A power-series description of strong subgroups of finite algebra groups (preliminary report)

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Definition

Let $F$ be a field of characteristic $p$ and order $q$. Let $J$ be a finite-dimensional, nilpotent, associative $F$-algebra. Define $G = 1 + J$ (formally). Then $G$ is a finite $p$-group. Groups of this form are called $F$-algebra groups. *We will assume this notation throughout.*

Example

Unipotent upper-triangular matrices over $F$

Theorem (Isaacs (1995))

*All irreducible characters of algebra groups have $q$-power degree.*
Subgroups: $1 + X$ where $X \subseteq J$ is closed under the operation $(x, y) \mapsto x + y + xy$

$X$ need not be an algebra.

**Definitions**

- If $L$ is a subalgebra of $J$, then $1 + L$ is an algebra subgroup of $G = 1 + J$.

- If $H \leq G$ such that $|H \cap K|$ is a $q$-power for all algebra subgroups $K$ of $G$, then $H$ is a strong subgroup of $G$.

**Fact**

*Algebra subgroups are strong.*
The subgroup \[ H = \left\{ \begin{pmatrix} 1 & \alpha & \binom{\alpha}{2} \\ 0 & 1 & \alpha \\ 0 & 0 & 1 \end{pmatrix} \middle| \alpha \in F \right\} \]
is a strong subgroup (but not an algebra subgroup) of the algebra group of unipotent $3 \times 3$ upper-triangular matrices over $F$.

(Here \( \binom{\alpha}{2} = \frac{\alpha(\alpha-1)}{2} \) is the generalized binomial coefficient.)
Why study strong subgroups?

Strong subgroups play an important role in the following:

  (Strong subgroups are central to the proof of main theorem mentioned above.)

  (Similar results for character degrees of certain $p$-subgroups of the symplectic, orthogonal, and unitary groups (for $p$ odd). The key to Previtali’s proof is that certain sections of the groups are strong.)

  (Gets stronger and more general versions of the above results using the fact that certain fixed-point subgroups are strong.)
Strong subgroups as point stabilizers

**Theorem (Isaacs 1995)**

Under certain conditions, character stabilizers are strong.

- If $J^p = 0$, $N \trianglelefteq G$ is an ideal subgroup, and $\theta \in \text{Irr}(N)$, then the stabilizer in $G$ of $\theta$ is strong.
- If $N \trianglelefteq G$ is an ideal subgroup, and $\lambda$ is a linear character of $N$, then the stabilizer in $G$ of $\lambda$ is strong.

**Theorem (2010)**

Let $H$ be an algebra subgroup of $G = 1 + J$.

- If $J^{p+1} = 0$, then $N_G(H)$ is strong.
- If $J^{p+1} \neq 0$, then examples exist for which $|N_G(H)| = p \cdot q^a$, and so $N_G(H)$ need not be strong.
Example when $J^p = 0$: $F$-exponent subgroups

- If $J^p = 0$, define $\exp: J \to 1 + J$ and $\log: 1 + J \to J$ by the usual power series.

Definitions

- For $x \in J$ and $\alpha \in F$, define $(1 + x)^\alpha = \exp(\alpha \log(1 + x))$.
- We define an $F$-exponent subgroup to be a subgroup of the following form:
  - $(1 + x)^F = \{(1 + x)^\alpha | \alpha \in F\}$
  - or equivalently
  - $\exp(F\hat{x}) = \{\exp(\alpha\hat{x}) | \alpha \in F\}$

Fact

$F$-exponent subgroups are strong.
Intersections of strong subgroups need not be strong.

Next, we construct an example to show that the collection of strong subgroups is not closed under intersection.

Suppose

- $q > p > 2$
- $x \in J$ with $x^p = 0$ but $x^2 \neq 0$
- $\epsilon : F \rightarrow F$ is a nonzero additive map with $\epsilon(1) = 0$
- $H = \{(1 + x)^\alpha (1 + x^2)^{\epsilon(\alpha)} \mid \alpha \in F\}$

Then

- $H$ is an abelian subgroup of $G$ of order $q$
- $1 + x \in H$ since $\epsilon(1) = 0$
- $H$ is strong, as we will show.
\[ H = \left\{ (1 + x)^\alpha (1 + x^2)^{\epsilon(\alpha)} \mid \alpha \in F \right\} \]

To show \( H \) is strong (i.e., \( |H \cap K| \) is a \( q \)-power \( \forall \) algebra subgroups \( K \)):

- Let \( A \subseteq J \), subalgebra, such that \( H \cap (1 + A) \neq 1 \).
- Then \( \exists \alpha_0 \in F, \alpha_0 \neq 0 \) such that \( (1 + x)^{\alpha_0} (1 + x^2)^{\epsilon(\alpha_0)} \in 1 + A \).
- I.e., \( (1 + \alpha_0 x + \cdots)(1 + \epsilon(\alpha_0)x^2 + \cdots) \in 1 + A \)
- So \( \alpha_0 x + \cdots \in A \)
- This generates the algebra \( xF[x] \), so \( xF[x] \subseteq A \).
- So \( H \cap (1 + A) = H \).
- Therefore, \( \forall \) algebra subgroups \( K \) of \( G \), \( H \cap K = 1 \) or \( H \cap K = H \)
- Conclude: \( H \) is a strong subgroup of \( G \).
Now consider the $F$-exponent group $(1 + x)^F = \{ (1 + x)\alpha \mid \alpha \in F \}$:

- A strong subgroup of $G$ of order $q$.
- Distinct from $H$ since $\epsilon$ is not the zero map.

Then we have

- $1 + x \in (1 + x)^F \cap H$.
- So $1 < \left| (1 + x)^F \cap H \right| < q$.
- Thus, $(1 + x)^F \cap H$ is not strong.
- Conclude: the intersection of strong subgroups need not be strong.
Ideal frames

Definition

Let \( J \) be a nilpotent \( F \)-algebra with \( \dim_F(J) = n \). An ideal frame of \( J \) is a basis \( \{ v_1, \ldots, v_n \} \) of \( J \) satisfying

\[
v_i J, Jv_i \subseteq \text{Span} \{ v_{i+1}, \ldots, v_n \}
\]

for all \( i = 1, \ldots, n \).

Such bases always exist. For example,

- Refine the chain \( J \supset J^2 \supset \cdots \supset J^{m-1} \supset J^m = 0 \) to a maximal flag \( J = V_1 \supset V_2 \supset \cdots \supset V_{n-1} \supset V_n \supset 0 \).
- Choose \( v_i \in V_i \setminus V_{i+1} \) for all \( i = 1, \ldots, n \).
- Then \( \{ v_1, \ldots, v_n \} \) is an ideal frame of \( J \).
Stringent power series

**Definition**

We call a power series $J \to 1 + J$ **stringent** if it is of the form

$$x \mapsto 1 + x + \alpha_2 x^2 + \alpha_3 x^3 + \cdots$$

where $\alpha_2, \alpha_3, \ldots \in F$.

**Example**

$$\exp(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

is stringent.

Let $S : J \to 1 + J$ be a stringent power series.

- Then $S(\alpha x)^{-1} = 1 - \alpha x + \cdots$.
- Define $S(Fx) = \{S(\alpha x) | \alpha \in F\}$.
- $S(Fx)$ is a subset, but not necessarily a subgroup, of $G$. 
Expressing elements of $G$ in terms of $f(Fx)$

**Lemma**

For a finite $F$-algebra group $G = 1 + J$, suppose

- $\{v_1, \ldots, v_n\}$ is an ideal frame of $J$ where $n = \dim_F(J)$
- $S : J \rightarrow 1 + J$ is a stringent power series
- $V_i = \text{Span}\{v_i, \ldots, v_n\}$

Then

- $1 + V_i = S(Fv_i)(1 + V_{i+1})$ for all $i = 1, \ldots, n - 1$;
- Every element of $G$ has a unique representation of the form $S(\alpha_1 v_1)S(\alpha_2 v_2) \cdots S(\alpha_n v_n)$ where $\alpha_1, \ldots, \alpha_n \in F$.
- In particular, $G = \prod_{i=1}^n S(Fv_i)$. 
Main Theorem

For a finite $F$-algebra group $G = 1 + J$, suppose

- $\{v_1, \ldots, v_n\}$ is an ideal frame of $J$ where $n = \dim_F(J)$
- $S : J \to 1 + J$ is a stringent power series
- $V_i = \text{Span}\{v_i, \ldots, v_n\}$.

If $H$ is a strong subgroup of $G$, then there exist

- a partition $I \cup \hat{I} = \{1, \ldots, n\}$
- functions $\epsilon_{ij} : F \to F$ for all $i \in I$ and $j \in \hat{I}$ with $j > i$

such that, $\forall \alpha \in F$, $h_i(\alpha) = S(\alpha v_i) \prod_{j \in \hat{I}} S(\epsilon_{ij}(\alpha) v_j)$ is an element of $H$.

Moreover, every $h \in H$ has a unique representation of the form $h = \prod_{i \in I} h_i(\alpha_i)$ where $\alpha_i \in F$. 
Example

Suppose $H$ is a strong subgroup of $G + 1 + J$ where $J^p = 0$, $\dim_F(J) = 6$, $S(\alpha x) = (1 + x)^\alpha$, and the partition given by the theorem turns out to be $I = \{1, 3, 5\}$ and $\hat{I} = \{2, 4, 6\}$.

Then there are functions $\epsilon_{1,2}, \epsilon_{1,4}, \epsilon_{1,6}, \epsilon_{3,4}, \epsilon_{3,6}, \epsilon_{5,6} : F \to F$ so that

\[
\begin{align*}
    h_1(\alpha_1) &= (1 + v_1)^{\alpha_1} (1 + v_2)^{\epsilon_{1,2}(\alpha_1)} (1 + v_4)^{\epsilon_{1,4}(\alpha_1)} (1 + v_6)^{\epsilon_{1,6}(\alpha_1)} \\
    h_3(\alpha_3) &= (1 + v_3)^{\alpha_3} (1 + v_4)^{\epsilon_{3,4}(\alpha_3)} (1 + v_6)^{\epsilon_{3,6}(\alpha_3)} \\
    h_5(\alpha_5) &= (1 + v_5)^{\alpha_5} (1 + v_6)^{\epsilon_{5,6}(\alpha_5)}
\end{align*}
\]

are all elements of $H$.

Moreover, every $h \in H$ is of the form $h = h_1(\alpha_1) h_3(\alpha_3) h_5(\alpha_5)$ for unique $\alpha_1, \alpha_3, \alpha_5 \in F$. 