MATH 11012
Circle one: 8:50 5:30

Intuitive Calculus

Spring 2011
Ms. Kracht

Name: KEY

Score: /100
(110 pts available)

EXAM 2: Version A

NO CALCULATORS.

Multiple Choice: 10 questions at 3 points each.

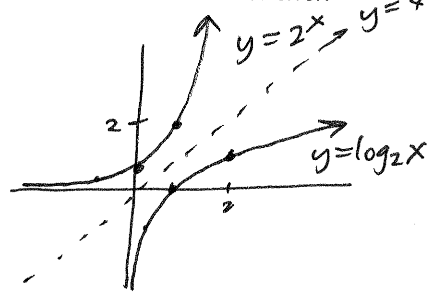
Circle the letter of the best response.

Although your work will not be graded, you should write out complete and careful solutions to reduce the chance of error.

1. The graph $y = \log_2 x$ may be obtained by reflecting the graph $y = 2^x$ across

   (a) the y-axis.
   (b) the x-axis.
   (c) the line $y = x$.
   (d) the origin.
   (e) none of these

These functions are inverses of one another. (This is how we defined the log functions.)



2. Express as a power of e: $\frac{e\,e^{-5}}{e^{-2}e^4}$. $= \frac{e^{-4}}{e^2} = e^{-4}e^{-2} = e^{-6}$

   (a) e^0
   (b) e^{-2}
   (c) e^6
   (d) e^{-6}
   (e) none of these

3. True or False: If $A, B > 0$, then $\ln(A + B) = \ln A + \ln B$.

   (a) True
   (b) False

$\ln(AB) = \ln A + \ln B$
product sum

4. What is $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$?

   (a) 0
   (b) ∞
   (c) e
   (d) does not exist
   (e) none of these

This was our definition of e.

5. Find an expression for the value of \$300 deposited in a bank at 2.75% interest for 8 years compounded quarterly.

(a) $300\left(1+\frac{0.0275}{4}\right)^{8}$

(b) $300\left(1+\frac{0.0275}{4}\right)^{32}$

(c) $300\left(1+\frac{2.75}{4}\right)^{32}$

(d) $300\left(1+0.0275\right)^{32}$

(e) none of these

$FV = PV\left(1+\frac{r}{n}\right)^{nt}$

here
$PV = \$300$
$r = 0.0275$
$n = 4$
$t = 8$

---

6. A rich uncle wants to give you a nice graduation present. How much money must he deposit in a trust fund paying 5.75% compounded monthly at the time you enter college to yield \$5,000 when you graduate exactly 4 years later?

(a) $5000e^{-0.0575\times 4}$

(b) $5000\left(1.0575\right)^{-48}$

(c) $5000\left(1+\frac{0.0575}{12}\right)^{-48}$

(d) $5000\left(1-\frac{0.0575}{12}\right)^{48}$

(e) none of these

$FV = PV\left(1+\frac{r}{n}\right)^{nt}$

so

$FV\left(1+\frac{r}{n}\right)^{-nt} = PV$

here $FV = \$5000$
$r = 0.0575$
$n = 12$
$t = 4$

---

7. Evaluate: $\ln(\ln e)$. $= \ln 1 = 0$

(a) 0

(b) 1

(c) e

(d) undefined

(e) none of these

8. An investment grows by 3.5% compounded continuously. How many years will it take to increase by 50%?

(a) $\frac{\ln 1.5}{0.035}$

(b) $\frac{\ln 0.5}{0.035}$

(c) $\frac{\ln 1.5}{3.5}$

(d) $\ln\left(\frac{1.5}{0.035}\right)$

(e) none of these

$A(t) = Pe^{0.035t}$

What is t when $A(t) = 1.5P$?

$Pe^{0.035t} = 1.5P$

$e^{0.035t} = 1.5$

$\ln e^{0.035t} = \ln 1.5$

$0.035t = \ln 1.5$

$t = \frac{\ln 1.5}{0.035}$.

---

9. Suppose that at a price of \$4.50 per widget, elasticity of demand for Slimy Green Widgets is 1.2. If the price is increased, then demand should

(a) increase weakly.

(b) increase strongly.

(c) decrease weakly.

(d) decrease strongly.

(e) none of these

$1.2 > 1$, so demand is elastic (reacts strongly to price changes)

---

10. Suppose that at a price of \$137 per widget, elasticity of demand for Prickly Pink Widgets is 0.75. To increase revenue, the producer should

(a) raise the price.

(b) lower the price.

(c) leave the price alone; revenue is maximal for the current price.

(d) none of these

$0 < 0.75 < 1$, so demand is inelastic (reacts weakly to price changes)

Long Answer

Write all work carefully and neatly for full credit.

11. (20 pts) Find the derivative of each function, simplifying your answer.

(a) $f(x) = \ln(1 - x^3)$.

Chain Rule

$$f'(x) = \frac{1}{1-x^3} \frac{d}{dx}(1-x^3)$$

$$= \frac{-3x^2}{1-x^3}$$

$$= \frac{3x^2}{x^3-1}.$$

(b) $K(u) = \sqrt{\ln u}$ $= (\ln u)^{1/2}$

parens needed!

Chain rule

$$K'(u) = \frac{1}{2}(\ln u)^{-1/2} \frac{d}{du}(\ln u)$$

$$= \frac{1}{2}(\ln u)^{-1/2}\left(\frac{1}{u}\right)$$

$$= \frac{1}{2u\sqrt{\ln u}}.$$

(c) $y = x \ln x$

Product Rule

$$\frac{dy}{dx} = 1 \cdot \ln x + x\left(\frac{1}{x}\right)$$

$$= \ln x + 1.$$

(d) $g(t) = \frac{2 - e^t}{3 + e^t}$

Quotient Rule

$$g'(t) = \frac{-e^t(3+e^t) - e^t(2-e^t)}{(3+e^t)^2}$$

$$= \frac{-e^t[(3+e^t) + (2-e^t)]}{(3+e^t)^2}$$

$$= \frac{-e^t[3+e^t+2-e^t]}{(3+e^t)^2}$$

$$= \frac{-5e^t}{(3+e^t)^2}.$$

12. (5 pts) Find the **second** derivative of $f(x) = e^{x^2}$.

$$f'(x) = e^{x^2} \frac{d}{dx}(x^2)$$ Chain Rule

$$= 2x e^{x^2}$$

Product rule

$$f''(x) = \left(\frac{d}{dx} 2x\right) e^{x^2} + 2x \frac{d}{dx} e^{x^2}$$

$$= 2e^{x^2} + (2x)\left[e^{x^2} \cdot 2x\right]$$ Chain Rule

$$= 2e^{x^2} + 4x^2 e^{x^2}$$

$$= 2e^{x^2}(1 + 2x^2)$$

---

13. (5 pts) Let $f(t) = \sqrt{t-3}$, where t represents time in years. Find the relative rate of change of f at time $t = 8$ years. (Express it as a percentage.)

$$f(t) = (t-3)^{1/2}$$

$$f'(t) = \frac{1}{2}(t-3)^{-1/2}(1)$$ Chain Rule

Relative Rate of Change:

$$\frac{f'(t)}{f(t)} = \frac{\frac{1}{2}(t-3)^{-1/2}}{(t-3)^{1/2}}$$

$$= \frac{1}{2(t-3)^1}$$

@ $t=8$:

$$\frac{f'(8)}{f(8)} = \frac{1}{2(8-3)} = \frac{1}{10} = 10\%$$ per year

14. (5 pts) Suppose the demand function for Sparkling Purple Widgets is $D(p) = 5000e^{-0.03p}$ widgets at price p dollars per widget. Find the elasticity of demand, $E(p)$, at a price of \$10 per widget.

Chain Rule

$$D'(p) = 5000e^{-0.03p}(-0.03)$$

$$E(p) = \frac{-p\, D'(p)}{D(p)}$$

$$= \frac{-p\, 5000e^{-0.03p}(-0.03)}{5000e^{-0.03p}}$$

$$= 0.03p$$

$$E(10) = 0.03(10)$$
$$= \underline{\underline{0.3}},$$ (there are no units here)

---

15. (5 pts) An investment of P dollars is made with an annual rate r compounded annually. After 1 year, the amount in the account is

$$P + rP = P(1+r).$$

DERIVE the formula for the amount in the account after 2 years. (That is, show where this formula comes from.)

$$\begin{pmatrix}\text{amount at} \\ \text{beginning} \\ \text{of 2nd} \\ \text{year}\end{pmatrix} + \begin{pmatrix}\text{interest} \\ \text{earned during} \\ \text{2nd year}\end{pmatrix}$$

$$= [P(1+r)] + r\,[P(1+r)]$$

$$= [P(1+r)]\,[1+r]$$

$$= P(1+r)^2.$$

16. (15 points) Let f be **continuous** on $\mathbb{R}$ with $f'(-1) = 0$, $f'(2)$ undefined, $f''(-1) = 0$, and $f''(2)$ undefined.

(a) Fill in the blanks in the sign charts with the terms "increasing," "decreasing," "concave up," or "concave down." *Write them out!*

interval	$(-\infty, -1)$	$(-1, 2)$	$(2, \infty)$
sign of $f'(x)$	$-$	$-$	$+$
behavior of f	decreasing	decreasing	increasing

interval	$(-\infty, -1)$	$(-1, 2)$	$(2, \infty)$
sign of $f''(x)$	$+$	$-$	$-$
behavior of f	concave up	concave down	concave down

(b) Give the x-**coordinates** of all of the following (or indicate "none").

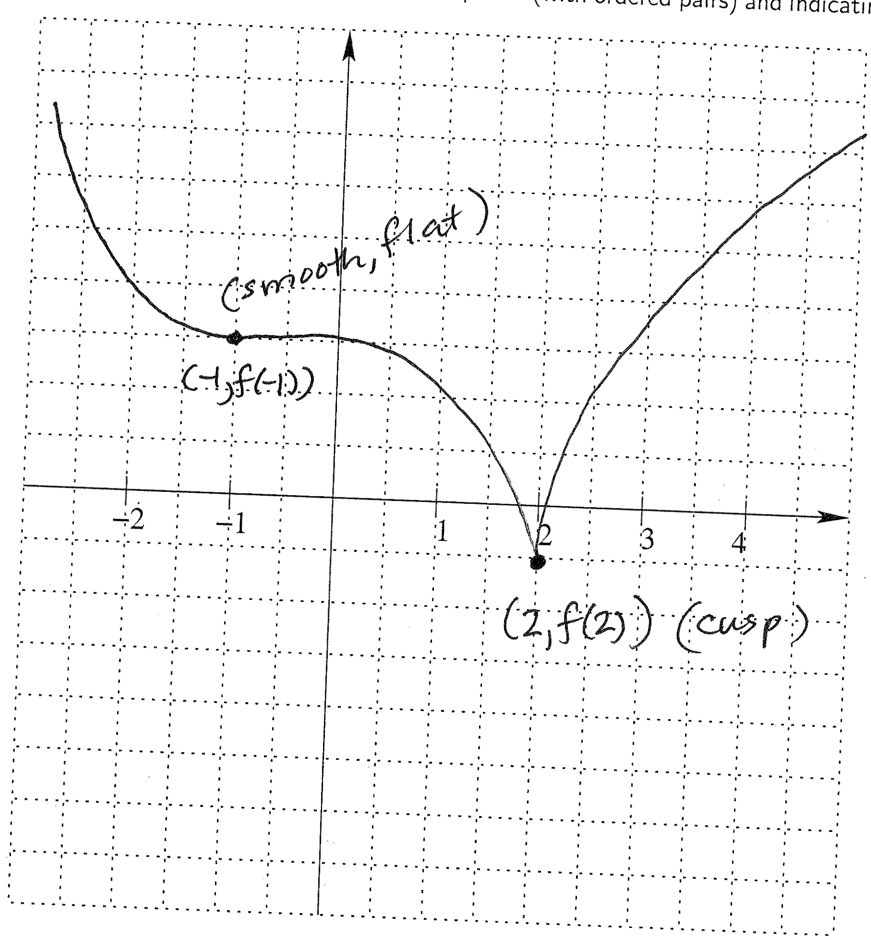
i. critical points of f: $x =$ $-1, 2$

ii. inflection points of f: $x =$ -1

iii. relative maximum points of f: $x =$ none

iv. relative minimum points of f: $x =$ 2

(c) Sketch a possible graph of f labeling all important points (with ordered pairs) and indicating all behavior clearly.



17. (25 points) Let $f(x) = x^3 - 3x^2 - 9x$.

(a) Find the y-intercept(s) of f. *Show work. Give the ordered pair(s).*

$f(0) = 0^3 - 3 \cdot 0^2 - 9 \cdot 0 = 0$

$(0,0)$

(b) Find $f'(x)$.

$f'(x) = 3x^2 - 6x - 9$

(c) Find the critical number(s) of f. *Show work.*

$\boxed{f'(x) = 0} \leftarrow$ key

$3x^2 - 6x - 9 = 0$

$3(x^2 - 2x - 3) = 0$

$3(x-3)(x+1) = 0$

$x - 3 = 0$ or $x + 1 = 0$

$x = 3$ $\quad x = -1$

(d) Find the y-coordinates of all critical points of f. *Show work.*

$f(-1) = (-1)^3 - 3(-1)^2 - 9(-1)$ $\quad (-1, 5)$

$= -1 - 3 + 9 = 5$

$f(3) = 3^3 - 3 \cdot 3^2 - 9 \cdot 3$ $\quad (3, -27)$

$= -27$

(e) Construct a sign chart **(with all factors as in class)** for $f'(x)$ and indicate the corresponding behavior of f.

	$(-\infty, -1)$	$(-1, 3)$	$(3, \infty)$
3	+	+	+
$x-3$	−	− 0	+
$x+1$	− 0	+	+
$f'(x)$	+ 0	− 0	+
$f(x)$	inc	dec	inc

(f) Find $f''(x)$.

$f''(x) = 6x - 6$

(g) Find the potential inflection point(s) of f. *Show work.*

$\boxed{f''(x) = 0} \leftarrow$ key

$6x - 6 = 0$

$6(x - 1) = 0$

$x - 1 = 0$

$x = 1$

(h) Find the y-coordinates of all potential inflection points of f. *Show work.*

$f(1) = 1^3 - 3 \cdot 1^2 - 9 \cdot 1$ $\quad (1, -11)$

$= 1 - 3 - 9$

$= -11$

(i) Construct a sign chart **(with all factors as in class)** for $f''(x)$ and indicate the corresponding behavior of f.

	$(-\infty, 1)$	$(1, \infty)$
6	+	+
$x-1$	− 0	+
$f''(x)$	− 0	+
$f(x)$	c. down	c. up

(j) Sketch the graph $y = f(x)$. Plot and label (with ordered pairs) all important points. Illustrate all aspects of the behavior of f found above clearly.