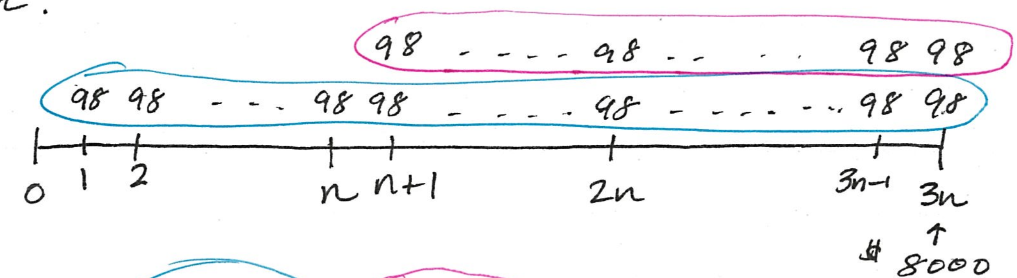
Theory of Interest Spring '15

Hand-in HW #3

1. One approach is to regard this as a $3n$-year annuity-immediate with \$98 payments starting at $t=0$ and a $2n$-year annuity-immediate w/ \$98 payments beginning at $t=n$.



Then $$98\, s_{\overline{3n}|} + 98\, s_{\overline{2n}|} = 8000$$

$$98\left(s_{\overline{3n}|} + s_{\overline{2n}|}\right) = 8000$$

$$98\left[\frac{(1+i)^{3n}-1}{i} + \frac{(1+i)^{2n}-1}{i}\right] = 8000$$

Given: $(1+i)^n = 2.0$

$$\frac{2^3-1}{i} + \frac{2^2-1}{i} = \frac{8000}{98}$$

$$\frac{10}{i} = \frac{8000}{98}$$

$$i = \frac{98}{800}$$

$$i = 0.1225 = 12.25\% \quad \blacksquare$$

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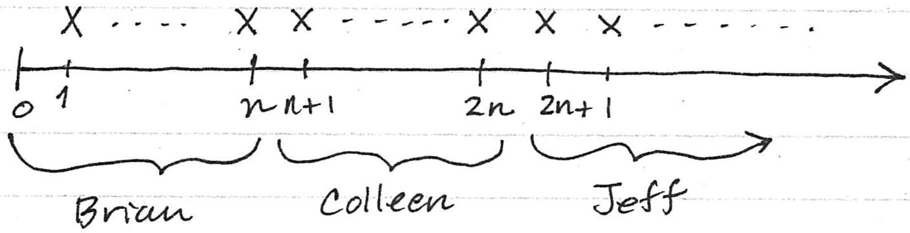
2.



Present value of the perpetuity: $X a_{\overline{\infty}|}$

" " Brian's portion: $X a_{\overline{n}|}$

" " Colleen's portion: $X a_{\overline{n}|} v^n$

" " Jeff's portion: $X a_{\overline{\infty}|} v^{2n}$

Given: $\dfrac{X a_{\overline{n}|}}{X a_{\overline{\infty}|}} = 0.4$ so $a_{\overline{n}|} = 0.4 a_{\overline{\infty}|}$.

That is, $\dfrac{1-v^n}{i} = \dfrac{0.4}{i}$

$$1 - v^n = 0.4$$
$$v^n = 1 - 0.4 = 0.6$$

We want to find Jeff's portion as share K of the whole present value:

$$K = \frac{X a_{\overline{\infty}|} v^{2n}}{X a_{\overline{\infty}|}} = v^{2n} = (v^n)^2 = 0.6^2 = 0.36 = \underline{36\%}$$

give Jeff's share as a decimal or percentag because that's how they gave Brian's share in the statement of the problem.

Alternate approach:

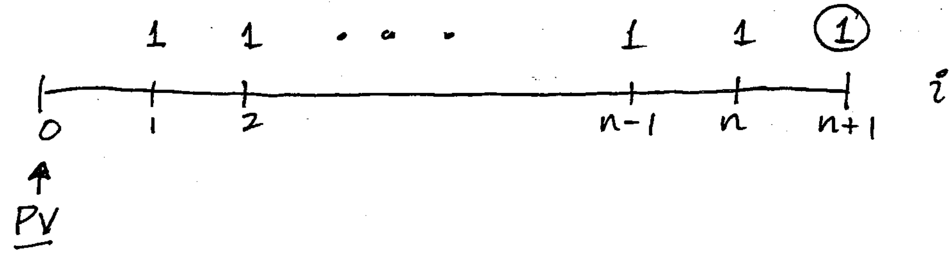
$$X a_{\overline{n}|} + X a_{\overline{n}|} v^n + X a_{\overline{\infty}|} v^{2n} = X a_{\overline{\infty}|}$$
$$0.4 a_{\overline{\infty}|} + 0.4 a_{\overline{\infty}|} v^n + a_{\overline{\infty}|} v^{2n} = a_{\overline{\infty}|}$$
$$0.4 + 0.4 v^n + v^{2n} = 1$$

etc...

HW 3

-3- annuity immediate:

3. (a)

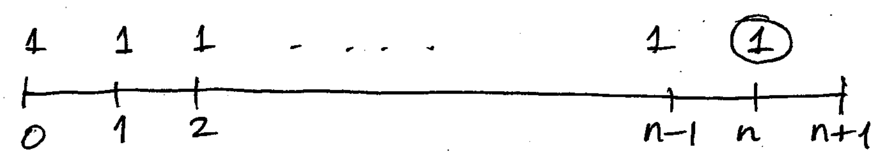


The symbol $a_{\overline{n+1}|}$ represents the present value of the payment stream depicted above, while $a_{\overline{n}|}$ is the present value of that sequence of payments, except for the last payment (at $t=n+1$). So the difference, $a_{\overline{n+1}|} - a_{\overline{n}|}$ represents the present value of the last payment, that is, the present value of \$1, discounted n+1 years. So

$$a_{\overline{n+1}|} - a_{\overline{n}|} = v^{n+1}.$$

(b) Similarly, $\ddot{a}_{\overline{n+1}|} - \ddot{a}_{\overline{n}|}$ represents the present value of of \$1, discounted n years, as pictured below.

annuity due:



So $\ddot{a}_{\overline{n+1}|} - \ddot{a}_{\overline{n}|} = v^n$. So

$$v = \frac{v^{n+1}}{v^n} = \frac{a_{\overline{n+1}|} - a_{\overline{n}|}}{\ddot{a}_{\overline{n+1}|} - \ddot{a}_{\overline{n}|}} = \frac{0.177208656}{0.185248436}$$

cont'd ...

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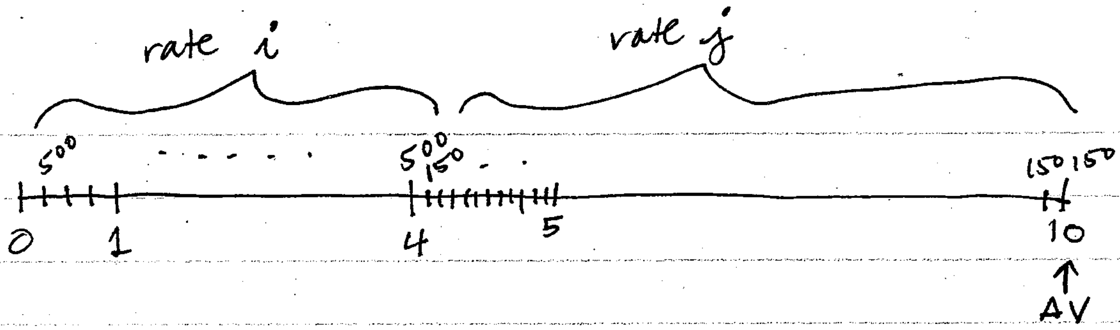
Now
$$v^n = 0.185248436$$
$$\ln v^n = \ln 0.185248436$$
$$n \ln v = \ln 0.185248436$$
$$n = \frac{\ln 0.185248436}{\ln v}$$

$$n = \frac{\ln 0.185248436}{\ln\left(\frac{0.177208656}{0.185248436}\right)}$$

$$n \approx \underline{38} \text{ yrs.} \blacksquare$$

4.



$$AV = (4\times 500)\, s^{(4)}_{\overline{4}|i}\,(1+j)^6 + (12\times 150)\, s^{(12)}_{\overline{6}|j}$$

$$= 2000\, s^{(4)}_{\overline{4}|i}\,(1+j)^6 + 1800\, s^{(12)}_{\overline{6}|j} \, ,$$

If $i = 0.045$, then $\left(1+\frac{i^{(4)}}{4}\right)^4 = 1.045$ and so

$$i^{(4)} = 4\left(1.045^{1/4}-1\right) \approx 0.0443 .$$

If $j = 0.0525$, then $\left(1+\frac{j^{(12)}}{12}\right)^{12} = 1.0525$ and so

$$j^{(12)} = 12\left(1.0525^{1/12}-1\right) \approx 0.05128 .$$

So in this case,

$$AV = 2000\,\frac{\left(1.045^4-1\right)}{0.0443\cdots}\,(1.0525)^6 + 1800\,\frac{\left(1.0525^6-1\right)}{0.05128\cdots}$$

$$\approx \$24{,}440.07 . \quad \blacksquare$$

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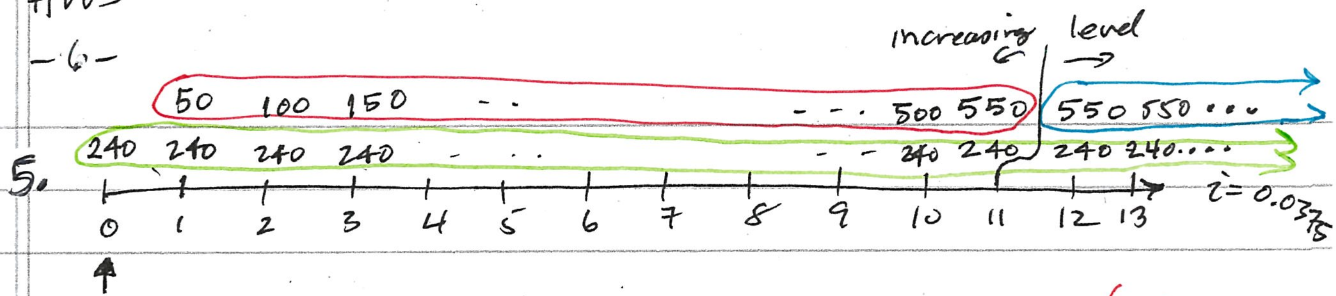
5.



$$PV = 240\,\ddot{a}_{\overline{\infty}|} + 50(Ia)_{\overline{11}|} + 550\,a_{\overline{\infty}|}\,v^{11}$$

(all are taken back to time 0.)

$$= \frac{240}{d} + 50\left(\frac{\ddot{a}_{\overline{11}|} - 11v^{11}}{i}\right) + \frac{550}{i}\,v^{11}$$

$$= \frac{240}{\frac{0.0375}{1.0375}} + 50\left(\frac{\frac{1-(1.0375)^{-11}}{\frac{0.0375}{1.0375}} - 11(1.0375)^{-11}}{0.0375}\right) + \frac{550(1.0375)^{-11}}{0.0375}$$

$$\approx \$18{,}923.72.$$

Instead of punching this all in to your calculator, I recommend storing d and v in memory first

* There were many other ways to break this apart – just be sure to clarify where the first + last payment for each part is and where the value is being evaluated.

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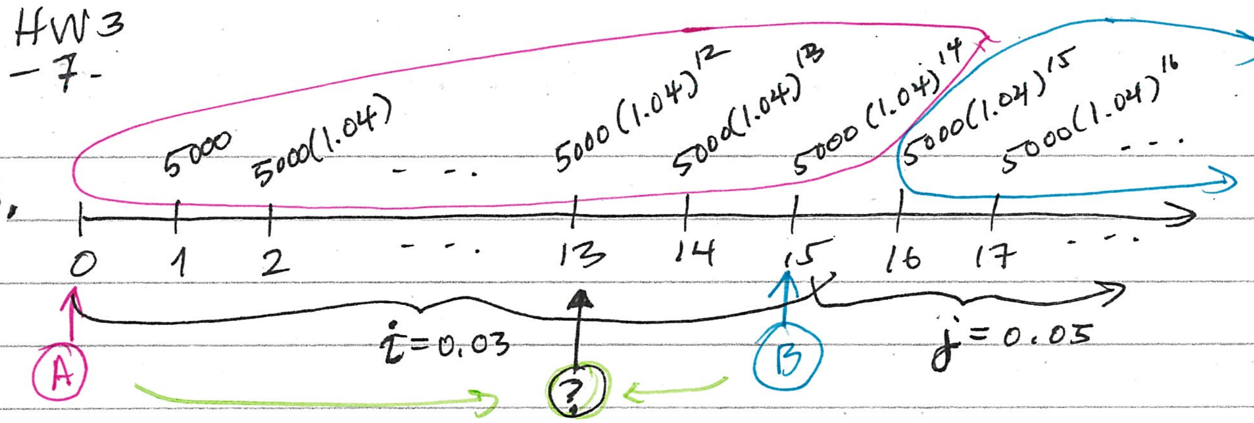
6.



I decided to separate the timeline into two pieces, depending on the interest rate: Let A be the value of the 1st 15 payments at $t=0$. Then

$$A = 5000 v_i + 5000(1.04) v_i^2 + \cdots + 5000(1.04)^{14} v_i^{15}$$

$$= 5000 v_i \left[1 + \frac{1.04}{1.03} + \cdots + \left(\frac{1.04}{1.03}\right)^{14} \right]$$

$$= \frac{5000}{1.03} \left[\frac{1-\left(\frac{1.04}{1.03}\right)^{15}}{1-\left(\frac{1.04}{1.03}\right)} \right] \approx \$77{,}978.5528\ldots$$

Let B be the value of the remaining payments at $t=15$. Then

$$B = 5000(1.04)^{15} v_j + 5000(1.04)^{16} v_j^2 + \cdots$$

$$= 5000(1.04)^{15} v_j \left[1 + \frac{1.04}{1.05} + \left(\frac{1.04}{1.05}\right)^2 + \cdots \right]$$

$$= \frac{5000(1.04)^{15}}{1.05} \left[\frac{1}{1-\left(\frac{1.04}{1.05}\right)} \right] \approx \$900{,}471.7528\ldots$$

So at $t=13$, the value is

$$A(1.03)^{13} + B\left(\frac{1}{1.03}\right)^2 \approx \$963{,}295.12.$$ ∎

Hw3

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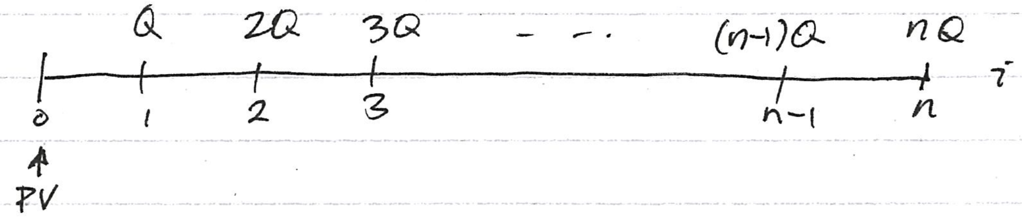
7.



We have, from the time line,

$$\begin{aligned} (1+i)PV &= Q + 2Qv + 3Qv^2 + \cdots + nQv^{n-1} \\ -\ PV &= \quad -Qv - 2Qv^2 - \cdots - (n-1)Qv^{n-1} - nQv^n \end{aligned}$$

So subtracting, we obtain

careful!

$$(1+i)PV - PV = Q + Qv + Qv^2 + \cdots + Qv^{n-1} - nQv^n$$

$$iPV = Q\left(1 + v + v^2 + \cdots + v^{n-1}\right) - Qnv^n$$

So
$$PV = \frac{Q\,\ddot{a}_{\overline{n}|} - Qnv^n}{i}$$

$$= Q\left[\frac{\ddot{a}_{\overline{n}|} - nv^n}{i}\right]$$

$$= Q\,(Ia)_{\overline{n}|i}\ .\ \blacksquare$$

Again, this is not the best way to show this, but I wanted you to practice this method.