Exam FM/2 Interest Theory Formulas

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The author of this study sheet is using some notation that is unique so that no designation will repeat. Each designation has only one meaning throughout the sheet.

Fundamentals of Interest Theory and Time Value of Money

$FV = PV(1+i)^n$ $\qquad$ $PV = \dfrac{FV}{(1+i)^n}$

$d = \dfrac{i}{(1+i)}$ $\qquad$ $d = 1 - v$ $\qquad$ $i - d = id$

$v = \dfrac{1}{(1+i)}$ $\qquad$ $v = 1 - d$ $\qquad$ $d = iv$

$a(t)$ $\equiv$ The amount an initial investment of 1 grows to by time t

$A(t)$ $\equiv$ The amount an initial investment of $A(0)$ grows to by time t

$a(t) = (1+i)^t = e^{t \cdot ln(1+i)}$ $\qquad$ $A(t) = A(0)(1+i)^t = A(0)e^{t \cdot ln(1+i)}$

$\delta = ln(1+i)$ $\qquad$ $a(t) = e^{\delta \cdot t}$

$v^n = (1+i)^{-n} = e^{-\delta n}$

$\delta(t) = \dfrac{a'(t)}{a(t)}$ $\qquad$ $e^{\int_0^t \delta(u)du} = a(t)$ $\qquad$ $A(0)e^{\int_0^t \delta(u)du} = A(t)$

Effective interest rate with nominal rate $i^{(m)}$ convertible m-thly

$$i = \left(1 + \frac{i^{(m)}}{m}\right)^m - 1$$

Effective discount rate with nominal rate $d^{(p)}$ convertible p-thly

$$1 - d = \left(1 - \frac{d^{(p)}}{p}\right)^p$$

<u>Nominal Rate Equivalence</u>

$$1 + i = e^{\delta} = \frac{1}{v} = \frac{1}{1-d} = \left(1 + \frac{i^{(m)}}{m}\right)^m = \left(1 - \frac{d^{(p)}}{p}\right)^{-p}$$

Effective annual rate i_t during the t-th year is given by:

$$i_t = \frac{amount\ \ earned}{beginning\ \ amount} = \frac{a(t)-a(t-1)}{a(t-1)} = \frac{A(t)-A(t-1)}{A(t-1)}$$

Note that the t-th year is given by the time period $[t-1, t]$

Therefore, the interest earned during the t-th year is given by:

$$A(t-1)\cdot i = A(t) - A(t-1)$$

For equivalent measures of interest we have the following relationship:

$$d < d^{(2)} < d^{(3)} < \cdots < \delta < \cdots < i^{(3)} < i^{(2)} < i$$

Annuities

Annuity Immediate— payments are made at the end of the period
Annuity Due— payments are made at the beginning of the period

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Annuity Immediate

$$a_{\overline{n}|i} = v + v^2 + \cdots + v^n = \frac{1-v^n}{i} \qquad s_{\overline{n}|i} = (1+i)^{n-1} + (1+i)^{n-2} + \cdots + 1 = \frac{(1+i)^n - 1}{i}$$

$$a_{\overline{n}|i} = v^n \cdot s_{\overline{n}|i} \qquad s_{\overline{n}|i} = (1+i)^n \cdot a_{\overline{n}|i}$$

Annuity Due

$$\ddot{a}_{\overline{n}|i} = 1 + v + \cdots + v^{n-1} = \frac{1-v^n}{d} \qquad \ddot{s}_{\overline{n}|i} = (1+i)^n + (1+i)^{n-1} + \cdots + (1+i) = \frac{(1+i)^n - 1}{d}$$

$$\ddot{a}_{\overline{n}|i} = v^n \cdot \ddot{s}_{\overline{n}|i} \qquad \ddot{s}_{\overline{n}|i} = (1+i)^n \cdot \ddot{a}_{\overline{n}|i}$$

Identities for Annuity Immediate and Annuity Due

$$\ddot{a}_{\overline{n}|} = \frac{i}{d} a_{\overline{n}|} = (1+i) a_{\overline{n}|} \qquad \ddot{s}_{\overline{n}|} = \frac{i}{d} s_{\overline{n}|} = (1+i) s_{\overline{n}|} \qquad \ddot{a}_{\overline{n}|} = 1 + a_{\overline{n-1}|} \qquad \ddot{s}_{\overline{n}|} = s_{\overline{n+1}|} - 1$$

Perpetuity

$$a_{\overline{\infty}|i} = \lim_{n\to\infty} a_{\overline{n}|i} = v + v^2 + v^3 + \cdots = \frac{1}{i} \qquad \ddot{a}_{\overline{\infty}|i} = \lim_{n\to\infty} \ddot{a}_{\overline{n}|i} = \frac{1}{d}$$

Continuous Annuities

$$\bar{a}_{\overline{n}|i} = \frac{1-v^n}{\delta} = \frac{i}{\delta} a_{\overline{n}|i} \qquad \bar{s}_{\overline{n}|i} = \frac{(1+i)^n - 1}{\delta} = \frac{i}{\delta} s_{\overline{n}|i} \qquad \bar{a}_{\overline{n}|i} = \int_0^n v^t\, dt$$

$$PV = \int_0^n e^{-\int_0^t \delta(u)\,du} p(t)\, dt \qquad FV = \int_0^n e^{\int_t^n \delta(u)\,du} p(t)\, dt$$ where $p(t)$ = payment function

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Increasing Annuities— Payments are 1, 2, … , n

$$(Ia)_{\overline{n}|i} = \frac{\ddot{a}_{\overline{n}|} - nv^n}{i} \qquad (I\ddot{a})_{\overline{n}|i} = \frac{i}{d}(Ia)_{\overline{n}|i} = (1+i)(Ia)_{\overline{n}|i} = \frac{\ddot{a}_{\overline{n}|} - nv^n}{d}$$

$$(Is)_{\overline{n}|i} = (1+i)^n (Ia)_{\overline{n}|i} = \frac{\ddot{s}_{\overline{n}|} - n}{i} \qquad (I\ddot{s})_{\overline{n}|i} = \frac{i}{d}(Is)_{\overline{n}|i} = (1+i)^n (I\ddot{a})_{\overline{n}|i} = \frac{\ddot{s}_{\overline{n}|} - n}{d}$$

$$(Ia)_{\overline{\infty}|i} = \lim_{n\to\infty} (Ia)_{\overline{n}|i} = \frac{1}{di} = \frac{1}{i} + \frac{1}{i^2} \qquad (I\ddot{a})_{\overline{\infty}|i} = \lim_{n\to\infty} (I\ddot{a})_{\overline{n}|i} = \frac{1}{d^2}$$

Decreasing Annuities— Payments are n, n-1,…, 2, 1

$$(Da)_{\overline{n}|i} = \frac{n - a_{\overline{n}|i}}{i} \qquad (D\ddot{a})_{\overline{n}|i} = \frac{i}{d}(Da)_{\overline{n}|i} = (1+i)(Da)_{\overline{n}|i} = \frac{n - a_{\overline{n}|i}}{d}$$

$$(Ds)_{\overline{n}|i} = (1+i)^n (Da)_{\overline{n}|i} = \frac{n(1+i)^n - s_{\overline{n}|i}}{i} \qquad (D\ddot{s})_{\overline{n}|i} = (1+i)^n (D\ddot{a})_{\overline{n}|i}$$

Present Value of the annuity with terms $X,\ X+Y,\ X+2Y, \ldots,\ X+(n-1)Y$

$$X \cdot a_{\overline{n}|i} + Y\left(\frac{\ddot{a}_{\overline{n}|} - nv^n}{i}\right)$$

Present Value of the perpetuity with terms $X,\ X+Y,\ X+2Y, \ldots$

$$\frac{X}{i} + \frac{Y}{i^2}$$

Annuities with Terms in Geometric Progression— $1, (1+q), (1+q)^2, \ldots, (1+q)^{n-1}$

Present Value is $V(0) = 1\cdot v + (1+q)\cdot v^2 + (1+q)^2 \cdot v^3 + \cdots + (1+q)^{n-1}\cdot v^n = \dfrac{1-(1+q)^n v^n}{i-q}$

Useful Identities

$$a_{\overline{n+k}|} = a_{\overline{n}|} + v^n a_{\overline{k}|} \qquad v^n - v^m = i\left(a_{\overline{m}|} - a_{\overline{n}|}\right) \qquad (Da)_{\overline{n}|} + (Ia)_{\overline{n}|} = (n+1)\, a_{\overline{n}|}$$

$$1 = v^n + i\, a_{\overline{n}|} \qquad \frac{\ddot{a}_{\overline{2n}|}}{\ddot{a}_{\overline{n}|}} = \frac{a_{\overline{2n}|}}{a_{\overline{n}|}} = \frac{1-v^{2n}}{1-v^n} = 1+v^n \qquad \frac{\ddot{s}_{\overline{2n}|}}{\ddot{s}_{\overline{n}|}} = \frac{s_{\overline{2n}|}}{s_{\overline{n}|}} = \frac{(1+i)^{2n}-1}{(1+i)^n-1} = (1+i)^n + 1$$

If the interest rate varies:

$$a_{\overline{n}|} = \frac{1}{a(1)} + \frac{1}{a(2)} + \cdots + \frac{1}{a(n)} \qquad s_{\overline{n}|} = \frac{a(n)}{a(1)} + \frac{a(n)}{a(2)} + \cdots + \frac{a(n)}{a(n)}$$

If the compounding frequency of the interest exceeds the payment frequency of k years—

Use an equivalent interest rate over k years: $j = (1+i)^k - 1$

If the payment frequency exceeds the compounding frequency of the interest—

(1) Use an m-thly annuity

$$a^{(m)}_{\overline{n}|} = \frac{i}{i^{(m)}}\, a_{\overline{n}|} \qquad s^{(m)}_{\overline{n}|} = \frac{i}{i^{(m)}}\, s_{\overline{n}|} \qquad \ddot{a}^{(m)}_{\overline{n}|} = \frac{d}{d^{(m)}}\, \ddot{a}_{\overline{n}|} \qquad \ddot{s}^{(m)}_{\overline{n}|} = \frac{d}{d^{(m)}}\, s_{\overline{n}|}$$

(2) Use an equivalent interest rate effective over the payment period: $j = (1+i)^{1/m} - 1$

$$a^{(m)}_{\overline{n}|i} = a_{\overline{n}|j} \qquad s^{(m)}_{\overline{n}|i} = s_{\overline{n}|j} \qquad \ddot{a}^{(m)}_{\overline{n}|i} = \ddot{a}_{\overline{n}|j} \qquad \ddot{s}^{(m)}_{\overline{n}|i} = \ddot{s}_{\overline{n}|j}$$

If the payments are $\dfrac{1}{m}, \dfrac{2}{m}, \ldots, \dfrac{n}{m}$, then the present value is $(Ia)^{(m)}_{\overline{n}|i} = \dfrac{\ddot{a}_{\overline{n}|i} - nv^n}{i^{(m)}}$

If the payments are $\dfrac{1}{m^2}, \dfrac{2}{m^2}, \ldots, \dfrac{n}{m^2}$, then the present value is $\left(I^{(m)}a\right)^{(m)}_{\overline{n}|i} = \dfrac{\ddot{a}^{(m)}_{\overline{n}|i} - nv^n}{i^{(m)}}$

Loan Repayment— Amortization

Amortization Method— when a payment is made, it must be first applied to pay interest due and then any remaining part of the payment is applied to pay principle

Notation

L	$\equiv$	amount of the loan
n	$\equiv$	number of payment periods
P_A	$\equiv$	amount of level payment at the end of the period (amortized payment)
$P_{(k)}$	$\equiv$	loan payment at time k
i	$\equiv$	effective interest rate per payment period
B_k	$\equiv$	balance at time k, balance after k-th payment. Note that $B_0 = L$
P_k	$\equiv$	principle paid in payment $P_{(k)}$
I_k	$\equiv$	interest paid in payment $P_{(k)}$

Useful Equations for Level Payments

$$L = P_A \cdot a_{\overline{n|}i} \qquad P_A = \frac{L}{a_{\overline{n|}i}}$$

Prospective Method $\quad B_k = P_A \cdot a_{\overline{n-k|}i}$

Retrospective Method $\quad B_k = L(1+i)^k - P_A \cdot s_{\overline{k|}i}$

$B_{k+t} = B_k(1+i)^t$ and $P_{k+t} = P_k(1+i)^t$

$$P_A = P_k + I_k \qquad I_k = i \cdot B_{k-1} = P_A\left(1 - v^{n-k+1}\right) \qquad P_k = P_A - I_k = P_A \cdot v^{n-k+1}$$

Useful Equations for Non-Level Payments

$$L = P_{(1)}v + P_{(2)}v^2 + \cdots + P_{(n)}v^n \qquad B_k = P_{(k+1)}v + P_{(k+2)}v^2 + \cdots + P_{(n)}v^{n-k} = B_{k-1}(1+i) - P_{(k)}$$

$$I_k = i \cdot B_{k-1} \qquad P_k = P_{(k)} - I_k = B_{k-1} - B_k$$

Loan Repayment— Sinking Fund

Sinking Fund Loan (SFL)— accumulate money in a separate fund by making a payment, in addition to the regular interest payment, every period.

Notation

L	$\equiv$	amount of the loan
n	$\equiv$	number of payment periods
i	$\equiv$	effective interest rate per payment period by the borrower to the lender
j	$\equiv$	effective interest rate earned by the borrower in the sinking fund
D_S	$\equiv$	periodic sinking fund deposit (SFD), assumed to be level
P_S	$\equiv$	periodic outlay by the borrower = interest payment to lender + SFD
S_k	$\equiv$	sinking fund balance after k-th deposit
L_k	$\equiv$	net loan balance at time k

Useful Equations

$$L = D_S \cdot s_{\overline{n}|j} \qquad D_S = \frac{L}{s_{\overline{n}|j}} \qquad P_S = Li + D_S = Li + \frac{L}{s_{\overline{n}|j}}$$

$$S_k = D_S \cdot s_{\overline{k}|j} = L\frac{s_{\overline{k}|j}}{s_{\overline{n}|j}} \qquad L_k = L - D_S \cdot s_{\overline{k}|j}$$

Net Principal Paid $\quad S_k - S_{k-1} = D_S \cdot s_{\overline{k}|j} - D_S \cdot s_{\overline{k-1}|j} = D_S(1+j)^{k-1}$

Net Interest Paid $\quad Li - jS_{k-1} = Li - jD_S \cdot s_{\overline{k-1}|j}$

Notes on Loans

Amortized Loan— over time interest paid decreases and principal paid increases

SFL— for each outlay interest paid to lender is constant

Installment Loan— over time interest paid decreases while the principal paid is constant

Bonds

Bonds— interest bearing securities; basically loans from lenders perspective
Callable Bond— a bond that can be paid off (called) before maturity

Notation

F ≡ par value

r ≡ coupon rate (interest rate of bond)

Fr ≡ coupon amount (payment to lender)

C ≡ redemption value (usually = F)

n ≡ number of coupon periods to maturity

P ≡ market price of the bond

BV_k ≡ book value of the bond (bond amortized balance after k-th payment)

i ≡ yield per period to investor at price P

$$v_i = \frac{1}{1+i}$$

$K = Cv_i^n$ ≡ Present value of the redemption value

$g = \frac{Fr}{C}$ ≡ modified coupon rate

Premium— If $i > r$ then the bond is priced at a premium. $P > C$, and $P - C$ is the amount of the premium.

$$Premium \equiv P - C = (Fr - iC)\, a_{\overline{n|}i}$$

$$P - C = P_k\left(v^{k-1} + v^{k-2} + \cdots + v + 1 + (1+i) + \cdots + (1+i)^{n-k}\right) = P_k\left(a_{\overline{k-1|}i} + s_{\overline{n-k+1|}i}\right)$$

Discount— If $i < r$ then the bond is priced at a discount. $P < C$, and $C - P$ is the amount of the discount

$$Discount \equiv C - P = (iC - Fr)\, a_{\overline{n|}i}$$

Par— If $i = r$ the bond is selling at the price $P = C$ we say that it sells at par.

Price and Premium-Discount Formula

$$P = Fra_{\overline{n|}i} + K$$

$$P = C\left(1 + (g-i)a_{\overline{n|}i}\right) \qquad \text{if} \quad F = C, \text{ then } \quad P = F\left(1 + (r-i)a_{\overline{n|}i}\right)$$

Bond Amortized

$$BV_k = Fra_{\overline{n-k|}i} + Cv_i^{n-k} \qquad BV_k = BV_m(1+i)^{k-m} - Fr \cdot s_{\overline{k-m|}i} \qquad Fr = I_k + P_k$$

$$I_k = i \cdot BV_{k-1} = Fr\left(1 - v^{n-k+1}\right) + iC\,v^{n-k+1} \qquad P_k = Fr\,v^{n-k+1} - iC\,v^{n-k+1}$$

If $F = C$, then $\dfrac{P_{k+t}}{P_k} = (1+i)^t$

Write-Up during the first k years (Discount) $\equiv BV_k - P$
Write-Down during the first k years (Premium) $\equiv P - BV_k$
Write-Up/Write-Down in general during time m to time $k, (k > m)$ $\equiv BV_k - BV_m$

$$WD_k = (Fr - iC)v^{n-k+1} \qquad WU_k = (iC - Fr)v^{n-k+1}$$

Makeham's Formula

$$P = K + \frac{g}{i}(C - K) \qquad \text{if} \quad F = C, \text{ then } \quad P = K + \frac{r}{i}(F - K)$$

Maturity to use in Pricing a Callable Bond

Type of Bond	Take N using…
Premium Bond	Earliest Possible Redemption Date
Discount Bond	Latest Possible Redemption Date

Price Between Payment Dates

$$t = \frac{\text{number of days from last coupon date to settlement date}}{\text{number of days in the bond period}}$$

Price Plus Accrued $\equiv P_0(1+i)^t$ Accrued Interest $\equiv t(Fr)$

$$P \equiv \text{Price Plus Accrued} - \text{Accrued Interest} = P_0(1+i)^t - t(Fr)$$

Yield Rate of an Investment

Internal Rate of Return (IRR)— the rate of interest at which the present value of all amounts invested is equal to the present value of all the amounts paid back to the investor

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Internal Rate of Return (IRR)

Given investment cash flows $C_0, C_1, C_2, \ldots, C_n$, the IRR is a solution for i of the equation

$$C_0 + \frac{C_1}{(1+i)} + \frac{C_2}{(1+i)^2} + \ldots + \frac{C_n}{(1+i)^n} = 0 \qquad \text{or} \qquad C_0 + C_1 v + C_2 v^2 + \ldots + C_n v^n = 0$$

Time Weighted Rates of Interest (TWR)

C'_k $\equiv$ Contribution at time t_k

B'_k $\equiv$ Fund value at time t_k before the contribution C'_k is made

j_k $\equiv$ Effective rate over $[t_{k-1}, t_k]$

$$1 + j_k = \frac{B'_k}{B'_{k-1} + C'_{k-1}} \qquad \text{TWR} \quad \rightarrow \quad 1 + i = (1+j_1)(1+j_2)\cdots(1+j_m)$$

Dollar Weighted Rates of Interest (DWR)

A $\equiv$ Initial fund balance

B $\equiv$ Final fund balance

I $\equiv$ Interest earned

C_t $\equiv$ Contribution or withdrawal at time t (cash flows)

C_{Net} $\equiv$ Net contribution

$$C_{Net} = \sum C_t \qquad B = A + C_{Net} + I \quad \Rightarrow \quad I = B - A - C_{Net}$$

$$\text{DWR} \quad \rightarrow \quad i = \frac{I}{A + \sum C_t (1-t)}$$

Term Structure of Interest Rates

Spot Rates

Denoted by $s_n \equiv$ the n-year spot rate

(1) The annual interest rate on the n-year Treasury STRIP is called the n-year spot rate, and the series of spot rates over time is called the yield curve.

(2) To value a bond, take the present value of each payment at the appropriate yield curve rate and sum the present values.

$$P = \frac{P_{(1)}}{(1+s_1)} + \frac{P_{(2)}}{(1+s_2)^2} + \cdots + \frac{P_{(n)}}{(1+s_n)^n} = \frac{P_{(1)}}{(1+f_1)} + \frac{P_{(2)}}{(1+f_1)(1+f_2)} + \cdots + \frac{P_{(n)}}{\prod_{i=1}^{n}(1+f_i)}$$

(3) Once we have found the price of a bond using the yield curve we can find the yield to maturity as the constant yield on the bond at that price.

For example—

Purchasing a bond with coupons has cash flows given by $-P,\ P_{(1)},\ P_{(2)},\ldots,\ P_{(n)}$

If payments $P_{(1)},\ P_{(2)},\ldots,\ P_{(n-1)}$ are not level

Using the BA-II Plus—

1. $\rightarrow$CF Worksheet $\rightarrow$ Set CFo $= -P$, C01 $= P_{(1)}$, C02 $= P_{(2)}, \ldots,$ CN $= P_{(n)}$
2. $\rightarrow$IRR $\rightarrow$CPT $\Rightarrow$ Constant Yield on the Bond = IRR

If payments $P_{(1)},\ P_{(2)},\ldots,\ P_{(n-1)}$ are level

Using the BA-II Plus

1. $\rightarrow$TVM Worksheet $\rightarrow$ Set PV $= -P$, N $= n$, PMT $= P_{(n-1)}$, FV $= C$
2. $\rightarrow$I/Y $\rightarrow$CPT $\Rightarrow$ Constant Yield on the Bond = I/Y

Forward Rates

Denoted by $f_n \equiv$ the n year forward rate

The rate agreed upon today for a one-year loan to be made n years in the future

$$1+f_n = \frac{(1+s_n)^n}{(1+s_{n-1})^{n-1}} \qquad \Rightarrow \qquad (1+s_n)^n = (1+f_n)(1+s_{n-1})^{n-1}$$

Duration

Duration— a measure of sensitivity of a financial asset to changes in interest rates

Investment Cash Flows $C_1, C_2, \ldots, C_n$

Investment Price $P(i) = vC_1 + v^2C_2 + \cdots + v^nC_n = \sum_{t>0} v^tC_t$

Weights for Macaulay Duration $w_t = \frac{v^tC_t}{P(i)} = \frac{v^tC_t}{vC_1 + v^2C_2 + \cdots + v^nC_n} = \frac{v^tC_t}{\sum_{t>0} v^tC_t}$

Macaulay Duration $D_M = 1 \cdot w_1 + 2 \cdot w_2 + \cdots + n \cdot w_n = \frac{\sum_{t>0} tv^tC_t}{\sum_{t>0} v^tC_t}$

Modified Duration $D = -\frac{d}{di}P(i) = \frac{1}{1+i}D_M$

Duration of a Level Payment Investment $D = \frac{(Ia)_{\overline{n|}i}}{a_{\overline{n|}i}}$

Macaulay Duration of a coupon bond with face value *F* and coupon *Fr* for *n* periods and redemption value *C*

$$D_M = \frac{Fr(Ia)_{\overline{n|}i} + nCv^n}{Fra_{\overline{n|}i} + Cv^n}$$

The Duration of a Zero-Coupon Bond payable in *n* periods is *n*

Modified Duration of a Portfolio with Investments X_k

$$D_{Tot} = W_1D_1 + W_2D_2 + \cdots + W_nD_n \qquad \text{where} \qquad W_k = \frac{X_k}{X_1 + X_2 + \cdots + X_n}$$

Convexity and Approximations in Change of Price

Convexity and Estimate $\mathcal{C} = \frac{1}{P(i)}\frac{d^2P(i)}{di^2} \approx \frac{P(i-\Delta i) - 2P(i) + P(i+\Delta i)}{P(i)\cdot(\Delta i)^2}$

Change in Price

$$\Delta P = P(i+\Delta i) - P(i) \approx P'(i)\Delta i = \frac{P'(i)}{P(i)}P(i)\Delta i = -D \cdot P(i)\Delta i$$

$$\Delta P \approx P'(i)\Delta i + \frac{P''(i)}{2}\Delta i^2 = -D \cdot P(i)\Delta i + \mathcal{C} \cdot \frac{P(i)}{2}\Delta i^2$$

Immunization

Notation

$A(i) \equiv$ Present Value of Assets

$A_t \equiv$ Asset Amount at time *t*

$L(i) \equiv$ Present Value of Liabilities

$L_t \equiv$ Liability Amount at time *t*

$S(i) \equiv$ Surplus

$S(i) = A(i) - L(i)$

Conditions for Immunization

To achieve immunization we must have $S(i_o) = 0$, $S'(i_o) = 0$, and $S''(i_o) > 0$

Immunization in terms of duration and convexity we need…

(1) PV Matching $\quad A(i_o) = L(i_o)$

(2) Duration Matching $\quad \left.\frac{d}{di}A(i)\right|_{i_0} = \left.\frac{d}{di}L(i)\right|_{i_0}$

(3) Greater Convexity for Assets $\quad \left.\frac{d^2}{di^2}A(i)\right|_{i_0} > \left.\frac{d^2}{di^2}L(i)\right|_{i_0}$

Immunization in terms of the asset and liability amounts at time *t*...

(1) PV Matching $\quad \sum_{t>0} A_t v_{i_0}^t = \sum_{t>0} L_t v_{i_0}^t$

(2) Duration Matching $\quad \sum_{t>0} tA_t v_{i_0}^t = \sum_{t>0} tL_t v_{i_0}^t$

(3) Greater Convexity for Assets $\quad \sum_{t>0} t^2 A_t v_{i_0}^t = \sum_{t>0} t^2 L_t v_{i_0}^t$

Special Cases

Yield Rate Reinvestments

Notation

y	$\equiv$	annual yield of total investment (IRR)
n	$\equiv$	number of years
k	$\equiv$	number of payments
i	$\equiv$	k effective interest in fund X
j	$\equiv$	k effective interest in fund Y

General Case—

Suppose you make an initial investment of C_0. The yield rate y is the actual rate of return you are receiving on the investment. AV is the accumulated value of your investment.

$$C_0(1+y)^n = AV$$

- Suppose you are investing payments into a fund X at the end of each k period
- …and reinvesting the interest accrued each k period into fund Y

$$\underbrace{a_{\overline{n}|y}(1+y)^n}_{AV \text{ of initial investment}} = s_{\overline{n}|y} = \underbrace{k + i(Is)_{\overline{k}|j}}_{AV \text{ of reinvestment}}$$

- Suppose you make an initial investment of C_0 into fund X at $t=0$
- You reinvest interest accrued in fund X after each k period into fund Y starting at $t=1$
- You reinvest interest accrued in fund Y after each k period into fund Z starting at $t=2$

$$C_0(1+y)^n = \underbrace{C_0 + k(i_X C_0) + i_Y i_X C_0 (Is)_{\overline{k-1}|\, i_Z}}_{\substack{\text{Sum of principal and interest} \\ \text{after k periods}}}$$

$$\Rightarrow \qquad (1+y)^n = 1 + k i_X + i_Y i_X (Is)_{\overline{k-1}|\, i_Z}$$

Bond Reinvestments

This refers to the case where we have bought a bond for a price of $P = Fra_{\overline{n}|i} + K$ and we reinvest the coupon payments Fr into a separate account at the time they are received.

Notation

y	$\equiv$	annual yield of total investment
n	$\equiv$	number of years
k	$\equiv$	number of payments the bond pays

$Frs_{\overline{k}|i} + C$ is the AV of the account and the price P is the initial investment.

$$P\left(1+y\right)^n = P\left(1+\frac{y^{(m)}}{m}\right)^{m\cdot n} = Frs_{\overline{k}|i} + C \qquad \Rightarrow \qquad \left(1+y\right)^n = \frac{Frs_{\overline{k}|i} + C}{Fra_{\overline{k}|i} + K}$$

Of course we can have more than one bond involved. If that is the case we just need to combine prices and coupon payments accordingly.

Matching Liabilities Using Bonds

We are going to cover the case that liability frequency matches the coupon frequency. (e.g. We would not have a liabilities at year 1 and year 2 with coupons semiannually).

Let F_1, r_1 and C_1 denote the par value, coupon rate and redemption value, respectively, for the bond with the longest duration. Denote F_2, r_2 and C_2 for the bond with the next longest duration, and so on.

Step1- Purchase $\dfrac{C_1}{F_1 r_1 + C_1}$ of the bond. This is a percentage.

Step2- This gives $F_1 r_1 \cdot \dfrac{C_1}{F_1 r_1 + C_1}$, a fractional amount of the coupon the period before.

Step 3- Determine the amount left we need to match. $C_2 - F_1 r_1 \cdot \dfrac{C_1}{F_1 r_1 + C_1}$

Step 4- Purchase $\dfrac{C_2 - F_1 r_1 \cdot \dfrac{C_1}{F_1 r_1 + C_1}}{F_2 r_2 + C_2}$ of the bond.

Price of the bond to match liabilities is: $\dfrac{C_2 - F_1 r_1 \cdot \dfrac{C_1}{F_1 r_1 + C_1}}{F_2 r_2 + C_2} P_2 + \dfrac{C_1}{F_1 r_1 + C_1} P_1$. This matches liabilities at time 1 and time 2.

Stupid Yield Curve Stuff

(2) To value a bond, take the present value of each payment at the appropriate yield curve rate and sum the present values.

$$P = \frac{P_{(1)}}{\left(1+s_1\right)} + \frac{P_{(2)}}{\left(1+s_2\right)^2} + \cdots + \frac{P_{(n)}}{\left(1+s_n\right)^n} = \frac{P_{(1)}}{\left(1+f_1\right)} + \frac{P_{(2)}}{\left(1+f_1\right)\left(1+f_2\right)} + \cdots + \frac{P_{(n)}}{\prod_{i=1}^{n}\left(1+f_i\right)}$$