MATH 11012
Circle one: 8:50 5:30

Intuitive Calculus

Spring 2011
Ms. Kracht

Name: KEY

Score: /100
(110 pts available)

EXAM 1: Version A

Show your reasoning for full credit.

Some Formulas

Product Rule: If $p(x) = f(x) \cdot g(x)$, then $p'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x)$.

Quotient Rule: If $q(x) = \dfrac{f(x)}{g(x)}$, then $q'(x) = \dfrac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{[g(x)]^2}$.

Generalized Power Rule: If $h(x) = [g(x)]^n$, then $h'(x) = n[g(x)]^{n-1} \cdot g'(x)$.

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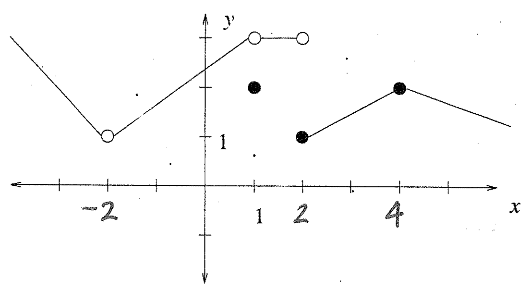
1. (10 points) The function f is depicted below.



(a) Find each of the following or state "does not exist" ("dne").

$\lim_{x \to -2^-} f(x) =$ 1	$\lim_{x \to 1^-} f(x) =$ 3	$\lim_{x \to 2^-} f(x) =$ 3	$\lim_{x \to 4^-} f(x) =$ 2
$\lim_{x \to -2^+} f(x) =$ 1	$\lim_{x \to 1^+} f(x) =$ 3	$\lim_{x \to 2^+} f(x) =$ 1	$\lim_{x \to 4^+} f(x) =$ 2
$\lim_{x \to -2} f(x) =$ 1	$\lim_{x \to 1} f(x) =$ 3	$\lim_{x \to 2} f(x) =$ dne	$\lim_{x \to 4} f(x) =$ 2
$f(-2) =$ dne	$f(1) =$ 2	$f(2) =$ 1	$f(4) =$ 2

(b) Circle YES or NO for each of the following.

i. Is f continuous at -2? YES NO

ii. Is f continuous at 1? YES NO

iii. Is f continuous at 2? YES NO

iv. Is f continuous at 4? YES NO

Multiple Choice: 10 questions at 3 points each.

Circle the letter of the best response

2. Evaluate $\lim_{x\to 2}(3x^2+5)$. $= (3)(2)^2+5$ $= 12+5$ $= 17$

(a) 0
(b) 11
(c) 17
(d) 41
(e) none of these

3. Evaluate $\lim_{x\to 6}\frac{x-6}{x^2-36}$. $= \lim_{x\to 6}\frac{x-6}{(x-6)(x+6)}$ $= \lim_{x\to 6}\frac{1}{x+6}$ $= \frac{1}{6+6} = \frac{1}{12}$

(a) $\frac{1}{12}$
(b) 12
(c) 0
(d) does not exist
(e) none of these

4. Let $f(x) = \begin{cases} x & \text{if } x \le 1; \\ 2x & \text{if } x > 1. \end{cases}$ Is $f(x)$ continuous at $x=1$?

$\lim_{x\to 1^-} f(x) = 1$

$\lim_{x\to 1^+} f(x) = 2\cdot 1 = 2$

so $\lim_{x\to 1} f(x)$ dne

(a) Yes
(b) No, since $f(1)$ is undefined.
(c) No, since $f(x)$ is a piecewise-defined function
(d) No, since $\lim_{x\to 1} f(x)$ does not exist.
(e) none of these

5. Let $f(x) = x^3+2x+1$. What is the average rate of change in $f(x)$ between the values $x=1$ and $x=5$?

(a) 29 because $f'(3) = 29$
(b) 41 because $\frac{f'(1)+f'(5)}{2} = 41$
(c) 33 because $\frac{f(5)-f(1)}{4} = 33$
(d) 132 because $f(5)-f(1) = 132$
(e) none of these

6. Find the instantaneous rate of change of the function $f(x) = 10x^4+5$ at $x=-2$.

$f'(x) = 40x^3$

$f'(-2) = 40(-2)^3$ $= 40(-8)$ $= -320$

(a) -80
(b) -320
(c) 320
(d) -315
(e) none of these

7. Find the slope of the tangent line to $f(x) = \frac{1}{x}$ at $x=3$.

$f(x) = x^{-1}$

$f'(x) = -x^{-2}$ $= -\frac{1}{x^2}$

$f'(3) = -\frac{1}{3^2} = -\frac{1}{9}$

(a) -1
(b) $-\frac{1}{3}$
(c) -3
(d) $-\frac{1}{9}$
(e) none of these

8. Find the derivative of $g(x) = 5x^{100}$.

(a) $g'(x) = 5x^{99}$
(b) $g'(x) = 500x^{99}$
(c) $g'(x) = 500x$
(d) $g'(x) = 500x^{101}$
(e) none of these

9. Use the product rule to find the derivative of $h(t) = (7t-3)(2t+5)$. *The answer need not be simplified.*

(a) $h'(t) = 7t(2t+5)+2t(7t-3)$
(b) $h'(t) = 7(2t+5)+2(7t-3)$
(c) $h'(t) = 14t^2$
(d) $h'(t) = 14$
(e) none of these

10. Suppose a company's revenue function is $R(x) = 25x + 4\sqrt{x}$, in dollars, where x is the number of widgets produced. Find and interpret $R(100)$.

(a) The company's revenue from the 100th widget is \$25.40.

(b) The company's total revenue from the first 100 widgets is \$25.20.

(c) The company's total revenue from the first 100 widgets is \$2540.

(d) The company's average revenue from the first 100 widgets is \$25.40 per widget.

(e) none of these

$R(100) = 25(100) + 4\sqrt{100}$
$= 2500 + 40$
$= \$2540$

11. Suppose a company's revenue function is $R(x) = 25x + 4\sqrt{x}$, in dollars, where x is the number of widgets produced. Find and interpret $R'(100)$.

(a) The company's revenue from the 100th widget is approximately \$25.20.

(b) The company's total revenue from the first 100 widgets is \$25.40.

(c) The company's average revenue from the first 100 widgets is \$25.40 per widget.

(d) The company's average revenue from the first 100 widgets is \$25.20 per widget.

(e) none of these

$R(x) = 25x + 4x^{1/2}$
$R'(x) = 25 + 4\left(\frac{1}{2}x^{-1/2}\right)$
$= 25 + \frac{2}{\sqrt{x}}$
$R'(100) = 25 + \frac{2}{\sqrt{100}} = 25 + \frac{2}{10} = \$25.20/\text{widget}$

Long Answer

Write all work carefully and neatly for full credit.

12. (15 pts) Find each derivative, simplifying your answer. *(You may use short-cuts.)*

(a) Find $\frac{d}{dz}\sqrt{9z^2 - 25}$.

$= \frac{d}{dz}(9z^2 - 25)^{1/2}$
$= \frac{1}{2}(9z^2-25)^{-1/2}(9 \cdot 2z)$
$= \dfrac{9z}{\sqrt{9z^2-25}}$

(b) Find $\frac{dy}{dx}$ where $y = x^2(3x-1)^5$

$\frac{dy}{dx} = 2x(3x-1)^5 + x^2\left[5(3x-1)^4(3)\right]$
$= 2x(3x-1)^5 + 15x^2(3x-1)^4$
$= x(3x-1)^4\left[2(3x-1) + 15x\right]$
$= x(3x-1)^4\left[6x - 2 + 15x\right]$
$= x(3x-1)^4(21x-2).$

(c) Find $g'(x)$ where $g(x) = \frac{x^2+3}{x^2-3}$

$g'(x) = \dfrac{2x(x^2-3) - (x^2+3)(2x)}{(x^2-3)^2}$
$= \dfrac{2x\left[x^2-3-(x^2+3)\right]}{(x^2-3)^2}$
$= \dfrac{2x\left[x^2-3-x^2-3\right]}{(x^2-3)^2}$
$= \dfrac{2x(-6)}{(x^2-3)^2}$
$= \dfrac{-12x}{(x^2-3)^2}$

$\boxed{21}$

13. (10 pts) A rocket rises to a height $h(t) = 1.5t^2 + 4t$ feet in t seconds. *Include units with each of the following.*

(a) Find the height of the rocket after 10 seconds.

$h(10) = 1.5(10)^2 + 4(10) = 150 + 40 = 190$ ft

(b) Find the velocity of the rocket after t seconds.

$v(t) = h'(t) = 3t + 4$ ft/s

(c) Find the velocity of the rocket after 10 seconds.

$v(10) = 3(10) + 4 = 34$ ft/s

(d) Find the acceleration of the rocket after t seconds.

$a(t) = v'(t) = 3$ ft/s^2

(e) Find the acceleration of the rocket after 10 seconds.

$a(10) = 3$ ft/s^2

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14. (5 pts) Evaluate $\frac{d^3}{dx^3}\sqrt[3]{x^5}$.

$$= \frac{d^3}{dx^3} x^{5/3} = \frac{d^2}{dx^2}\left(\frac{5}{3}x^{2/3}\right) = \frac{d}{dx}\left(\frac{5}{3}\cdot\frac{2}{3}x^{-1/3}\right) = \frac{d}{dx}\left(\frac{10}{9}x^{-1/3}\right) = \frac{10}{9}\left(-\frac{1}{3}\right)x^{-4/3} = \frac{-10}{27}x^{-4/3}$$

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15. (10 points) Find an equation of the line tangent to the graph of the function $f(x) = x^3 + 4x^2 - 5$ at the point where $x = -1$ by following the given steps. *Show your reasoning.*

2 (a) Find the y-coordinate of the point on the curve $y = f(x)$ where $x = -1$.

$f(-1) = (-1)^3 + 4(-1)^2 - 5 = -1 + 4 - 5 = -2$. $(-1, -2)$

3 (b) Find the derivative $f'(x)$. *You may use short-cuts.*

$f'(x) = 3x^2 + 8x$

2 (c) Find the slope of the line tangent to the curve $y = f(x)$ at the point where $x = -1$.

$m = f'(-1) = 3(-1)^2 + 8(-1) = 3 - 8 = -5$

3 (d) Find the equation of the line tangent to the curve $y = f(x)$ at the point where $x = -1$. *Write it in the form "$y = mx + b$."*

$y - y_1 = m(x - x_1)$

$y - (-2) = -5(x - (-1))$

$y + 2 = -5(x + 1)$

$y + 2 = -5x - 5$

$y = -5x - 5 - 2$

$\underline{y = -5x - 7}$.

16. (15 points) Use the **definition of derivative** to find and simplify the derivative, $f'(x)$, for the function $f(x) = 6 - 2x^2$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{[6 - 2(x+h)^2] - [6 - 2x^2]}{h}$$

$$= \lim_{h \to 0} \frac{6 - 2(x^2 + 2hx + h^2) - 6 + 2x^2}{h}$$

$$= \lim_{h \to 0} \frac{6 - 2x^2 - 4hx - 2h^2 - 6 + 2x^2}{h}$$

$$= \lim_{h \to 0} \frac{-4hx - 2h^2}{h}$$

$$= \lim_{h \to 0} \frac{h(-4x - h)}{h}$$

$$= \lim_{h \to 0} (-4x - h) = -4x - 0 = -4x.$$

Bonus. (10 points) Use the **definition of derivative** to prove that if f and g are differentiable functions, then

$$(f - g)'(x) = f'(x) - g'(x)$$

$$(f-g)'(x) = \lim_{h \to 0} \frac{(f-g)(x+h) - (f-g)(x)}{h}$$

$$= \lim_{h \to 0} \frac{[f(x+h) - g(x+h)] - [f(x) - g(x)]}{h}$$

$$= \lim_{h \to 0} \frac{f(x+h) - g(x+h) - f(x) + g(x)}{h}$$

$$= \lim_{h \to 0} \frac{f(x+h) - f(x) - g(x+h) + g(x)}{h}$$

$$= \lim_{h \to 0} \left[\frac{f(x+h) - f(x)}{h} - \frac{g(x+h) - g(x)}{h} \right]$$

$$= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} - \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}$$

$$= f'(x) - g'(x)$$