Some Subgroups of Finite Algebra Groups: Normalizers of Algebra Subgroups

(preliminary report)

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Definition

Let $F$ be a field of characteristic $p$ and order $q$. Let $J$ be a finite-dimensional, nilpotent, associative $F$-algebra. Define $G = 1 + J$ (formally). Then $G$ is a finite $p$-group. Groups of this form are called $F$-algebra groups. We will assume this notation throughout.

Example

Unipotent upper-triangular matrices over $F$

Theorem (Isaacs (1995))

All irreducible characters of algebra groups have $q$-power degree.
Algebra subgroups and strong subgroups

- Subgroups: $1 + X$ where $X \subseteq J$ is closed under the operation $(x, y) \mapsto x + y + xy$
- $X$ need not be an algebra.

Definitions

- If $L$ is a subalgebra of $J$, then $1 + L$ is an algebra subgroup of $G = 1 + J$.
- If $H \leq G$ such that $|H \cap K|$ is a $q$-power for all algebra subgroups $K$ of $G$, then $H$ is a strong subgroup of $G$.

Fact

Algebra subgroups are strong.
Strong subgroups as point stabilizers

Theorem (Isaacs (1995))

Under certain conditions, character stabilizers are strong.

- If $J^p = 0$, $N \trianglelefteq G$ is an ideal subgroup, and $\theta \in \text{Irr}(N)$, then the stabilizer in $G$ of $\theta$ is strong.
- If $N \trianglelefteq G$ is an ideal subgroup, and $\lambda$ is a linear character of $N$, then the stabilizer in $G$ of $\lambda$ is strong.

Question

Are normalizers of algebra subgroups strong?
When $J^p = 0$: The Exponential Map

- If $J^p = 0$, define $\exp: J \to 1 + J$ and $\log: 1 + J \to J$ by the usual power series.

Definitions

- For $x \in J$ and $\alpha \in F$, define $(1 + x)^\alpha = \exp(\alpha \log(1 + x))$.
- We define an $F$-exponent subgroup to be a subgroup of the following form:
  - $(1 + x)^F = \{(1 + x)^\alpha | \alpha \in F\}$
  - or equivalently
  - $\exp(F \hat{x}) = \{\exp(\alpha \hat{x}) | \alpha \in F\}$

Fact

$F$-exponent subgroups are strong.
Exponentially closed subgroups

**Definition**

The subgroup $H$ is said to be **exponentially closed** if $\exp(Fx) \subseteq H$ whenever $\exp(x) \in H$.

- Also called **partitioned** subgroups

**Fact**

*Exponentially closed subgroups are strong.*
Suppose $J^p = 0$. If $H$ is an algebra subgroup of $G = 1 + J$, then $N_G(H)$ is exponentially closed (hence strong).

**Sketch of proof.**

- Let $H = \exp(L)$ be an algebra subgroup of $G$.
- To show: if $\exp(x) \in N_G(H)$, then $\exp(\alpha x) \in N_G(H)$ for all $\alpha \in F$.
- Key: $N_G(H) = \exp N_J(L)$ where $N_J(L) = \{ x : [L, x] \subseteq L \}$.
  - $y^{(\exp x)^{-1}} = y^{\exp(-x)} = \exp(\text{ad } x)(y)$.
  - $\exp(x) \in N_G(H) \iff \exp(\text{ad } x)$ stabilizes $L$
  - $\iff \text{ad } x$ stabilizes $L$
A generalization of the exponential map.

**Goal**

*To find an analog of \(\exp\) that works if \(x^p \neq 0\).*

- From the study of Witt rings and \(p\)-adic analysis:

**Definition**

Fix a prime \(p\) and a nilpotent algebra \(X\) over a field of characteristic 0. The *Artin-Hasse exponential function*, \(\text{hexp}: X \to 1 + X\), is defined by

\[
\text{hexp}(x) = \exp \left( x + \frac{x^p}{p} + \frac{x^{p^2}}{p^2} + \frac{x^{p^3}}{p^3} + \cdots \right) \\
= \exp (x) \exp \left( \frac{x^p}{p} \right) \exp \left( \frac{x^{p^2}}{p^2} \right) \exp \left( \frac{x^{p^3}}{p^3} \right) \cdots
\]
Another formula for \( \text{hexp} \)

**Miracle**

*The coefficients in \( \text{hexp}(x) \) are \( p \)-integral.*

**Sketch of proof.**

\[ \text{hexp}(x) = \sum \frac{| \bigcup \text{Syl}_p(S_n) |}{n!} x^n \]

- Frobenius: The highest power of \( p \) that divides \( n! = |S_n| \) also divides \( | \bigcup \text{Syl}_p(S_n) | \).

- \( \text{hexp} \) makes sense in characteristic \( p \)
hexp lacks some nice properties of exp

- Suppose $xy = yx$. Then $\exp(x) \exp(y) = \exp(x + y)$.
- Does $\hexp(x) \hexp(y) = \hexp(x + y)$?
- Not usually:
  $$(\exp \left( x + \frac{x^p}{p} + \frac{x^p^2}{p^2} + \cdots \right)) \left( \exp \left( y + \frac{y^p}{p} + \frac{y^p^2}{p^2} + \cdots \right) \right) \neq \left( \exp \left( (x + y) + \frac{(x+y)^p}{p} + \frac{(x+y)^p^2}{p^2} + \cdots \right) \right)$$
- We have

  $$\hexp(x) \hexp(y) = \hexp(s_1) \hexp(s_p) \hexp(s_p^2) \cdots$$

  where $s_1 = x + y$

  $$s_p = \frac{x^p + y^p - (x + y)^p}{p}$$

  and the remaining polynomials $s_{p^n}$ can be shown to have $p$-integral coefficients.
Hexponent and hexponentially closed subgroups

**Definitions**

- A **hexponent subgroup** is defined to be a subgroup of the following form: \( \text{hexp}(Fx) \text{hexp}(Fx^p) \text{hexp}(Fx^{p^2}) \cdot \cdots \).
- A subgroup \( H \) is said to be **hexponentially closed** if \( \text{hexp}(\gamma x) \in H \) for all \( \gamma \in F \) whenever \( \text{hexp}(x) \in H \).

**Facts**

- **Hexponent and hexponentially closed subgroups are strong.**
- **Hexponent subgroups are not necessarily hexponentially closed.**
- **If** \( J^{2p-1} = 0 \), **then** \( \text{hexp}(Fx) \text{hexp}(Fx^p) \) **is hexponentially closed.**
Question

Can we use the hexponential map to show normalizers of algebra subgroups are strong?

Answer

Only if $J^{p+1} = 0$.

Theorem

Let $H$ be an algebra subgroup of $G = 1 + J$.

- If $J^{p+1} = 0$, then $N_G(H)$ is hexponentially closed (hence strong).
- If $J^{p+1} \neq 0$, then examples exist for which $|N_G(H)| = p \cdot q^a$, and so $N_G(H)$ need not be strong.
Sketch of proof

- We find a function $\text{had}$ analogous to $\text{ad}$ so that $\text{hexp}(x) \in N_G(H)$
  \[ \iff \text{hexp}(\text{had} x) \text{ stabilizes } L \iff \text{had} x \text{ stabilizes } L \]

- $\text{had} x = \text{ad} x + \delta_x$ where $\delta_x$ is not linear

- If $J^{2p-1} = 0$, $\delta_x = \frac{L_x^p - R_x^p - (L_x - R_x)^p}{p}$, where $L_x, R_x$ are left and right multiplication by $x$, respectively.

- $\delta_x(y) \in J^{p+1}$

- $J^{p+1} = 0 \implies$
  - $\text{had} x = \text{ad} x$
  - $\text{had} x$ stabilizes $L \iff \text{had}(\alpha x) \text{ stabilizes } L \text{ for all } \alpha \in F$
  - $N_G(H)$ is hexponentially closed (hence strong)
Suppose $J^{p+1} \neq 0$.

$\text{hexp}(\alpha x) \in N_G(H) \iff \text{had}(\alpha x) = \alpha \text{ad} x + \alpha^p \delta_x$ stabilizes $L$

Now $\text{had} x = \text{ad} x + \delta_x$ stabilizes $L \iff \alpha^p \text{had} x = \alpha^p (\text{ad} x + \delta_x) = \alpha^p \text{ad} x + \alpha^p \delta_x$ stabilizes $L$ for all $\alpha$.

So $\text{had}(\alpha x)$ stabilizes $L \iff (\alpha^p - \alpha) \text{ad} x$ stabilizes $L$

Examples exist (for all $p$) for which this happens $\iff \alpha \in GF(p)$ and so $|N_G(H)| = p \cdot q^a$.

So if $J^{p+1} \neq 0$ and $|F| = q > p$, examples exist for which normalizers of algebra subgroups are not strong.
Other uses for $\text{hexp}$?

- Algebra subgroups are strong, but is the converse true? Are strong subgroups at least isomorphic to algebra subgroups?
- Not necessarily.
- Suppose $x^{p+1} = 0$, but $x^p \neq 0$; $H = \text{hexp}(Fx) \text{hexp}(Fx^p)$.
- $H$ is a strong subgroup of exponent $p^2$.
- Assume there is an isomorphism $H \rightarrow 1 + A$, some $F$-algebra group.
- $\dim_F(A) = 2$
- There is some $u \in A$ with $o(1 + u) = p^2$, so $u, u^2, \ldots, u^p$ are non-zero, linearly independent, a contradiction, if $p$ is odd.
- Similar example exists for $p = 2$. 