Exam 2: Thursday, March 13, 2014

Show all work. Give exact values, simplified. Use proper notation. NO CALCULATORS.

1. (15 points)

(a) DERIVE the Pythagorean Identity involving sine and cosine from the definitions of the trig functions. Sketch a diagram. Give a reason for each step.

By the Pythagorean theorem, for the equation of the circle,

\[ x^2 + y^2 = r^2 \]

\[ \begin{align*}
\frac{x^2}{r^2} + \frac{y^2}{r^2} &= 1 \\
\left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2 &= 1 \\
\cos^2 \theta + \sin^2 \theta &= 1.
\end{align*} \]

(b) Use your answer above and other basic identities to DERIVE either of the other two Pythagorean Identities. Give a reason for each step.

\[ \sin^2 \theta + \cos^2 \theta = 1 \quad \text{(proved in (a))} \]

\[ \frac{1}{\cos^2 \theta} \left( \sin^2 \theta + \cos^2 \theta \right) = \frac{1}{\cos^2 \theta} \cdot 1 \quad \text{(algebra)} \]

\[ \frac{\sin^2 \theta}{\cos^2 \theta} + \frac{\cos^2 \theta}{\cos^2 \theta} = \frac{1}{\cos^2 \theta} \quad \text{(algebra)} \]

\[ \left( \frac{\sin \theta}{\cos \theta} \right)^2 + 1 = \left( \frac{1}{\cos \theta} \right)^2 \quad \text{(algebra)} \]

\[ (\tan \theta)^2 + 1 = (\sec \theta)^2 \quad \text{(ratio + reciprocal identities)} \]

\[ \tan^2 \theta + 1 = \sec^2 \theta. \]
2. (10 points)

(a) State the Sum formula for sine.

\[
\sin(u + v) = \sin u \cos v + \cos u \sin v
\]

(b) Use your answer above, basic identities, and algebra to DERIVE the Difference formula for sine, that is, the familiar formula for \(\sin(u - v)\). Give a reason for each step.

\[
\begin{align*}
\sin(u - v) &= \sin(u + (-v)) \\
&= \sin u \cos(-v) + \cos u \sin(-v) & \text{sum fun} \\
&= \sin u \cos v - \cos u \sin v & \text{add/ren} \\
&= \sin u \cos v - \cos u \sin v & \text{alg}
\end{align*}
\]

3. (10 points) Use Sum or Difference Formulas to evaluate each of the following. Write out each step.

(a) Find the exact value, simplified: \(\sin \frac{7\pi}{12} = \sin \left( \frac{2\pi}{12} + \frac{4\pi}{12} \right)\)

\[
= \sin \left( \frac{\pi}{4} + \frac{\pi}{3} \right)
\]

\[
= \sin \frac{\pi}{4} \cos \frac{\pi}{3} + \cos \frac{\pi}{4} \sin \frac{\pi}{3}
\]

\[
= \frac{\sqrt{2}}{2} \cdot \frac{1}{2} + \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2}
\]

\[
= \frac{1 + \sqrt{3}}{2\sqrt{2}}
\]

(b) Find the exact value: \(\cos 94^\circ \cos 4^\circ + \sin 94^\circ \sin 4^\circ = \cos (94^\circ - 4^\circ)\)

\[
= \cos 90^\circ
\]

\[
= 0
\]
4. (10 points) Use the following Double Angle Formula for cosine to DERIVE another Double Angle Formula involving only the sine function. Provide a reason for each step.
\[ \cos 2x = \cos^2 x - \sin^2 x \]

Solution:
\[
\cos 2x = \cos^2 x - \sin^2 x \quad \text{given}
\]
\[
= (1-\sin^2 x) - \sin^2 x \quad \text{Pythag Identity, } \sin^2 x + \cos^2 x = 1
\]
\[
= 1 - \sin^2 x - \sin^2 x
\]
\[
= 1 - 2\sin^2 x
\]

5. (10 points) DERIVE the Cofunction Identity for \( \sec \left( \frac{\pi}{2} - \theta \right) \) from the right triangle definitions of the trig functions.
Sketch a diagram. Explain your reasoning in full sentences.

If \( \theta \) is an acute angle, then \( \frac{\pi}{2} - \theta \) is another acute angle, forming a right triangle together with the right angle \( \frac{\pi}{2} \). Labelling as shown, we have

\[ \sec \left( \frac{\pi}{2} - \theta \right) = \frac{c}{a} = \csc \theta \]
6. (10 points) Verify the following identity. Give a reason for each step. Do not skip steps.

\[
\tan^2 \theta (1 + \cot^2 \theta) = \frac{1}{1 - \sin^2 \theta}.
\]

Proof:

\[
\begin{align*}
\tan^2 \theta (1 + \cot^2 \theta) &= \tan^2 \theta + \tan^2 \theta \cot^2 \theta \\
&= \frac{\sin^2 \theta}{\cos^2 \theta} + \frac{\sin^2 \theta}{\cos^2 \theta} \cdot \frac{\cos^2 \theta}{\sin^2 \theta} \\
&= \frac{\sin^2 \theta + \cos^2 \theta}{\cos^2 \theta} \\
&= \frac{1}{\cos^2 \theta} \quad \text{Pythagorean Identity} \\
&= \frac{1}{1 - \sin^2 \theta} \\
&= \text{ RHS}. \quad \square
\end{align*}
\]

7. (15 points) Suppose sec \( \theta = -5 \) and the terminal side of \( \theta \) lies in quadrant III. Find the values of the other 5 trig functions of \( \theta \). Show your work clearly. Give exact answers, simplified.

\[
\cos \theta = \frac{1}{\sec \theta} = -\frac{1}{5}
\]

\[
\begin{align*}
\sin \theta &= \tan \theta = -\sqrt{\frac{24}{5}} = -\frac{2\sqrt{5}}{5} \\
\sin \theta &= \tan \theta \cos \theta = -\frac{2\sqrt{5}}{5} \cdot \left(-\frac{1}{5}\right) = \frac{2\sqrt{3}}{5} \\
\cot \theta &= \frac{1}{\tan \theta} = -\frac{1}{2\sqrt{3}} \\
\cos \theta &= -\frac{1}{5} \\
\tan \theta &= \pm \sqrt{24} = \pm 2\sqrt{6} \\
\tan \theta &= -2\sqrt{6} \quad \text{in Q III} \\
\end{align*}
\]

(a) \( \sin \theta = -\frac{2\sqrt{5}}{5} \)  (c) \( \tan \theta = 2\sqrt{6} \)  (e) \( \sec \theta = -5 \)

(b) \( \cos \theta = -\frac{1}{5} \)  (d) \( \cot \theta = \frac{1}{2\sqrt{6}} \)  (f) \( \csc \theta = -\frac{5}{2\sqrt{6}} \)
8. (30 points) Consider the function \( f(x) = 1 - 2 \sin \left( \frac{2}{3} x - \frac{\pi}{6} \right) = 1 - 2 \sin \left[ \frac{2}{3} \left( x - \frac{\pi}{4} \right) \right] \).

(a) Find each of the following.

i. (3 pts) amplitude: \( \left| -2 \right| = 2 \)

\( \frac{2\pi}{\frac{2}{3}} = 3\pi \)

\( \frac{2}{3} x - \frac{\pi}{6} = 0 \)

\( \frac{2}{3} x = \frac{\pi}{6} \)

\( x = \frac{\pi}{6} \cdot \frac{3}{2} = \frac{\pi}{4} \)

ii. (3 pts) period: \( 3\pi \)

iii. (3 pts) phase shift (horizontal translation): \( -\frac{\pi}{4} \)

iv. (3 pts) vertical translation: \( 1 \) up

(b) (12 pts) Sketch the graph \( y = f(x) \). Fill out the whole coordinate grid.

(c) (6 pts) Plot and label (with their ordered pairs) the 5 important points in one period.

\[
\begin{align*}
(f(\frac{\pi}{4})) &= 1 - 2 \sin \left( \frac{2}{3} \cdot \frac{\pi}{4} - \frac{\pi}{6} \right) \\
&= 1 - 2 \sin \left( \frac{\pi}{6} - \frac{\pi}{6} \right) \\
&= 1 - 2 \sin 0 \\
&= 1 - 2 \\
&= -1
\end{align*}
\]

\[
\begin{align*}
(f(\frac{5\pi}{2})) &= 1 - 2 \sin \left( \frac{2}{3} \cdot \frac{5\pi}{2} - \frac{\pi}{6} \right) \\
&= 1 - 2 \sin \left( \frac{10\pi - \pi}{6} \right) \\
&= 1 - 2 \sin \left( \frac{9\pi}{6} \right) \\
&= 1 - 2 \sin \left( \frac{3\pi}{2} \right) \\
&= 1 - 2 \cdot 1 \\
&= -1
\end{align*}
\]

\[
\begin{align*}
(f(\frac{7\pi}{4})) &= 1 - 2 \sin \left( \frac{2}{3} \cdot \frac{7\pi}{4} - \frac{\pi}{6} \right) \\
&= 1 - 2 \sin \left( \frac{14\pi - \pi}{6} \right) \\
&= 1 - 2 \sin \left( \frac{13\pi}{6} \right) \\
&= 1 - 2 \sin \left( \frac{7\pi}{2} \right) \\
&= 1 - 2 \cdot 1 \\
&= -1
\end{align*}
\]

\[
\begin{align*}
(f(\frac{13\pi}{4})) &= 1 - 2 \sin \left( \frac{2}{3} \cdot \frac{13\pi}{4} - \frac{\pi}{6} \right) \\
&= 1 - 2 \sin \left( \frac{26\pi - \pi}{6} \right) \\
&= 1 - 2 \sin \left( \frac{25\pi}{6} \right) \\
&= 1 - 2 \sin \left( \frac{4\pi}{2} \right) \\
&= 1 - 2 \cdot 1 \\
&= -1
\end{align*}
\]