\)  
(c) \(-315\)  
(d) \(-80\)  
(e) none of these

\[ f'(x) = 40x^3 + 0 \]

\[ f'(-2) = 40(-2)^3 = -320 \]

6. Find the slope of the tangent line to \( f(x) = \frac{1}{x} \) at \( x = 3 \).

(a) \(\frac{1}{9}\)  
(b) \(-1\)  
(c) \(-\frac{1}{3}\)  
(d) \(-3\)  
(e) none of these

\[ f'(x) = -\frac{1}{x^2} \]

\[ f'(3) = -\frac{1}{3^2} \]

\[ = -\frac{1}{9} \]

7. Evaluate \( \frac{d}{dx} \sqrt[3]{x^3} \bigg|_{x=1} \).

(a) \(\frac{3}{6}\)  
(b) \(1\)  
(c) \(\frac{2}{3}\)  
(d) \(0\)  
(e) none of these

\[ \frac{d}{dx} x^{\frac{3}{3}} \bigg|_{x=1} = \frac{5}{3} x^{\frac{2}{3}} \bigg|_{x=1} = \frac{5}{3} (1)^{\frac{2}{3}} \]

\[ = \frac{5}{3} \cdot 1 \]

\[ = \frac{5}{3} \]

8. A rocket rises to a height \( h(t) = t^3 + 0.5t^2 \) feet in \( t \) seconds. Find the rocket's velocity at time \( t = 10 \) seconds.

(a) \(40 \text{ ft/s}\)  
(b) \(61 \text{ ft/s}\)  
(c) \(310 \text{ ft/s}\)  
(d) \(1050 \text{ ft/s}\)  
(e) none of these

\[ v(t) = h'(t) = 3t^2 + 0.5(2t) = 3t^2 + t \text{ ft/s} \]

\[ v(10) = 3(10^2) + 10 = 300 + 10 = 310 \text{ ft/s} \]

9. A rocket rises to a height \( h(t) = t^3 + 0.5t^2 \) feet in \( t \) seconds. Find the rocket's acceleration at time \( t = 10 \) seconds.

(a) \(40 \text{ ft/s}^2\)  
(b) \(61 \text{ ft/s}^2\)  
(c) \(310 \text{ ft/s}^2\)  
(d) \(1050 \text{ ft/s}^2\)  
(e) none of these

\[ a(t) = v'(t) = h''(t) = 6t + 1 \text{ ft/s}^2 \]

\[ a(10) = 6(10) + 1 = 61 \text{ ft/s}^2 \]

10. If \( f(x) = x^5 - 2x^4 + 10x^3 - 5x^2 + 7x - 1 \), evaluate \( f'''(-1) \).

(a) \(0\)  
(b) \(72\)  
(c) \(48\)  
(d) \(168\)  
(e) none of these

\[ f'(x) = 5x^4 - 8x^3 + 30x^2 - 10x + 7 \]

\[ f''(x) = 20x^3 - 24x^2 + 60x - 10 \]

\[ f'''(x) = 60x^2 - 48x + 60 \]

\[ f'''(-1) = 60(-1)^2 - 48(-1) + 60 = 60 + 48 + 60 = 168 \]
11. Find the derivative of \( g(x) = 5x^{100} \).
   (a) \( g'(x) = 500x \)
   (b) \( g'(x) = 5x^{99} \)
   (c) \( g'(x) = 500x^{99} \)
   (d) \( g'(x) = 500x^{101} \)
   (e) none of these

\[
g'(x) = 5 \cdot 100x^{99} = 500x^{99}
\]

13. Find the derivative of \( y = \frac{x^4 + 7x^3}{x^2} \).
   (a) \( \frac{dy}{dx} = \frac{4x^3 + 21x^2}{2x} \)
   (b) \( \frac{dy}{dx} = 4x^3 + 7 \)
   (c) \( \frac{dy}{dx} = 23x^2 \)
   (d) \( \frac{dy}{dx} = 2x + 7 \)
   (e) none of these

\[
y = \frac{x^4}{x^2} + \frac{7x^3}{x^2} = x^2 + 7x
\]

\[
\frac{dy}{dx} = 2x + 7
\]

12. Find the derivative of \( w(x) = \frac{4}{\sqrt{x}} \).
   (a) \( w'(x) = \frac{-2}{\sqrt{x}^3} \)
   (b) \( w'(x) = \frac{-2}{\sqrt{x}^2} \)
   (c) \( w'(x) = \frac{2}{\sqrt{x}} \)
   (d) \( w'(x) = \frac{-4}{\sqrt{x}} \)
   (e) none of these

\[
w'(x) = -2 \cdot \frac{1}{2} x^{-3/2} = -2x^{-3/2}
\]

14. Use the product rule to find the derivative of \( h(t) = (7t - 3)(2t + 5) \). The answer need not be simplified.
   (a) \( h'(t) = 14 \)
   (b) \( h'(t) = 14t^2 \)
   (c) \( h'(t) = 7(2t + 5) + 2(7t - 3) \)
   (d) \( h'(t) = 7t(2t + 5) + 2t(7t - 3) \)
   (e) none of these
15. Suppose a company's revenue function is $R(x) = 25x + 4\sqrt{x}$, in dollars, where $x$ is the number of widgets produced. Find and interpret $R(100)$.

(a) The company's total revenue from the first 100 widgets is $25.20.

(b) The company's revenue from the 100th widget is $25.40.

(c) The company's total revenue from the first 100 widgets is $2540.

(d) The company's average revenue from the first 100 widgets is $25.40 per widget.

(e) none of these

\[ R(100) = 25(100) + 4\sqrt{100} \]
\[ = 2500 + 4 \cdot 10 \]
\[ = 2540 \]

16. Suppose a company's revenue function is $R(x) = 25x + 4\sqrt{x}$, in dollars, where $x$ is the number of widgets produced. Find and interpret $R'(100)$.

(a) The company's total revenue from the first 100 widgets is $25.40.

(b) The company's revenue from the 100th widget is approximately $25.20.

(c) The company's average revenue from the first 100 widgets is $25.40 per widget.

(d) The company's average revenue from the first 100 widgets is $25.20 per widget.

(e) none of these

\[ R(x) = 25x + 4\sqrt{x} \]
\[ \text{So } R'(x) = 25 + 2 \cdot \frac{1}{2} x^{-\frac{1}{2}} \]
\[ = 25 + \frac{1}{\sqrt{x}} \]
\[ \text{So } R'(100) = 25 + \frac{1}{\sqrt{100}} \]
\[ = 25 + \frac{1}{10} \]
\[ = 25.20 / \text{widget} \]

17. Find the derivative of $f(z) = \sqrt{9z^2 - 25}$.

(a) $f'(z) = \frac{9z}{\sqrt{9z^2 - 25}}$

(b) $f'(z) = 3z - 5$

(c) $f'(z) = 3$

(d) $f'(z) = \frac{1}{2\sqrt{9z^2 - 25}}$

(e) none of these

\[ f'(z) = \frac{9z}{\sqrt{9z^2 - 25}} \]
\[ = \frac{9z}{\sqrt{9z^2 - 25}} \cdot \left( \frac{18z}{18z} \right) \]
\[ = 9z \cdot \frac{18z}{\sqrt{9z^2 - 25}} \]
\[ = \frac{9z}{\sqrt{9z^2 - 25}} \]

18. Find the derivative of $y = x^2 (3x - 1)^5$ and simplify your answer.

(a) $\frac{dy}{dx} = 10x(3x - 1)^4$

(b) $\frac{dy}{dx} = 30x(3x - 1)^4$

(c) $\frac{dy}{dx} = x(3x - 1)^4(11x - 1)$

(d) $\frac{dy}{dx} = x(3x - 1)^4(21x - 2)$

(e) none of these

\[ \frac{dy}{dx} = 2x(3x - 1)^5 + x^2 \left[ 5(3x - 1)^4 \cdot 3 \right] \]
\[ = 2x(3x - 1)^5 + 15x^2 (3x - 1)^4 \]
\[ = x(3x - 1)^4 \left[ 2(3x - 1) + 15x \right] \]
\[ = x(3x - 1)^4 \left[ 6x - 2 + 15x \right] \]
\[ = x(3x - 1)^4 (21x - 2) \]
Long Answer
Write all work carefully and neatly for full credit.

19. (6 points) Find and simplify the derivative of \( f(x) = \frac{x^2 + 4}{x^2 - 4} \).

\[
\begin{align*}
f'(x) &= \frac{2x(x^2 - 4) - (x^2 + 4)(2x)}{(x^2 - 4)^2} \\
&= \frac{2x \left[ x^2 - 4 - (x^2 + 4) \right]}{(x^2 - 4)^2} \\
&= \frac{2x \left[ x^2 - 4 - x^2 - 4 \right]}{(x^2 - 4)^2} \\
&= \frac{2x \left[ -8 \right]}{(x^2 - 4)^2} \\
&= \frac{-16x}{(x^2 - 4)^2}.
\end{align*}
\]

20. (12 points) Find an equation of the line tangent to the graph of the function \( f(x) = x^3 + 5x^2 - 1 \) at the point where \( x = -1 \) by following the given steps. Show your reasoning.

(a) Find the \( y \)-coordinate of the point on the curve \( y = f(x) \) where \( x = -1 \).

\[
f(-1) = (-1)^3 + 5(-1)^2 - 1 = -1 + 5 - 1 = 3
\]

(b) Find the derivative \( f'(x) \). You may use short-cuts.

\[
f'(x) = 3x^2 + 10x
\]

(c) Find the slope of the line tangent to the curve \( y = f(x) \) at the point where \( x = -1 \).

\[
m = f'(-1) = 3(-1)^2 + 10(-1) = 3 - 10 = -7
\]

(d) Find the equation of the line tangent to the curve \( y = f(x) \) at the point where \( x = -1 \). Write it in the form \( "y = mx + b." \)

\[
\begin{align*}
y - y_1 &= m (x - x_1) \\
y - 3 &= -7(x + 1) \\
y - 3 &= -7x - 7
\end{align*}
\]

\[
\begin{align*}
y - 3 &= -7x - 7 \\
y &= -7x - 7 + 3 \\
y &= -7x - 4
\end{align*}
\]
21. (10 points) Use the definition of derivative to find and simplify the derivative, $f'(x)$, for the function $f(x) = 2 - 3x^2$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{2 - 3(x+h)^2 - [2 - 3x^2]}{h}$$

$$= \lim_{h \to 0} \frac{2 - 3(x^2 + 2hx + h^2) - 2 + 3x^2}{h}$$

$$= \lim_{h \to 0} \frac{-6hx - 3h^2}{h} = \lim_{h \to 0} \frac{h(-6x - 3h)}{h} = \lim_{h \to 0} (-6x - 3h)$$

$$= -6x - 3(0) = -6x$$

**Bonus.** (10 points) Use the definition of derivative to prove that if $f$ and $g$ are differentiable functions, then

$$(f - g)'(x) = \lim_{h \to 0} \frac{(f - g)(x+h) - (f - g)(x)}{h}$$

$$= \lim_{h \to 0} \frac{f(x+h) - g(x+h) - [f(x) - g(x)]}{h}$$

$$= \lim_{h \to 0} \frac{f(x+h) - g(x+h) - f(x) + g(x)}{h}$$

$$= \lim_{h \to 0} \frac{f(x+h) - f(x) - g(x+h) + g(x)}{h}$$

$$= \lim_{h \to 0} \left[ \frac{f(x+h) - f(x)}{h} - \frac{g(x+h) - g(x)}{h} \right]$$

$$= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} - \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}$$

$$= f'(x) - g'(x)$$