MATH 11012 | Intuitive Calculus | Spring 2011
Circle one: 8:50 5:30 | Ms. Kracht

Name: KEY | Score: ____/100
(110 pts available)

EXAM 1: Version B

Show your reasoning for full credit.

Some Formulas

Product Rule: If $p(x) = f(x) \cdot g(x)$, then $p'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x)$.

Quotient Rule: If $q(x) = \dfrac{f(x)}{g(x)}$, then $q'(x) = \dfrac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{[g(x)]^2}$.

Generalized Power Rule: If $h(x) = [g(x)]^n$, then $h'(x) = n[g(x)]^{n-1} \cdot g'(x)$.

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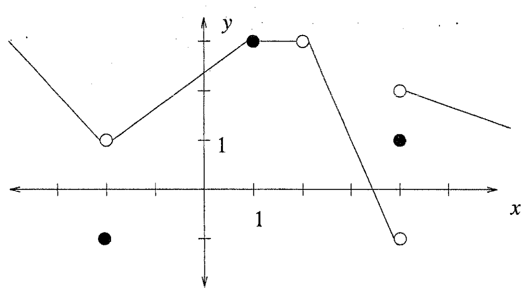
1. (10 points) The function f is depicted below.



(a) Find each of the following or state "does not exist" ("dne").

$\lim_{x\to -2^-} f(x) = 1$	$\lim_{x\to 1^-} f(x) = 3$	$\lim_{x\to 2^-} f(x) = 3$	$\lim_{x\to 4^-} f(x) = -1$
$\lim_{x\to -2^+} f(x) = 1$	$\lim_{x\to 1^+} f(x) = 3$	$\lim_{x\to 2^+} f(x) = 3$	$\lim_{x\to 4^+} f(x) = 2$
$\lim_{x\to -2} f(x) = 1$	$\lim_{x\to 1} f(x) = 3$	$\lim_{x\to 2} f(x) = 3$	$\lim_{x\to 4} f(x) =$ dne
$f(-2) = -1$	$f(1) = 3$	$f(2) =$ dne	$f(4) = 1$

(b) Circle YES or NO for each of the following.

i. Is f continuous at -2? YES NO

ii. Is f continuous at 1? YES NO

iii. Is f continuous at 2? YES NO

iv. Is f continuous at 4? YES NO

Multiple Choice: 10 questions at 3 points each.

Circle the letter of the best response

2. Evaluate $\lim_{x\to 2}(3x^2+5)$. $= 3\cdot 2^2+5$ $= 12+5$ $= 17$

   (a) 11
   (b) 17
   (c) 41
   (d) 0
   (e) none of these

3. Evaluate $\lim_{x\to 6}\frac{x-6}{x^2-36}$. $= \lim_{x\to 6}\frac{x-6}{(x-6)(x+6)}$ $= \lim_{x\to 6}\frac{1}{x+6}$ $= \frac{1}{6+6} = \frac{1}{12}$

   (a) 0
   (b) does not exist
   (c) $\frac{1}{12}$
   (d) 12
   (e) none of these

4. Let $f(x) = \begin{cases} x & \text{if } x \le 1; \\ 2x & \text{if } x > 1. \end{cases}$ Is $f(x)$ continuous at $x = 1$?

   $\lim_{x\to 1^-} f(x) = 1$
   $\lim_{x\to 1^+} f(x) = 2\cdot 1 = 2$
   so $\lim_{x\to 1} f(x)$ dne

   (a) Yes
   (b) No, since $f(x)$ is a piecewise-defined function
   (c) No, since $\lim_{x\to 1} f(x)$ does not exist.
   (d) No, since $f(1)$ is undefined.
   (e) none of these

5. Let $f(x) = x^3 + 2x + 1$. What is the average rate of change in $f(x)$ between the values $x = 1$ and $x = 5$?

   (a) 41 because $\frac{f'(1)+f'(5)}{2} = 41$
   (b) 33 because $\frac{f(5)-f(1)}{4} = 33$
   (c) 132 because $f(5) - f(1) = 132$
   (d) 29 because $f'(3) = 29$
   (e) none of these

6. Find the instantaneous rate of change of the function $f(x) = 10x^4 + 5$ at $x = -2$.

   $f'(x) = 40x^3$
   $f'(-2) = 40(-2)^3$ $= 40(-8)$ $= -320$

   (a) -320
   (b) 320
   (c) -315
   (d) -80
   (e) none of these

7. Find the slope of the tangent line to $f(x) = \frac{1}{x}$ at $x = 3$.

   $f(x) = x^{-1}$
   $f'(x) = -x^{-2}$ $= -\frac{1}{x^2}$
   $f'(3) = -\frac{1}{3^2} = -\frac{1}{9}$

   (a) $-\frac{1}{3}$
   (b) -3
   (c) $-\frac{1}{9}$
   (d) -1
   (e) none of these

8. Find the derivative of $g(x) = 5x^{100}$.

   (a) $g'(x) = 500x^{99}$
   (b) $g'(x) = 500x$
   (c) $g'(x) = 500x^{101}$
   (d) $g'(x) = 5x^{99}$
   (e) none of these

9. Use the product rule to find the derivative of $h(t) = (7t-3)(2t+5)$. *The answer need not be simplified.*

   (a) $h'(t) = 7(2t+5) + 2(7t-3)$
   (b) $h'(t) = 14t^2$
   (c) $h'(t) = 14$
   (d) $h'(t) = 7t(2t+5) + 2t(7t-3)$
   (e) none of these

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10. Suppose a company's revenue function is $R(x) = 25x + 4\sqrt{x}$, in dollars, where x is the number of widgets produced. Find and interpret $R(100)$.

   (a) The company's total revenue from the first 100 widgets is $25.20.

   (b) The company's total revenue from the first 100 widgets is $2540.

   (c) The company's average revenue from the first 100 widgets is $25.40 per widget.

   (d) The company's revenue from the 100th widget is $25.40.

   (e) none of these

$$R(100) = 25(100) + 4\sqrt{100}$$
$$= 2500 + 40$$
$$= \$2540$$

11. Suppose a company's revenue function is $R(x) = 25x + 4\sqrt{x}$, in dollars, where x is the number of widgets produced. Find and interpret $R'(100)$.

   (a) The company's total revenue from the first 100 widgets is $25.40.

   (b) The company's average revenue from the first 100 widgets is $25.40 per widget.

   (c) The company's average revenue from the first 100 widgets is $25.20 per widget.

   (d) The company's revenue from the 100th widget is approximately $25.20.

   (e) none of these

$$R(x) = 25x + 4x^{1/2}$$
$$R'(x) = 25 + 4\left(\tfrac{1}{2}x^{-1/2}\right)$$
$$= 25 + \frac{2}{\sqrt{x}}$$
$$R'(100) = 25 + \frac{2}{\sqrt{100}} = 25 + \frac{2}{10} = \$25.20/\text{widget}$$

Long Answer

Write all work carefully and neatly for full credit.

12. (15 pts) Find each derivative, simplifying your answer. *(You may use short-cuts.)*

   (a) Find $\frac{d}{dz}\sqrt{16z^2 - 49}$.

$$= \frac{d}{dz}(16z^2 - 49)^{1/2}$$
$$= \frac{1}{2}(16z^2-49)^{-1/2}(16 \cdot 2z)$$
$$= \frac{16z}{\sqrt{16z^2-49}}$$

   (b) Find $\frac{dy}{dx}$ where $y = x^2(2x-1)^7$

$$\frac{dy}{dx} = 2x(2x-1)^7 + x^2\left[7(2x-1)^6(2)\right]$$
$$= 2x(2x-1)^7 + 14x^2(2x-1)^6$$
$$= 2x(2x-1)^6\left[(2x-1) + 7x\right]$$
$$= 2x(2x-1)^6(9x-1)$$

   (c) Find $g'(x)$ where $g(x) = \frac{x^2-5}{x^2+8}$

$$g'(x) = \frac{2x(x^2+8) - (x^2-5)(2x)}{(x^2+8)^2}$$
$$= \frac{2x\left[x^2+8-(x^2-5)\right]}{(x^2+8)^2}$$
$$= \frac{2x\left[x^2+8-x^2+5\right]}{(x^2+8)^2}$$
$$= \frac{2x(13)}{(x^2+8)^2}$$
$$= \frac{26x}{(x^2+8)^2}.$$

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13. (10 pts) A rocket rises to a height $h(t) = 0.5t^2 + 8t$ feet in t seconds. *Include units with each of the following.*

(a) Find the height of the rocket after 10 seconds.

$h(10) = 0.5(10)^2 + 8(10) = 50 + 80 = 130$ ft

(b) Find the velocity of the rocket after t seconds.

$v(t) = h'(t) = 0.5(2t) + 8 = t + 8$ ft/s

(c) Find the velocity of the rocket after 10 seconds.

$v(10) = 10 + 8 = 18$ ft/s

(d) Find the acceleration of the rocket after t seconds.

$a(t) = v'(t) = 1$ ft/s^2

(e) Find the acceleration of the rocket after 10 seconds.

$a(10) = 1$ ft/s^2

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14. (5 pts) Evaluate $\dfrac{d^3}{dx^3}\sqrt[3]{x^5}$. $= \dfrac{d^3}{dx^3} x^{5/3}$

$= \dfrac{d^2}{dx^2}\left(\frac{5}{3}x^{2/3}\right)$

$= \dfrac{d}{dx}\left(\frac{5}{3}\cdot\frac{2}{3}x^{-1/3}\right)$

$= \dfrac{d}{dx}\left(\frac{10}{9}x^{-1/3}\right)$

$= \frac{10}{9}\left(-\frac{1}{3}\right)x^{-4/3} = \frac{-10}{27}x^{-4/3}$

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15. (10 points) Find an equation of the line tangent to the graph of the function $f(x) = x^3 + 4x^2 - 5$ at the point where $x = -1$ by following the given steps. *Show your reasoning.*

2 (a) Find the y-coordinate of the point on the curve $y = f(x)$ where $x = -1$.

$f(-1) = (-1)^3 + 4(-1)^2 - 5$
$= -1 + 4 - 5 = -2$ $(-1, -2)$

3 (b) Find the derivative $f'(x)$. *You may use short-cuts.*

$f'(x) = 3x^2 + 8x$

2 (c) Find the slope of the line tangent to the curve $y = f(x)$ at the point where $x = -1$.

$m = f'(-1) = 3(-1)^2 + 8(-1)$
$= 3 - 8 = -5$

3 (d) Find the equation of the line tangent to the curve $y = f(x)$ at the point where $x = -1$. *Write it in the form "$y = mx + b$."*

$y - y_1 = m(x - x_1)$

$y - (-2) = -5(x - (-1))$

$y + 2 = -5(x + 1)$

$y = -5x - 5 - 2$

$\underline{y = -5x - 7}$

$\boxed{25}$

16. (15 points) Use the **definition of derivative** to find and simplify the derivative, $f'(x)$, for the function $f(x) = 5-3x^2$.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{[5-3(x+h)^2]-[5-3x^2]}{h}$$

$$= \lim_{h\to 0} \frac{5-3(x^2+2hx+h^2)-5+3x^2}{h}$$

$$= \lim_{h\to 0} \frac{5-3x^2-6hx-3h^2-5+3x^2}{h}$$

$$= \lim_{h\to 0} \frac{-6hx-3h^2}{h}$$

$$= \lim_{h\to 0} \frac{h(-6x-3h)}{h}$$

$$= \lim_{h\to 0} (-6x-3h) = -6x-3(0) = \underline{\underline{-6x}}.$$

Bonus. (10 points) Use the **definition of derivative** to prove that if f and g are differentiable functions, then

$$(f-g)'(x) = f'(x) - g'(x)$$

$$(f-g)'(x) = \lim_{h\to 0} \frac{(f-g)(x+h)-(f-g)(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h)-g(x+h)-[f(x)-g(x)]}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h)-g(x+h)-f(x)+g(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h)-f(x)-g(x+h)+g(x)}{h}$$

$$= \lim_{h\to 0} \left[\frac{f(x+h)-f(x)}{h} - \frac{g(x+h)-g(x)}{h}\right]$$

$$= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} - \lim_{h\to 0} \frac{g(x+h)-g(x)}{h}$$

$$= f'(x) - g'(x).$$