$\frac{60}{60}$ good

MATH 30055 SPRING 2015 HW 2 NAME: [redacted]

DUE TUESDAY, 10 FEBRUARY 2015

1. Suppose $a(t) = \alpha t^2 + \beta t + \gamma$. If \$100 invested at time 0 accumulates to \$152 at time 4 and \$200 invested at time 0 accumulates to \$240 at time 2, find the accumulated value at time 8 of \$5000 invested at time 6.

SOLUTION.

If we first examine $a(0)$, we get

$$a(0) = \alpha(0)^2 + \beta(0) + \gamma = \gamma,$$

but $a(0) = 1$. So, $\gamma = 1$. To find α and β, the accumulation function tells us

$$\begin{cases} 100\left(\alpha(4)^2 + \beta(4) + 1\right) = 152 \\ 200\left(\alpha(2)^2 + \beta(2) + 1\right) = 240 \end{cases}$$

$$\implies - \begin{cases} 1600\alpha + 400\beta + 100 = 152 \\ 800\alpha + 400\beta + 200 = 240 \end{cases} \tag{1}$$

$$\begin{aligned} 800\alpha - 100 &= -88 \\ 800\alpha &= 12 \\ \alpha &= 0.015. \end{aligned}$$

Since $\alpha = 0.015$, we can substitute α in one of the equation in (1):

$$\begin{aligned} 800(0.015) + 400\beta + 200 &= 240 \\ 400\beta + 212 &= 240 \\ 400\beta &= 28 \\ \beta &= 0.07. \end{aligned}$$

So, $a(t) = 0.015t^2 + 0.07t + 1$. Let K be the accumulated value. Proportionally,

$$\frac{K}{5000} = \frac{a(8)}{a(6)} = \frac{0.015(8)^2 + 0.07(8) + 1}{0.015(6)^2 + 0.07(6) + 1}$$

$$\begin{aligned} K(1.96) &= 5000(2.52) \\ K &= \boxed{\$6,428.57.} \end{aligned}$$

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2. (a) At a par[redacted] interest, \$1400 invested at time $t = 0$ accumulates to \$1820 at time T years. Find the accumulated value at time $2T$ years of \$500 invested at time $t = 0$ at the same rate of simple interest.

SOLUTION.

Using the amount function for simple interest,

$$\begin{aligned} 1400(1 + iT) &= 1820 \\ 1 + iT &= 1.3 \\ iT &= .3 \\ T &= \frac{.3}{i}. \end{aligned}$$

Knowing T in terms of i gives us the ability to substitute into the new amount function at time $2T$ with a principle amount of \$500. So, \$500 will accumulate to

$$\begin{aligned} 500(1 + i2T) &= 500\left(1 + \cancel{i}(2)\left(\frac{.3}{\cancel{i}}\right)\right) \\ &= 500(1 + .6) \\ &= 500(1.6) \\ &= \boxed{\$800.} \end{aligned}$$

(b) At a particular rate of compound interest, \$1400 invested at time $t = 0$ accumulates to \$1820 at time T years. Find the accumulated value at time $2T$ years of \$500 invested at time $t = 0$ at the same rate of compound interest.

SOLUTION.

Using the amount function for compound interest,

$$\begin{aligned} 1400(1 + i)^T &= 1820 \\ (1 + i)^T &= 1.3 \\ 1 + i &= \sqrt[T]{1.3} \\ i &= \sqrt[T]{1.3} - 1. \end{aligned}$$

Using substitution for i to find the amount \$500 will accumulate in $2T$,

$$\begin{aligned} 500(1 + i)^{2T} &= 500(1 + \sqrt[T]{1.3} - 1)^{2T} \\ &= 500(\sqrt[T]{1.3})^{2T} \\ &= 500(1.3^{2T/T}) \\ &= 500(1.3)^2 \\ &= \boxed{\$845.} \end{aligned}$$

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3. (a) Find the nominal rate of discount convertible monthly that is equivalent to a nominal rate of interest of 7% per annum converted semiannually.

SOLUTION.

Since $\left(1+\frac{i^{(m)}}{m}\right)^m = \left(1-\frac{d^{(p)}}{p}\right)^{-p}$,

$$\left(1+\frac{.07}{2}\right)^2 = \left(1-\frac{d^{(12)}}{12}\right)^{-12}$$

$$\left(1+\frac{.07}{2}\right)^{-2/12} = \left(1-\frac{d^{(12)}}{12}\right)$$

$$(1.035)^{-1/6} = 1-\frac{d^{(12)}}{12}$$

$$0.9942828344 = 1-\frac{d^{(12)}}{12}$$

$$-0.0057171656 = -\frac{d^{(12)}}{12}$$

$$d^{(12)} = 0.0686059868$$

$$\approx \boxed{6.86\%.}$$

(b) Express $i^{(6)}$ in terms of $d^{(4)}$.

SOLUTION.

Since $\left(1+\frac{i^{(m)}}{m}\right)^m = \left(1-\frac{d^{(p)}}{p}\right)^{-p}$,

$$\left(1+\frac{i^{(6)}}{6}\right)^6 = \left(1-\frac{d^{(4)}}{4}\right)^{-4}$$

$$1+\frac{i^{(6)}}{6} = \left(1-\frac{d^{(4)}}{4}\right)^{-4/6}$$

$$\frac{i^{(6)}}{6} = \left(1-\frac{d^{(4)}}{4}\right)^{-2/3} - 1$$

$$i^{(6)} = \boxed{6\left[\left(1-\frac{d^{(4)}}{4}\right)^{-2/3} - 1\right].}$$

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4. Find n given that

$$1 - \frac{d^{(n)}}{n} = \frac{1 + \frac{i^{(7)}}{7}}{1 + \frac{i^{(6)}}{6}}.$$

First, we can manipulate this equality by taking the n^{th} powers

$$\left(1 - \frac{d^{(n)}}{n}\right)^n = \left[\frac{1 + \frac{i^{(7)}}{7}}{1 + \frac{i^{(6)}}{6}}\right]^n. \tag{2}$$

We know

$$\left(1 + \frac{i^{(n)}}{n}\right)^n = 1 + i$$

and

$$\left(1 - \frac{d^{(n)}}{n}\right)^n = 1 - d = \frac{1}{1+i} = (1+i)^{-1}$$

where i and d is the effective rates interest and discount, respectively, equivalent to $i^{(n)}$ and $d^{(n)}$. So we can rewrite (2) as

$$(1+i)^{-1} = \frac{\left(1 + \frac{i^{(7)}}{7}\right)^n}{\left(1 + \frac{i^{(6)}}{6}\right)^n}$$

$$(1+i)^{-1} = \frac{\left(1 + \frac{i^{(7)}}{7}\right)^{7(n/7)}}{\left(1 + \frac{i^{(6)}}{6}\right)^{6(n/6)}}$$

$$(1+i)^{-1} = \frac{\left[\left(1 + \frac{i^{(7)}}{7}\right)^7\right]^{n/7}}{\left[\left(1 + \frac{i^{(6)}}{6}\right)^6\right]^{n/6}}$$

$$(1+i)^{-1} = \frac{(1+i)^{n/7}}{(1+i)^{n/6}}$$

$$(1+i)^{-1} = (1+i)^{n/7 - n/6}$$

$$(1+i)^{-1} = (1+i)^{-n/42}.$$

Since exponential functions are one-to-one,

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$$-1 = -\frac{n}{42}$$

$$n = \boxed{42.}$$

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5. Maude deposits \$500 into a bank account. Her account is credited interest at a nominal rate of interest of 10% convertible quarterly. At the same time, Jeffrey deposits \$500 into a separate account. His account is credited interest at a constant force of interest of δ.

(a) Find δ if the value of Maude's account is the same as the value of Jeffrey's account after 3.5 years.

SOLUTION.

In this problem, we are given that Maude's and Jeffery's amounts are equal. Setting the amount functions equal to each other,

$$\cancel{500}\left(1+\frac{.1}{4}\right)^{4(3.5)} = \cancel{500}e^{(3.5)\delta}$$

$$(1.412973821) = e^{3.5\delta}$$

$$\ln(1.412973821) = 3.5\delta$$

$$\delta = \frac{\ln(1.412973821)}{3.5}$$

$$= \boxed{0.09877.}$$

(b) Find δ if the value of Maude's account is the same as the value of Jeffrey's account after 10.7 years.

SOLUTION.

~~Since the force of interest is constant,~~

$$\cancel{\boxed{\delta = 0.09877.}}$$

6. Fund A grows under simple interest with rate i. That is, $a(t) = 1 + it$ is the accumulation function. Fund B grows under simple discount with rate d. That is, $b(t) = \frac{1}{1-dt}$ is the accumulation function. Suppose $i > d$. Derive an expression for the time t at which the force of interest in the two is the same.

SOLUTION.

Fund A has the accumulation function of $a(t) = (1 + it)$, so $a'(t) = i$. Since $\delta_t = \frac{a'(t)}{a(t)}$,

$$d_t^{a} = \frac{i}{1+it}.$$

Fund B has the accumulation function of $b(t) = \frac{1}{1-dt}$, so $b'(t) = -(1-dt)^{-2} \cdot -d = \frac{d}{(1-dt)^2}$. Since $\delta_t = \frac{a'(t)}{a(t)}$,

indicate somehow that they're different

$$d_t^{b} = \frac{d}{(1-dt)^2} \cdot \cancel{(1-dt)} = \frac{d}{1-dt}.$$

see next page

now set them ~~equal & solve for t~~

ok sorry

These forces of interest are the same when

$$\frac{i}{1+it} = \frac{d}{1-dt}$$

$$i(1-dt) = d(1+it)$$

$$i - idt = d + idt$$

$$i = d + 2idt$$

$$i - d = 2idt$$

$$t = \boxed{\frac{i-d}{2id}.}$$

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