Confidence Intervals

Some of the following exercises are based on Example 3 of *Normal Distributions: Standard Normal* in your text (slides 123–129). Please review this example before attempting the exercises. Following the notation of this example, let $X$ be the random variable representing the salary of a randomly selected county administrator.

1. Suppose, as in Example 3, that the unhappy county administrator’s annual salary is $83,500 and that he wishes to find the 95% confidence interval for the mean of $X$.

   (a) Suppose that the county administrator takes a random sample of 50 people in similar positions around the country and finds the sample mean to be $\bar{x} = 88,989$ and sample standard deviation $s$ to be as given in each problem below. Use the formula developed in the example to find the 95% confidence interval for the mean of $X$ in each case. Then state, for each, whether the unhappy administrator can make a convincing case to the county board of supervisors that his salary is below the national average (and so he should get a raise).

   i. $s = 22,358$
   ii. $s = 25,476$
   iii. $s = 30,173$
   iv. $s = 20,008$
   v. $s = 16,945$
   vi. $s = 10,234$

   (b) Explain how the variance of $X$ affects the length of the 95% confidence interval for the mean of $X$. Justify your answer carefully.

   (c) Suppose that the county administrator takes a random sample of people in similar positions around the country and finds the sample mean to be $\bar{x} = 88,989$ and the sample standard deviation to be $s = 22,358$. Use the formula developed in the example to find the 95% confidence interval for the mean of $X$ if the size of the sample were the number $n$ given below. Then state, for each, whether the unhappy administrator can make a convincing case to the county board of supervisors that his salary is below the national average (and so he should get a raise).

   i. $n = 60$
   ii. $n = 100$
   iii. $n = 200$
   iv. $n = 35$
   v. $n = 10$
   vi. $n = 5$

   (d) Explain how the sample size $n$ affects the length of the 95% confidence interval for the mean of $X$.

2. Our disgruntled county supervisor knows that the county board of supervisors is a tough bunch. He now thinks that he better find a 99% confidence interval for the mean of $X$.

   (a) Follow the method of the example to develop a formula for the positive number $b$ with the property that $P(\bar{x} - b \leq \mu \leq \bar{x} + b) = 0.99$. Write out all steps carefully. Make sure you can justify each.

   (b) Suppose that the county administrator takes a random sample of 50 people in similar positions around the country and finds the sample mean to be $\bar{x} = 88,989$ and sample standard deviation $s$ to be as given in each problem below. Use your formula to find the 99% confidence interval for the mean of $X$ in each case. Then state, for each, whether the unhappy administrator can make a convincing case to the county board of supervisors that his salary is below the national average (and so he should get a raise).

   i. $s = 22,358$
   ii. $s = 16,945$
   iii. $s = 10,234$

   (c) Suppose that the county administrator takes a random sample of people in similar positions around the country and finds the sample mean to be $\bar{x} = 88,989$ and the sample standard deviation to be $s = 22,358$. Use your formula to find the 99% confidence interval for the mean of $X$ if the size of the sample were the number $n$ given below. Then state, for each, whether the unhappy administrator can make a convincing case to the county board of supervisors that his salary is below the national average (and so he should get a raise).

   i. $n = 60$
   ii. $n = 100$
   iii. $n = 200$
3. A new county board of supervisors has been elected. Most of them are good friends and relatives of the overworked county supervisor. He now thinks that an 85% confidence interval for the mean of $X$ will be adequate.

(a) Follow the method of the example to develop a formula for the positive number $c$ with the property that $P(\bar{x} - c \leq \mu_X \leq \bar{x} + c) = 0.85$. Write out all steps carefully. Make sure you can justify each.

(b) Suppose that the county administrator takes a random sample of 50 people in similar positions around the country and finds the sample mean to be $\bar{x} = 88.989$ and sample standard deviation $s$ to be as given in each problem below. Use your formula to find the 85% confidence interval for the mean of $X$ in each case. Then state, for each, whether the unhappy administrator can make a convincing case to the county board of supervisors that his salary is below the national average (and so he should get a raise).

\begin{align*}
i. & s = $22,358 \\
ii. & s = $25,476 \\
iii. & s = $30,173
\end{align*}

(c) Suppose that the county administrator takes a random sample of people in similar positions around the country and finds the sample mean to be $\bar{x} = 88.989$ and the sample standard deviation to be $s = 22.358$. Use your formula to find the 85% confidence interval for the mean of $X$ if the size of the sample were the number $n$ given below. Then state, for each, whether the unhappy administrator can make a convincing case to the county board of supervisors that his salary is below the national average (and so he should get a raise).

\begin{align*}
i. & n = 35 \\
ii. & n = 10 \\
iii. & n = 5
\end{align*}

4. The county administrator now thinks that an 97% confidence interval for the mean of $X$ will be adequate.

(a) Follow the method of the example to develop a formula for the positive number $d$ with the property that $P(\bar{x} - d \leq \mu_X \leq \bar{x} + d) = 0.97$.

(b) Use your formula to find the 97% confidence interval for the mean of $X$ for each situation given below. Then state, for each, whether the unhappy administrator can make a convincing case to the county board of supervisors that his salary ($83,500) is below the national average (and so he should get a raise).

\begin{align*}
i. & n = 50, \bar{x} = 88.989, s = 22.358 \\
ii. & n = 25, \bar{x} = 90.872, s = 17.056 \\
iii. & n = 175, \bar{x} = 85.122, s = 9.765
\end{align*}

5. The McNut Company grows nuts, packages them, and sells them to various suppliers. Macadamia nuts are packaged in 32-ounce bags. McNut wishes to minimize the number of ounces that each package goes over 32 ounces. At the same time, to give their customers the most value for their money, they wish to minimize the number of ounces that each package goes under 32 ounces. That is, the company wants each package to be as close as possible to the stated weight. It is too time consuming to test each package. Therefore, random samples of size 50 packages are collected periodically and a confidence interval for the true weight is computed from each sample.

Let $X$ be the random variable that represents the true weight, in ounces, of a randomly selected “32-ounce” bag of McNut macadamia nuts. Here are the observations (in ounces) from the sample from $X$ taken at the packaging plant today.

\begin{tabular}{llllllllll}
31.40 & 35.47 & 30.62 & 27.76 & 31.35 & 30.97 & 35.32 & 31.83 & 29.11 & 32.06 \\
29.44 & 27.63 & 28.62 & 30.86 & 31.26 & 35.94 & 28.78 & 30.95 & 30.31 & 31.35 \\
32.49 & 31.53 & 28.31 & 31.19 & 34.69 & 33.73 & 33.08 & 33.35 & 28.96 & 36.39 \\
34.55 & 34.19 & 30.04 & 32.27 & 31.83 & 36.75 & 33.80 & 31.24 & 31.27 & 28.52 \\
34.40 & 29.83 & 30.45 & 31.27 & 31.83 & 30.69 & 35.84 & 33.52 & 31.94 & 30.53
\end{tabular}

(a) **Statistics from the sample.** Find the mean, $\bar{x}$, and standard deviation, $s$, of today’s sample.

(b) **95% confidence interval.** Find the 95% confidence interval for the true mean weight of 32-oz. packages of macadamia nuts. (You can use some of the computations from the salary example for this.)

(c) **80% confidence interval.**

i. Follow the method of the examples in class and in the text to develop a formula for the positive number $z_0$ with the property that $P(\bar{x} - z_0 \leq \mu_X \leq \bar{x} + z_0) = 0.80$.

ii. Use your formula to find the 80% confidence interval for the mean of $X$.

continued
(d) **90% confidence interval.**

i. Follow the method of the examples in class and in the text to develop a formula for the positive number $z_0$ with the property that $P(\bar{x} - z_0 \leq \mu_X \leq \bar{x} + z_0) = 0.90$.

ii. Use your formula to find the 90% confidence interval for the mean of $X$.

(e) **Compare and contrast.** On a number line, plot the sample mean and each of the three confidence intervals computed above. Which interval is the longest? the shortest? Explain how this makes good intuitive sense.

(f) **Conclusion.** Do you think the McNut Company’s packages are being properly filled? Explain why or why not in the language of nuts and statistics (and nutty statistics).

6. Thirty-six observations of $X$, the lifetime of a flashlight battery, are given below.

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(a) **Statistics from the sample.** Find the mean, $\bar{x}$, and standard deviation, $s$, of the sample.

(b) **95% confidence interval.** Find the 95% confidence interval for the lifetime of the flashlight batteries.

(c) **80% confidence interval.** Find the 80% confidence interval for the lifetime of the flashlight batteries.

(d) **90% confidence interval.** Find the 90% confidence interval for the lifetime of the flashlight batteries.

(e) **75% confidence interval.**

i. Follow the method of the examples in class and in the text to develop a formula for the positive number $z_0$ with the property that $P(\bar{x} - z_0 \leq \mu_X \leq \bar{x} + z_0) = 0.75$.

ii. Use your formula to find the 75% confidence interval for the mean of $X$.

(f) **99.9% confidence interval.**

i. Follow the method of the examples in class and in the text to develop a formula for the positive number $z_0$ with the property that $P(\bar{x} - z_0 \leq \mu_X \leq \bar{x} + z_0) = 0.999$.

ii. Use your formula to find the 99.9% confidence interval for the mean of $X$.

(g) **Compare and contrast.** On a number line, plot the sample mean and each of the four confidence intervals computed above. Which interval is the longest? the shortest? Explain how this makes good intuitive sense.