MATH 11022–006
TR 11:00

Trigonometry

Dr. Kracht
Spring 2014

Name: KEY Exam Score: ____ /100

110 pts available

Exam 2: Thursday, March 13, 2014

Show all work. Give exact values, simplified. Use proper notation. NO CALCULATORS.

1. (15 points)

(8) (a) *DERIVE* the Pythagorean Identity involving sine and cosine from the definitions of the trig functions. *Sketch a diagram. Give a reason for each step.*

By the Pythagorean theorem or equation of the circle,

$$x^2 + y^2 = r^2$$

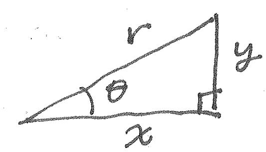
$$\frac{1}{r^2}(x^2 + y^2) = \frac{r^2}{r^2} \quad \text{(algebra)}$$

$$\frac{x^2}{r^2} + \frac{y^2}{r^2} = 1 \quad \text{(algebra)}$$

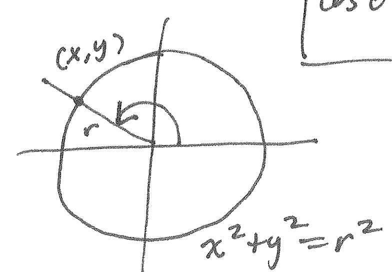
$$\left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2 = 1 \quad \text{(algebra)}$$

$$(\cos\theta)^2 + (\sin\theta)^2 = 1 \quad \text{(def}^{\underline{n}}\text{ of sine, cosine)}$$

$$\sin^2\theta + \cos^2\theta = 1. \ \blacksquare$$



So
$\sin\theta = \frac{y}{r}$
$\cos\theta = \frac{x}{r}$



(7) (b) Use your answer above and other basic identities to *DERIVE* either of the other two Pythagorean Identities. *Give a reason for each step.*

$$\sin^2\theta + \cos^2\theta = 1 \quad \text{(proved in (a))}$$

$$\frac{1}{\cos^2\theta}(\sin^2\theta + \cos^2\theta) = \frac{1}{\cos^2\theta}\cdot 1 \quad \text{(algebra)}$$

$$\frac{\sin^2\theta}{\cos^2\theta} + \frac{\cos^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta} \quad \text{(algebra)}$$

$$\left(\frac{\sin\theta}{\cos\theta}\right)^2 + 1 = \left(\frac{1}{\cos\theta}\right)^2 \quad \text{(algebra)}$$

$$(\tan\theta)^2 + 1 = (\sec\theta)^2 \quad \text{(ratio + reciprocal identities)}$$

$$\tan^2\theta + 1 = \sec^2\theta. \ \blacksquare$$

2. (10 points)

3 (a) State the Sum formula for sine.

$\sin(u+v) =$ $\sin u \cos v + \cos u \sin v$

7 (b) Use your answer above, basic identities, and algebra to *DERIVE* the Difference formula for sine, that is, the familiar formula for $\sin(u-v)$. *Give a reason for each step.*

$$\begin{aligned}\sin(u-v) &= \sin(u+(-v)) && \text{alg}\\ &= \sin u \cos(-v) + \cos u \sin(-v) && \text{sum form}\\ &= \sin u \cos v + \cos u(-\sin v) && \text{odd/even}\\ &= \sin u \cos v - \cos u \sin v && \text{alg.}\end{aligned}$$

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3. (10 points) Use Sum or Difference Formulas to evaluate each of the following. *Write out each step.*

(a) Find the exact value, simplified: $\sin\frac{7\pi}{12}$ $= \sin\left(\frac{3\pi}{12}+\frac{4\pi}{12}\right)$

$$\begin{aligned}&= \sin\left(\frac{\pi}{4}+\frac{\pi}{3}\right)\\ &= \sin\frac{\pi}{4}\cos\frac{\pi}{3} + \cos\frac{\pi}{4}\sin\frac{\pi}{3}\\ &= \frac{1}{\sqrt{2}}\cdot\frac{1}{2} + \frac{1}{\sqrt{2}}\cdot\frac{\sqrt{3}}{2}\\ &= \frac{1+\sqrt{3}}{2\sqrt{2}}\end{aligned}$$

(b) Find the exact value : $\cos 94^\circ \cos 4^\circ + \sin 94^\circ \sin 4^\circ$ $= \cos(94^\circ - 4^\circ)$

$$\begin{aligned}&= \cos 90^\circ\\ &= 0\end{aligned}$$

4. (10 points) Use the following Double Angle Formula for cosine to *DERIVE* another Double Angle Formula involving only the sine function. *Provide a reason for each step.*

$\cos 2x = \cos^2 x - \sin^2 x$

Solution:

$\cos 2x = \cos^2 x - \sin^2 x$ given

$= (1 - \sin^2 x) - \sin^2 x$ Pythag Identity $\sin^2 x + \cos^2 x = 1$

$= 1 - \sin^2 x - \sin^2 x$ alg

$= 1 - 2\sin^2 x$ alg.

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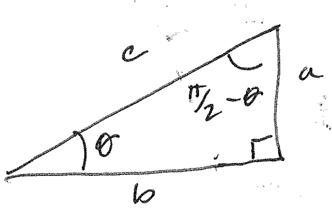
5. (10 points) *DERIVE* the Cofunction Identity for $\sec\left(\frac{\pi}{2} - \theta\right)$ from the right triangle definitions of the trig functions. *Sketch a diagram. Explain your reasoning in full sentences.*



If θ is an acute angle, then $\frac{\pi}{2} - \theta$ is another acute angle, forming a right triangle together with the right angle $\pi/2$. Labelling as shown, we have

$\sec\left(\frac{\pi}{2} - \theta\right) = \frac{c}{a} = \csc\theta.$

6. (10 points) Verify the following identity. Give a reason for each step. Do not skip steps.

$$\tan^2\theta\,(1+\cot^2\theta) = \frac{1}{1-\sin^2\theta}$$

Proof:

LHS $= \tan^2\theta(1+\cot^2\theta) = \tan^2\theta + \tan^2\theta\cot^2\theta$ alg

$= \frac{\sin^2\theta}{\cos^2\theta} + \frac{\sin^2\theta}{\cos^2\theta}\cdot\frac{\cos^2\theta}{\sin^2\theta}$ ratio

$= \frac{\sin^2\theta}{\cos^2\theta} + \frac{\cos^2\theta}{\cos^2\theta}$ alg

$= \frac{\sin^2\theta+\cos^2\theta}{\cos^2\theta}$ alg

$= \frac{1}{\cos^2\theta}$ Pythag Ident

$= \frac{1}{1-\sin^2\theta}$ Pythag.

$=$ RHS. ∎

better

$\tan^2\theta(1+\cot^2\theta) = \tan^2\theta\csc^2\theta$

$= \frac{1}{\cos^2\theta}$

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7. (15 points) Suppose $\sec\theta = -5$ and the terminal side of θ lies in quadrant III. Find the values of the other 5 trig functions of θ. *Show your work clearly. Give exact answers, simplified.*

$\cos\theta = \frac{1}{\sec\theta} = \frac{-1}{5}$

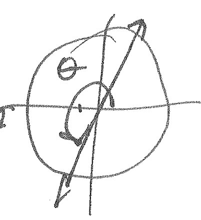
$1+\tan^2\theta = \sec^2\theta$

$\tan^2\theta = (-5)^2 - 1$

$\tan^2\theta = 24$

$\tan\theta = \pm\sqrt{24} = \pm 2\sqrt{6}$

$\tan\theta = +2\sqrt{6}$ in QIII



$\frac{\sin\theta}{\cos\theta} = \tan\theta$

$\sin\theta = \tan\theta\cos\theta$

$= 2\sqrt{6}\cdot\left(-\frac{1}{5}\right)$

$= -\frac{2\sqrt{3}}{5}$

$\cot\theta = \frac{1}{\tan\theta} = \frac{1}{2\sqrt{6}}$

(a) $\sin\theta = -\frac{2\sqrt{6}}{5}$

(b) $\cos\theta = -\frac{1}{5}$

(c) $\tan\theta = 2\sqrt{6}$

(d) $\cot\theta = \frac{1}{2\sqrt{6}}$

(e) $\sec\theta = -5$

(f) $\csc\theta = -\frac{5}{2\sqrt{6}}$

8. (30 points) Consider the function $f(x) = 1 - 2\sin\left(\frac{2}{3}x - \frac{\pi}{6}\right) = 1 - 2\sin\left[\frac{2}{3}\left(x - \frac{\pi}{4}\right)\right]$.

(a) Find each of the following.

i. *(3 pts)* amplitude: $|-2| = 2$

ii. *(3 pts)* period: 3π

iii. *(3 pts)* phase shift (horizontal translation): $\frac{\pi}{4}$

iv. *(3 pts)* vertical translation: 1 up

$$\frac{2\pi}{2/3} = 3\pi$$

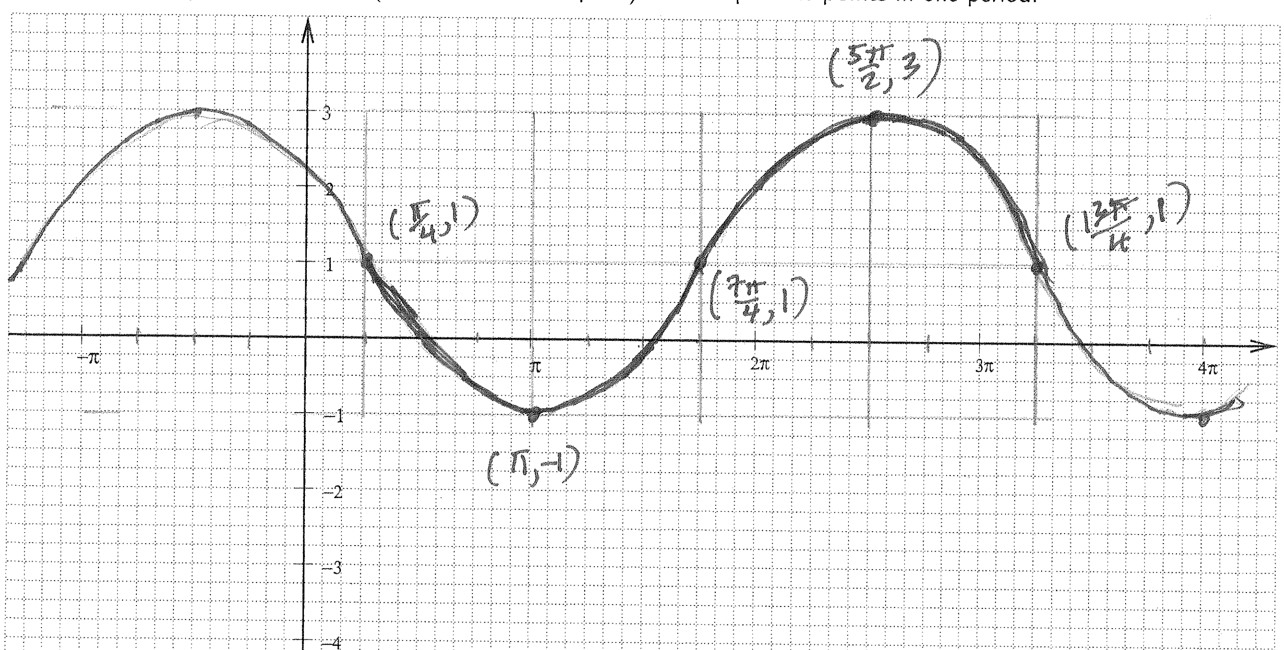
$$\frac{2}{3}x - \frac{\pi}{6} = 0$$

$$\frac{2}{3}x = \frac{\pi}{6}$$

$$x = \frac{\pi}{6} \cdot \frac{3}{2} = \frac{\pi}{4}$$

(b) *(12 pts)* Sketch the graph $y = f(x)$. *Fill out the whole coordinate grid.*

(c) *(6 pts)* Plot and label (with their ordered pairs) the 5 important points in one period.



$f(\frac{\pi}{4}) = 1 - 2\sin\left(\frac{2}{3}\cdot\frac{\pi}{4} - \frac{\pi}{6}\right)$
$= 1 - 2\sin\left(\frac{\pi}{6} - \frac{\pi}{6}\right)$
$= 1 - 2\sin 0$
$= 1 - 2 \cdot 0$
$= 1$

$f(\frac{7\pi}{4}) = 1 - 2\sin\left(\frac{2}{3}\cdot\frac{7\pi}{4} - \frac{\pi}{6}\right)$
$= 1 - 2\sin\left(\frac{7\pi}{6} - \frac{\pi}{6}\right)$
$= 1 - 2\sin(\pi)$
$= 1 - 2 \cdot 0$
$= 1$

$f(\frac{13\pi}{4}) = 1 - 2\sin\left(\frac{2}{3}\cdot\frac{13\pi}{4} - \frac{\pi}{6}\right)$
$= 1 - 2\sin\left(\frac{13\pi - \pi}{6}\right)$
$= 1 - 2\sin(2\pi)$
$= 1 - 2 \cdot 0$
$= 1$

$f(\pi) = 1 - 2\sin\left(\frac{2}{3}\pi - \frac{\pi}{6}\right)$
$= 1 - 2\sin\left(\frac{4\pi - \pi}{6}\right)$
$= 1 - 2\sin\left(\frac{\pi}{2}\right)$
$= 1 - 2 \cdot 1$
$= -1$

$f(\frac{5\pi}{2}) = 1 - 2\sin\left(\frac{2}{3}\cdot\frac{5\pi}{2} - \frac{\pi}{6}\right)$
$= 1 - 2\sin\left(\frac{10\pi - \pi}{6}\right)$
$= 1 - 2\sin\left(\frac{3\pi}{2}\right)$
$= 1 - 2(-1)$
$= 3$