Development of Students' Understanding of Cosets, Normality, and Quotient Groups

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Abstract

This paper reports on a continuing development of an abstract algebra course that was first implemented in the summer of 1990. This course was designed to address discrepancies between how students learn and how they were traditionally being taught. Based on results from the first implementation, pedagogical changes were made, including increased computer programming activities and other exercises which were designed to give the students the opportunity to build experience to draw on in order to construct understanding of the topics in class. A second experimental course was run. To assess the impact of these methods, and to continue to better understand how students go about learning, test results from the latter class and interviews with students from both experimental courses and a lecture-based class were analyzed. The students in the second experimental course demonstrated a deep understanding of the title concepts, especially cosets and normality. We discuss the details of the revised experimental course; the epistemological theory behind its design; and the framework used to analyze the results. We demonstrate through examples from interviews and test results the applicability of this analysis to the data, and the strides made by the students in comparison with the students from the lecture-based course, and with the students from the first experimental course. We hope to illustrate difficulties students face in learning abstract algebra, and to discuss instructional strategies to help students overcome these difficulties.

Contents

1 Introduction

This paper reports on a study of abstract algebra students' understanding of cosets, normality and quotient groups. The study was carried out according to a very specific methodology for research and curriculum development designed by the members of the Research in Undergraduate Mathematics Education Community, or *RUMEC*, for the purpose of studying and improving the learning of collegiate mathematics.

We find in this study that our theoretical perspective is useful for describing mental constructions that students might make in learning cosets, normality and quotients. The instructional treatments that were designed to get students to make these constructions and which were based on a general pedagogical strategy (the ACE Teaching Cycle, described below) appear to be effective in helping students to make substantial progress towards understanding the three concepts.

1.1 Overall framework

Our framework for conducting research and curriculum development has three components: a theoretical analysis, an instructional treatment, and empirical data. These components form a three-step cycle which guides our approach. We will describe our framework only briefly; for more information, the reader is referred to Asiala et al. (1996)

The first step in our framework is to make an initial theoretical analysis of the epistemology of the concept of interest. The purpose of this theoretical analysis is to propose a *genetic decomposition* which is a model of cognition: that is, a description of specific mental constructions that a learner might make in order to develop her or his understanding of the concept. These constructions are called actions, processes, objects, and schemas, so that the theory is sometimes called the *APOS* Theory.

An *action* is a transformation of mathematical objects that is performed by an individual according to some explicit algorithm and is seen by the subject as externally driven. When the individual reflects on the action and constructs an internal operation that performs essentially the same transformation then we say that the action has been *interiorized* to a *process*. When it becomes necessary to perform actions on a process, the subject must *encapsulate* the process as a totality to create an *object*. In many mathematical operations, it is necessary to *de-encapsulate* an object and work with the process from which it came. A *schema* is a coherent collection of processes, objects and other schemas that is invoked to deal with a mathematical problem situation. A schema can be *thematized* to become another kind of object; a thematized schema can also be unpacked to access the underlying components of the schema.

Returning to our overall framework, we note that the initial analysis is based primarily on the researchers' understanding of the concept, on their experiences as learners and teachers of the concept, and on other published research. The resulting *genetic decomposition* for the concept forms the basis for the second step which is the design and implementation of instruction that is intended to help students make the proposed mental constructions. The pedagogical method that drives this instructional treatment is referred to as the *ACE* teaching cycle (Activities, Class discussion, and Exercises); the main strategies of this method include having students construct mathematical ideas on the computer using a mathematical programming language, and having them work in cooperative learning groups for problem solving and discussion of the results of the computer activities.

Implementing the instruction provides an opportunity for gathering data, which is the third step in our framework. There are two ways in which the data are related to the theoretical analysis. First, the theoretical analysis directs the analysis of data by asking the question: Did the proposed mental constructions appear to be made by the students? Second, the results of this analysis may lead to revisions in the genetic decomposition.

This three-step cycle is then repeated. The initial theoretical analysis is replaced by the analysis resulting from the previous cycle, the instruction is accordingly revised and implemented again, data is gathered and analyzed leading again to revisions. The repetitions continue for as long as is necessary for the researcher to come to a deeper understanding of how the concept might develop in a student's mind. In each cycle, additional data is gathered on the performance of students on mathematical tasks related to the concept in question. Our analysis of this latter data is expressed in mathematical terms, rather than in terms of what mental constructions might, or might not have been made.

Thus the outcome of this approach is, by nature, two-fold. One result of the research is the deepening of the researcher's understanding of the epistemology of the concept. The second result is the creation of pedagogical strategies which are better aligned with the way we believe that students can come to understand the concept; these improved strategies should thus lead to increased learning by the students. As a consequence, many of the studies conducted by *RUMEC* contain findings both about epistemological issues and about pedagogical issues, as well as the relations between them.

1.2 This study

This paper can be seen as a report on a second iteration of the cycles in our framework as regards cosets, normality, and quotient groups. The first cycle is reported by Dubinsky et al. (1994) which began with an initial epistemological analysis based mainly on the general theoretical framework, the researchers' understanding of the mathematics involved and their many years of experience in teaching abstract algebra. This analysis was applied in a summer institute for high school mathematics teachers taking a short, intensive course in abstract algebra. A pedagogical approach was used that was a first approximation to applying our theoretical perspective. Data was gathered and analyzed resulting in revisions of the epistemological analyses. These revised analyses are taken as the starting point in the present study.

The goals of this present study (and possible continuations) are: to determine to what extent the *APOS* theoretical perspective is useful for understanding the mental constructions made by students learning about cosets, normality and quotient groups; to increase our understanding of how learning about these topics might take place and apply that understanding to instruction; to evaluate the extent to which our instructional treatment leads to students successfully performing mathematical tasks that require an understanding of these concepts; and to develop a base of information which sheds light on the epistemology and pedagogy associated with these topics.

The organization of the paper is to treat each of cosets, normality, and quotient groups separately. After a brief review of the literature on these topics and a description of the students that were studied, the overall pedagogical strategy, and the instruments that were used to gather data, we follow the structure of the framework and begin by providing an epistemological analysis of each of the three concepts. This is followed by a description of the instructional treatment, analyses of the mental constructions the students appeared to be making, and details on their performances with mathematical tasks. Then we return to a reconsideration of the epistemological analyses of the three concepts and make revisions based on analyses of the data. Next, we summarize what our data seems to be telling us about the learning that may have taken place. Finally, we consider some pedagogical recommendations and open questions for future study.

1.3 Comments on the literature

Our literature search revealed two major categories of articles which are related to the current paper in distinct ways: research articles reporting on students' understanding of various topics in abstract algebra and articles related to the teaching of abstract algebra. Additionally, we will mention other, smaller categories of papers. These include papers which focus on teaching strategies in abstract algebra which are explicitly linked to research in learning and historical articles dealing with group theory. Each of these categories is surveyed below.

As noted in section 1.2, this paper builds directly upon an earlier study by Dubinsky, Dautermann, Leron and Zazkis (1994) where the authors conducted an initial investigation into the conceptual development of a number of topics in abstract algebra, including cosets, normality, and quotient groups. Another follow-up study examining the mental constructions students make in developing understanding of binary operations, groups and subgroups is reported in Brown, Devries, Dubinsky, and Thomas (1997). Related work specifically on symmetry groups and the role of visualization in developing such understanding has been reported in Zazkis and Dubinsky (1996) and in Zazkis, Dubinsky, and Dautermann (1996). The development of students' conception of group isomorphism has been studied by Leron, Hazzan and Zazkis (1995).

Moving away from individual topics, a study by Hart (1994) analyzed the proof-writing performance of students using elementary group theory as a subject domain, although none of the proofs studied involved cosets, normality, or quotients. Selden and Selden (1987) classified errors and misconceptions in college-level theorem proving. Their analysis suggests that many errors result not only from the complexity of the subject matter domain, but can also be due to students' weak mathematical backgrounds.

A variety of teaching innovations in abstract algebra courses have been developed to deal with the perceived ineffectiveness of the lecture-listen format for a large number of students. Freedman (1983) quotes Paul Halmos as writing: "A good lecture is usually systematic, complete, precise-and dull; it is a bad teaching instrument" and E. E. Moise: "It is simplistic to suppose that people remember what they are told and understand the things that are explained to them clearly." Freedman goes on to describe a method of teaching an abstract algebra course incorporating a student seminar in which the students share with their instructor the responsibility of selecting, preparing, and teaching the course material. The essence of this method, as of many of the other methods surveyed, is to "encourage students to play a more active part in their own education...".

One way to enhance student involvement with mathematical content reported in the literature is to require them to write mathematics. Brown (1990), Czerwinski (1994), Kiltinen and Mansfield (1990) and Leganza (1995) all give examples of specific writing assignments in abstract algebra courses and describe to varying degrees the instructors' and students' responses to such assignments. Although each of the above four articles give writing assignments in abstract algebra courses, these assignments are fundamentally no different than writing assignments one would use in other mathematics courses at the same level.

A second technique for enhancing student involvement with mathematical content that is reported in the literature involves the use of computers. An early example of this was described in 1976 by Gallian where students participated in projects using computer programs written in Fortran to investigate finite groups. More recently, specialized software such as *Exploring Small Groups* (Geissinger, 1989) and CAYLEY (O'Bryan & Sherman, 1992) have been developed and used as computing environments for student experimentation in abstract algebra. Additionally, less specialized software packages such as MATLAB have recently been reported as useful for computing in abstract algebra, see for example Makiw (1996). Hodgson (1995) reports positive results from abstract algebra students writing simple programs in the mathematical programming language **ISETL**.

Despite the apparent interest in making abstract algebra more interesting and meaningful for students, our literature search found only a single research study, Clark, Hemenway, St. John, Tolias and Vakil (in press), which focused on affective issues in abstract algebra. Clearly, more needs to be done.

Most of the teaching innovations described above are completely disconnected from research findings in student learning. The one isolated exception is the constructivist approach to teaching an introductory course in abstract algebra described in Leron and Dubinsky (1995) and also described by Hodgson (1995). In Leron and Dubinsky, the authors give a description of some of the difficulties which students have in abstract algebra and interpret these difficulties in terms of a specific learning theory model of cognitive development. They then provide examples of computer activities which are specifically designed to help foster the mental constructions proposed by the theoretical analysis. The authors of Leron and Dubinsky are active in both research and curriculum development, and have produced an undergraduate text (Dubinsky & Leron, 1994) which includes the major topics in elementary group and ring theory. Included in this text are specific activities designed to help students to construct meaning for the concepts of cosets, normality, and quotient groups. The reader who is interested in seeing more examples of one approach to integrating research findings into the teaching practice is referred to Asiala, Brown, DeVries, et al. (1996).

Finally, several articles in the literature provide a historical context for our research. Kleiner (1986) surveys the long history of the development of the group concept. Nicholson (1993) describes the important historical antecedents of the relationship between abstract groups and quotient groups.

2 Description of this Study

2.1 Subjects

The subjects in this study were undergraduate students at a large midwestern university who had taken or were taking a first course in abstract algebra which was designed for mathematics majors but was not the honors course. The main group of subjects consisted of 31 students who were taking an experimental version of the course during the fall of 1991; a description of the instructional treatment used in this course is given in the next section and in the sections devoted to specific topics. The students were mostly pre-service secondary mathematics teachers, and the instructor is one of the authors of this study.

In addition, there were 20 students who had taken an abstract algebra course taught using lecture-based methods at various times. Specifically, 5 of these students took the course in the fall of 1991, 8 in the spring of 1991, and 3 in the fall of 1990. One student had taken it in the spring of 1990 but was taking a second undergraduate course in abstract algebra at the time the data for this study was collected. There were 3 students who were not asked when they took the course.

It is possible that some of the students in either group were taking other courses that touched on concepts in abstract algebra before or at the time the data was collected. In particular, two of the 31 students in the experimental course had previously taken the standard course.

2.2 Instructional treatments

As mentioned earlier, the overall structure of the experimental course was based on the *ACE* teaching cycle; the instruction utilized both computers and cooperative groups. Students were grouped into permanent teams consisting of three or four members each, and the majority of course work was completed in groups. Material was broken into topical sections, each of which ran for approximately one week. One two-hour class session per week was spent in a computer laboratory, and two one-hour class sessions per week were held in a classroom with no computers. In the computer laboratory, students completed computer activities using the mathematical programming language **ISETL**. In order to stimulate reflection, the computer activities usually dealt with concepts that had not been formally studied in class. These concepts were then discussed in the successive class meetings. To encourage further exploration of the concepts, students were assigned homework to be completed outside of class; both computer exercises and traditional exercises were included in the assignments. The course used a textbook that was written explicitly to support this pedagogical approach, *Learning Abstract Algebra with ISETL*, by Dubinsky and Leron (1994).

There are several references to **ISETL** in this paper. In most cases, the meaning should be clear from the context. A detailed description of the syntax is given in Dautermann (1992). Readers interested in the historical development of **ISETL** are referred to Dubinsky (1995). Specific details on computer tasks used to help students make mental constructions related to cosets, normality and quotient groups are given in the sections in which these individual concepts are considered.

3 Instruments

This paper reports on a portion of a large-scale study of students learning abstract algebra. In the full study, there were a total of five instruments used to gather data: three written examinations in the course and two sets of interviews. All of the instruments were administered only to students in the experimental course except for the second interview which was given to both groups of students. The interviews were conducted by a team consisting of one of the authors of this paper and four research assistants. In the appendix, we list those examination and interview questions that are related to the topics of cosets, normality and quotient groups. We also give, for each question, an indication of what we expected to learn from the responses.

The first two instruments were two examinations given during the semester as part of the course. The students took the first exam in their permanently assigned cooperative groups. They were given unlimited time for this closed book examination and each group turned in one exam for which each member of the group received the same grade. The second exam was also closed book and unlimited time, but the students took it individually and did not communicate with each other during the exam. Each student received two grades for this exam: one was the score on her or his own paper and the other was the average of the scores received by all the members of the student's group.

The final examination for the experimental course was given in a traditional way in a two hour period with each student taking the exam individually and receiving only one grade. The exam had three parts: definitions, true/false questions, and a set of 11 propositions from which the student selected any two to prove.

Our data also includes two sets of interviews covering topics from abstract algebra. Audio-taped interviews with 24 of the 31 students in the experimental course were conducted during the last week of the Fall 1991 semester. A second set of audio-taped interviews were conducted during the following semester with 17 of the 31 students from that course, together with 20 students who had taken a standard course in abstract algebra. Transcripts of all sessions were produced to complement the record of written work which the student completed during the interview. The transcripts were carefully read and analyzed in order to produce a list of mathematical issues that arose during the interviews. Focusing on these issues, we obtained results about the mental constructions that students appear to have made, as well as a general statement on performance, all of which are reported, along with the results of the examinations, in the remainder of this paper.

4 Cosets

We took the analysis of cosets in Dubinsky et al. (1994) as our starting point for the present study of cosets. We revised our instruction to take into account what was learned in that investigation. After implementing this revised pedagogy, we again observed the students.

4.1 Epistemological analysis of cosets

The epistemological analysis which comes out of Dubinsky et al. (1994) is an action / process / object description.

An action conception of coset has to do with forming a coset in a familiar situation where formulas or recipes can be used such as the multiples of 3 in $\mathcal{Z}$ or in $\mathcal{Z}_{18}$. The action conception is not strong enough to handle formation of cosets in more complicated situations such as in S_n beyond the familiar examples of S_3 or S_4, where cosets are not generally representable by formulas or simple recipes.

A process conception of coset will allow an individual to think of the (left) coset of a subgroup by an element by imagining the product of that element with every member of the subgroup — without having to actually form the products. In a process conception of coset, the main thing one thinks about is the formation of the coset.

With an object conception of coset, the individual can think about, name and manipulate a coset without necessarily focusing on how it is formed. Actions can be performed on cosets such as comparing the cardinality of two cosets or counting the number of cosets, both of which arise in the proof of Lagrange's theorem which we will discuss in this section.

There are other actions which can be applied to cosets as objects, such as considering the relations among elements, subgroups and cosets expressed by the various equivalent conditions for normality or the formation of products of cosets. We will consider these actions in the next two sections on normality and quotient groups.

4.2 Instructional strategy specific to cosets

The instructional treatment of cosets in the experimental course takes place in three stages: preparation, followed by a major computer construction, and then a mathematical treatment of the relevant concepts by using specific computer constructions to implement the ideas of coset formation, coset product, normality and quotient groups.

The preparation consists of various computer activities that come early in the semester and which appear to the students as practice with learning the computer language and/or with modular arithmetic. One of these activities comes in the first 2–3 weeks of the course and asks the student to write a computer function which treats the sets $\{0, 1, \ldots, 11\}$, $\{0, 3, 6, 9\}$ as constants, accepts a parameter x and computes the sum mod 12 of x with each element in the second set. The sets are called $\mathcal{Z}_{12}$ and H respectively and the required function is to be given the name "coset". There is no explicit discussion of this example in the text. A week or so later, the students are asked to compute what amounts to right cosets (that name is actually used) of a subgroup of a group (they are free to choose their own examples) and to think about whether such an object is a subgroup.

The major computer construction comes at the end of the students' introductory study of groups and subgroups (about 8 weeks into the course). It is just at the beginning of the section on Lagrange's theorem. The students are asked to write a computer function `PR` which will accept a group and its operation as input and return a generalized binary operation. This operation will accept two inputs and decide amongst the four possibilities for these two inputs to be elements or subsets of the group and form the appropriate one of the products: product of two elements of the group, product of a fixed element of the group with every element in a subset of the group, product of every element in a subset of the group with a fixed element of the group, or the product of all pairs of elements where the first is taken from one subset of the group and the second from another subset. Thus, if the group is called `G` and its operation `o` then after running the following code,

```
oo := PR(G,o);
```

the operation `oo` can be used for products of elements, left cosets, right cosets or product of two cosets.

This is followed by a mathematical treatment of cosets and their formation with formal definitions, lemmas, examples, and culminating in the proof of Lagrange's theorem. During this period, students are writing various computer programs and using them to investigate specific examples and trying to discover various properties of cosets.

The instructional treatment is based on the conjecture that writing `PR` (which the students find to be *extremely* difficult) is the main disequilibrating/re-equilibrating experience that helps them construct cosets as processes and objects. Because students will often insist, during their struggles to construct `PR`, that the syntax of the programming language is too complex and presents them with unhelpful and unnecessary obstacles, it may be useful to see the following, fairly standard version of this program that most students, with varying amounts of prompting, eventually complete.

```
PR := func(G,o);
          return
              func(x,y);
                  if x in G and y in G then
                       return x .o y;
                  elseif x in G and y subset G then
                       return { x .o b :  b in y };
                  elseif x subset G and y in G then
                       return { a .o y :  a in x };
                  elseif x subset G and y subset G then
                       return { a .o b :  a in x, b in y };
                  end;
              end;
      end;
```

This program is a function that creates and returns a function. Such an operation is conceptually (but not syntactically) difficult but the students have had ample opportunity to gain experience with it in various contexts. The function that is to be returned is a straightforward decision tree with a main trunk, four branches and no looping.

4.3 Mental constructions

4.3.1 Test 1, Question 5. S_3 subgroup of S_4.

What we see here is, essentially, completed knowledge on the question asked. Once the subgroup is specified, an action conception of coset consists of listing the elements of each coset.

There is little opportunity here to indicate a process conception. One way might be to express the coset as a set former, although a student might think this was not giving enough information and, indeed, this response did not occur. Picking an appropriate element, say a in S_4 and writing something like $aH =$ the set with its elements listed might indicate at least a beginning of a process conception.

One can indicate an object conception of cosets by going beyond what the question actually required, for example by forming the set of all the cosets. A transition to object might be indicated by doing this, but within the set, giving each coset by its list of elements rather than by some appropriate name.

4.3.2 Test 1, Question 7. Statement and proof of Lagrange's theorem.

In this situation, writing out the elements of a subgroup and one of its cosets as in

$$\{h_1, h_2, \ldots, h_n\}$$

$$\{gh_1, gh_2, \ldots, gh_n\}$$

indicates an action conception that is moving towards a process. This is so because the student is dealing with an arbitrary subgroup and its coset by an arbitrary element. Hence, the actual details cannot be fully written out. Symbols must be selected and ellipses used.

A process conception may be indicated by using an expression like

$$aH = \{ah : h \in H\}$$

In giving an overview of the proof of Lagrange's theorem, there are several situations in which actions must be applied to cosets. This includes determining the number of elements, comparing the cardinality of two cosets and realizing that two cosets must be either identical or disjoint. Talking about these properties indicates an object conception.

The proof of Lagrange's theorem requires a de-encapsulation of the object conception to the process of formation of a coset so that the student can go through all of the details of the various proofs. Doing this explicitly indicates a fairly strong process and object understanding of coset.

4.3.3 Test 2, Question 4(a). Conditions for Normality.

There were four kinds of responses and they can be subjected to an action/process/object analysis.

1. The response of some students did not indicate very much understanding of cosets.

2. An action conception moving towards process may have been indicated by the student only dealing with gH or Hg by writing out all elements, e.g., $\{gh_1, gh_2, \ldots, gh_n\}$.

3. A process conception may have been indicated using a set former for cosets, e.g., $\{gh : h \in H\}$.

4. An object conception may have been indicated by performance of manipulations on cosets in calculations such as the following:

$$\begin{aligned} gHg^{-1} &\subset H \\ gHg^{-1}g &\subset Hg \\ gH &\subset Hg \end{aligned}$$

   In such cases the student might indicate a de-encapsulation explicitly by interspersing in the calculation a set former expression for one or more of the objects H, Hg, gH, gHg^{-1}

4.3.4 Test 2, Question 6. Normal subgroup of S_3 and quotient.

As far as cosets are concerned, the responses on these questions showed the students calculating the cosets of a particular subgroup, either A_3 or the trivial subgroup, $\{(1)\}$ of S_3.

4.3.5 Test 2, Question 7(a). Description of $2\mathcal{Z}/6\mathcal{Z}$.

Again, for cosets, the students only needed to perform appropriate calculations of cosets in the context of a ring.

4.3.6 Proofs on the Final. Normality of image.

Only one student chose this proof. He showed what appeared to be at least a process conception of coset in proving normality by checking that $f(xhx^{-1}) \in f(H)$ for $x \in G, h \in H$.

4.3.7 First interview, Questions 2 and 5. S_4/K.

The responses that the students gave have much more to say about the students' understanding of cosets. We examine them using an action/process/object analysis.

The names of the students have been changed, and the several people who conducted the interviews are all referred to as "I".

action. Almost all of the students appeared to have little or no difficulty with an action concept of coset, but Eli appears to be struggling to move beyond this point. In his written work, he indicates that he is able to take an element, in this case (1 2 3) (we go left to right - 1 goes to 2, 2 goes to 3, and 3 goes to 1) and multiply it with each element of K to obtain the coset $(1\ 2\ 3)K$. He has difficulty in describing this calculation when not actually performing it. Consider, for example, his attempt to explain the condition $pKp^{-1} \subset K$ for each $p \in S_4$.

Eli: K would be a subgroup, and S_4 would be... the group.

I: M-hmm.

Eli: Where every element, um, any element of S_4, okay, and comp... composition in order, you know, um, take any extra element and composition with the whole group... the entire group

I: You just run through the whole group?

Eli: Yeah. Um, just take that and composition again with the inverse of the element in S_4.

I: OK, and then what will have to be true about those compositions?

Eli: Um, it'll have to be... it'll be the reflection of that composition.

Eli's difficulty in bringing it all together to assert that the resulting set is contained in K, and his apparent confusion between operating on K and selecting elements from S_4 may be due to an inability to think about the total process in forming a coset. His comments, such as "... composition with the whole group... " suggest that he may be beginning to do this, but the clear statement, "You just run through the whole group?" comes from the interviewer.

A little later he does somewhat better in describing the coset $(3\ 4)K$ and seems to be able to think about combining (3 4) with each of the four elements of K.

Eli: And (3 4) composition with all those elements. Where, um, actually, um, the two case there would still be four different combinations of all the results.

It appears as though Eli is in transition from an action to process conception of cosets in S_4.

process. In the following excerpt, Kristen seems to show quite clearly that she has formed processes by interiorizing the actions of forming left cosets and right cosets.

I: OK. What I have for you is a subgroup of S_4. OK,which you may or may not recognize that particular one. But the question for you is, how would you go about finding out if the subgroup K is a normal subgroup? How do you figure that out? This is good review for the final.

Kristen: Yeah. K is normal if you take any other element. K is a subgroup of S_4, if you take any other element of S_4, call it s, such that $sK = Ks$.

I: OK. Can you show me what that looks like? Can you show me for example?

Kristen: Well it's true, right?

I: Well, I don't know. We're going to find out.

Kristen: Take (1 2) in S_4 and multiply it by K.

I: OK. (pause) Here's some more paper, if you run out.

Kristen: I'm driving you crazy, aren't I?

I: No, you're doing just fine.

Kristen: (long pause) OK. (1 2) times K is (1 2), (3 4), (1 4 2 3) and (1 3 2 4)

I: OK

Kristen: So K times (1 2) (pause)

I: You're starting off good.

Kristen: (long pause) and K times (1 2) is the set of cycle notations (1 2), (3 4), (1 4 2 3) and (1 3 2 4). So they're equal.

I: OK

Kristen: So that makes. Well if that were true for all s in S_4 then K would be normal.

In the end, Kristen has a sufficiently strong grasp of this process that she is able to think about doing it for all s in S_4.

object. Only a few students had difficulty in seeing cosets as objects. One student who did was Aimee who sees a representative of a coset as its first element, but seems to be unable to discuss manipulating cosets without returning to the process of coset formation.

I: Yeah. And what I'd like to talk about now with this is. . . what, how would you start going about making an operation table for this?

Aimee: I would just take all these first elements. (laughs)

I: And do what?

Aimee: And use them. That's called the representative, well, yeah, the representative element of the coset and uh, so, well, an operation table is gonna have. . . six columns. . . (draws table) 1, 2,. . . 3, 4, 5, 6. . . I can even count, and then. . . 1, 2, 3, 4, 5, 6. I'd say, OK, K is going to be your zero element and then um, (1 2) times K, (1 3). . . Hey! (laughter)

I: Logical first choices, you know.

Aimee: It's nice to know that I'm being logical sometime. . . (2 4) times K. . . those will be the same here. And then to find, like, this one, let's see, then to find, say, uh, (1 3), (1 4), you just take (1 3) times (1 4) and then it'll be... this is the first one you do and this is the second one... uh... one'll go to 3, which goes to 3, 3 goes to one and one goes to 4, so it'll be (1 3 4) times K, which is... Wait! Oh, yuck,it's gonna be, um ... (1 3 4)... check back to our handy-dandy list of sets and cosets or our coset list... $[1, 3, 4]$ which is in (1 2 3) so I can ...

I: OK. And you could form the whole chart if you sat around for awhile.

Aimee: Yeah, if I really wanted to. But I don't.

I: OK, so you just explained one way to...

Aimee: Fill in the chart.

Another indication that the process of forming a coset has been encapsulated to an object is the clarity with which students can see that the elements of the quotient group are constructed by iterating through the original group and, for each element in the group, constructing its coset by the subgroup. Subsumed in this, and accessible if necessary, is the process of forming each individual coset by iterating through the subgroup and forming the appropriate product of two elements.

Again we have a lack of clarity on this point — indicated in these interviews by confusion between the two iterations: through S_4 and through K and their relationship. For example, in the following excerpt, Lynette is definitely multiplying elements in S_4 by elements in K, but she does not have a clear conception of the iterations.

I: Can you compute the operation table for S_4 mod K?

Lynette: Um, probably not. Um.

I: Do you know what S_4 mod K is?

Lynette: The quotient group of S_4 on K

I: Yes.

Lynette: Um,

I: What do you need to go about getting K?

Lynette: You use the coset and then, OK, I don't know, I forget how you do how this but I would look, write out what S_4 is... go ahead and do that, this could take a while, K is, K is the same thing we just had... square root... and then you take one element out of K that says multiply it by these element

I: To get that here you're representing that...

Lynette: Yeah I mean you don't have it

I: ...put your, put your

Lynette: Yeah.

I: That's all the elements of that, and then you've got K written down, and so what do you say to do?

Lynette: You have to do the cosets so um

I: Tell me how do you go about finding one?

Lynette: Yeah, I think you take like this, an element in K and multiply it by everyone in S_4 or backwards. I think it would be that, it would be, I'm going to say every in S_4 by K, because you do like this, could you, or do you, could you know that, that you're taking (1 2) times this, and you write out, (1 2) take that times that

I: OK, so what you are saying is that picking out representative element out of, something out of S_4, and multiplying that,

Lynette: by K.

She chooses the right alternative in the end, but only after a leading question from the interviewer, and without any reason for it. It is almost a guess.

It is possible for this confusion to be fairly minor. Some students corrected themselves as they worked it out and others needed only a small prompt. In the following excerpt, after Duane explains his response to a request to compute the elements of S_4 mod K, the interviewer prompts him to think about what he is doing. As Duane works with the details he seems to realize that he needs to get different objects, whereas the way he is doing it gives him only one object. He then struggles a little before working out that, given his conception of coset formation, he needs to iterate over S_4 and form cosets each time.

I: Now, can you explain to me what you're doing here?

Duane: Um, the quotient group is each element in K operated on S_4 on, on the group S_4.

I: Well, now wait a moment. If you take an element in K and operate it on the group S_4, what will you get?

Duane: Uh, a group or set of elements back.

I: How many?

Duane: The number in S_4.

I: Do you happen to know how much that is?

Duane: Um, four factorial.

I: Right. Which is...

Duane: Four times three is twelve, er, twenty-four.

I: Twenty-four. OK, now if you took, so you get all of S_4 right?

Duane: Mm-hmm.

I: Now, if you took another element and operate it on S_4, what would you get?

Duane: Seems like you'd get S_4 back.

I: Right. (Chuckles)

Duane: So, I'm doing this in reverse.

I: So, what should you do?

Duane: You're taking the elements of S_4 and operating them on K.

I: Right, OK.

Duane: Hmm. (Pause) Well, I know if you take any element in, that any element that's in K and operated on K, you'll get K back.

I: OK.

Duane: But, I'm not sure how to find the other ones.

I: Uh, well, so, so, uh, sometimes you get K right?

Duane: Right.

I: So one of these elements in the quotient group is going to be K.

Duane: Right.

I: OK, how, how would you find another one?

Duane: Um, well you could take any element that's not in K and operate it on K and that would give you another one.

I: OK, why don't you try one?

Duane: Um, one-two-three. So, that's half of the four.

I: OK.

Duane: And...

I: So, which one did you take?

Duane: Um, the cycle (1 2 3).

I: OK, and, and you calculated it's coset?

Duane: Right. Well, I haven't calculated it, but I represented it.

I: I see. Oh, OK.

Duane: I just called it (1 2 3).

I: Uh-huh, uh-huh.

Duane: In order—the only way I could think to find the other two would be to actually calculate it and then pick an element that's not in either K or this coset and that would be another one. And then calculate that.

Most of the students were quite clear about the two iterations and, as appears in the following excerpt, this may indicate that the process of coset formation had, indeed, been encapsulated to an object.

I: Um, how would you go about first finding the elements in S_4 mod K?

Thomas: Um, it's just, it's going to be the set of aK where a is an element of S_4.

I: OK.

Thomas: So I just pull an element out of S_4, like (1 2) and then I'd multiply, well I'd do permutation composition with all of the elements in K and that would form my coset (1 2)K.

I: OK,let's go ahead and find that one.

Thomas: OK. So (1 2) with 1 is just going to be (1 2). (1 2) with the element (1 2)(3 4) That's just going to be the element (3 4). (1 2) with (1 4)(3 2) is. . . 1 goes to 2, 2 goes to 4, which goes to 2, which goes to 3, 4 goes to 1, so it's just the 4-cycle (1 4 2 3) That's all four so it would be that coset there.

I: OK. Now, I'm not going to make you do all of them, here are all the cosets. Um now as we construct this table what I am really interested in is how you take the product of 2 cosets.

Thomas: OK.

I: So let's just write out say the first portion of the table.

Thomas: OK. So that would be K, (1 2)K. (1 3)K,. . .

Once the cosets had been computed, most of the students showed reasonable process conceptions and many displayed explicitly what appeared to be a de-encapsulation of this object back to the process of coset formation. They could go back and forth between a coset as an object and a coset as a set of elements. The following is an example of a student who thinks of cosets as objects when multiplying them but, on request, can go back to thinking about a coset as a set of elements.

I: OK. That is good enough. Can you explain to me how you did the last one that you did?

Jocelyn: $K(1\ 2)$

I: $K(1\ 2)$ is right. You did $K(1\ 3)$ too, times $K(1\ 2)$, how did you get that answer?

Jocelyn: I picked representatives out of each coset. So out of $K(1\ 3)$ I picked the cycle $(1\ 3)$ and out of $K(1\ 2)$ I picked the cycle $(1\ 2)$ and multiplied them.

I: OK. Is that how a coset product is defined?

Jocelyn: It's uh, for subgroups one you can pick representatives and just multiply them and then your answer will be the coset that contains the product.

I: Right, but that is not the original definition of this.

Jocelyn: Why not?

I: Do you remember what the original definition is?

Jocelyn: Uh, I think we had to go through and multiply every single element in the first coset by every single element in the second coset.

4.3.8 Second interview, Question 3. Lagrange's Theorem.

Student responses on the statement of Lagrange's theorem and discussion of its applications did not tell us much about their mental constructions regarding cosets and so our consideration of these points will be deferred to the following section on performance. However, the third part of the question provided us with another picture of student conceptions of cosets through the ways in which they remembered this term, its meaning and its use in the proof of Lagrange's theorem.

We first consider the question of a student remembering or reconstructing her or his conception of coset. The following outcomes occurred.

1. Cosets were not mentioned by either the student or the interviewer.

2. Cosets were not mentioned by the student, the interviewer brought up the idea, but this did not result in any useful response from the student.

3. Cosets were not mentioned by the student, the interviewer brought up the idea, and it became part of the discussion.

4. The student brought up the term coset and it became part of the discussion.

We now consider only the responses of those students which involved cosets (about half of the group) and what their comments suggest about the mental constructions they might have made. Since cosets arise here as a tool in proving Lagrange's theorem, it is reasonable that they appear first as objects. For those students who were able to get into the proof, we can learn something about their de-encapsulation of the object conception, since the process of coset formation must be considered in the details of the proof of Lagrange's theorem.

We saw three kinds of response in these discussions.

1. Some students were unable to talk about cosets in any other terms than objects: they could not de-encapsulate cosets into the process of coset formation. For example, in the following excerpt, the interviewer mentions cosets, but Nelson does not indicate much understanding of what they are.

   I: That'd do it quick, wouldn't it? Um, OK, if I said the word coset, does that bring anything to mind?

   (pause)

   I: OK, coset, the way we define coset is if you take an element, say a, in a group, and you multiply it by everything in the subset, say your subset is S then you get the coset, aS. So is there any way you could use that?

   Nelson: Show that each order of the those cosets is, let's see, uh, using the cosets, the number of elements will produce a. . . if it's a coset. . . if you have a certain. . . the coset also has to be cyclic? Or does it?

   Even when the student is the one to bring up cosets, as in the following excerpt, the student may not appear to know very much about cosets beyond the name.

I: Um, what do you remember about Lagrange's Theorem?

Katelyn: I remember the name.

I: OK.

Katelyn: Did that have to do with cosets or not? I don't remember. I don't, don't know. I remember the name.

Katelyn does not bring up the subject again, but a few minutes later, the interviewer reminds her of it during a discussion of the proof of Lagrange's theorem. Her response does not help very much.

I: You said something when I first asked about Lagrange's Theorem...

Katelyn: Mmm-hmm.

I: ...about cosets.

Katelyn: Right.

I: And that's a good instinct, because the...

Katelyn: OK.

I: ...proof involves cosets.

Katelyn: Oh. Now, if I can remember what to do with cosets we're...in business but I...mmmm
...um...now...that's like...I just remember doing this thing where we'd start to name cosets and then pretty soon we'd be back and this would be the same one as this one.

It is not clear how much the students who reacted in this way understood about cosets. It might be no more than a name from the past, vaguely remembered.

2. In the second kind of response, we see a struggle to reconstruct knowledge of cosets and it appears as though the student is trying to de-encapsulate this object. In the following excerpt, we see the beginning of this struggle. At first, Jeff has little to offer except a remembered fragment relating cosets to normality. A few minutes later, when the interviewer reminds him of the idea, we see some progress.

Jeff: Well, Lagrange's theorem, what I remember, is, uh, has to do with cosets, left and right cosets. If, let me think, I can remember it, um...Something about, let me think. Left, OK. If the left cosets and the right cosets are equal... Ah... I know it has to do with cosets, at the basic... premise of the cosets. If I'm not mistaken, it leads into the normality, from Lagrange's theorem. Um... I remember that the proof is relatively simple, once we understood what the terminology was.

...

I: How about the thought of partitioning? Does that bring anything to mind? Using the cosets.

Jeff: OK. Let me think. OK. It's along the lines... I think now I think it's along the lines of the group can be divided into...

I: What do you mean by dividing a group?

Jeff: Breaking it up... OK. The subgroups. The subgroups of a group will contain the same amount... the same element... the same number of elements in each. So, if we had, like, a group of 24 elements, there might be 6 subgroups of 4, and those... Because I remember doing it with S_4. That's were I got the 24. You partition the group... break up... into it's cosets. I'm right, I'm like...

I: It sounds like it's just out of where you can reach it. But, it's there but you can't quite get a hold.

In the following excerpt, we again see a student who brings up cosets, and appears to only be able to discuss them as objects, but later, in discussing the proof and at the suggestion of the interviewer, she takes up the concept again and seems to be struggling to de-encapsulate the objects to processes. She has some success and even indicates an understanding of the relation between the iteration over the subgroup and the iteration over the original group.

I: OK. Can you tell me what the theorem's even about? What's it deal with?

Kathy: Lagrange's theorem?

I: Um-hmm.

Kathy: Well, Oh! Cosets?

I: Think about it for a minute.

Kathy: I know we did something with cosets.

I: OK. And I think that's in the proof.

Kathy: And, Lagrange's theorem, oh goodness. I remember my... there are short little things you have to remember. little definitions. Oh, goodness. Lagrange's theorem. (laughs) We had to work with cosets, I remember that. And we had to... oh my. I remember like gH equals H, big H, little g, or something.

I: Do you remember using it on any problem where it helped you out or an application?

Kathy: I remember using it, but I can't remember what we were doing. I think that it was when we were working with Lagrange's theorem. So, I mean I can't remember, I remember doing a proof for it, but I couldn't tell you what it was. And I remember writing it down. I could just show it to you in my notes. (laughs)

...

Kathy: We had a little subgroup, H. And we took... Oh, that's not even a coset Kathy. H, H is a subgroup of G. I was thinking an element of G, but that's different than a coset.

I: OK, and you did find cosets, why? What did that do for you?

Kathy: I know that we had... each coset only had one element of G.

...

I: If you compared cosets, could you make a statement?

Kathy: Cosets...

I: About the cosets, if you compared them.

Kathy: Well, if I compared them, I would say something about um the set of cosets is the same as G, the group, because G, all of G's elements are going to be in these cosets only once and they're going to be in G only once. And H, since it's a subset, or subgroup of G...

I: Um-hmm.

Kathy: It's going to generate all of the elements eventually, it's multiplied by...

I: OK.

Kathy: By all of little g's in big G.

3. Finally, we had students who brought up cosets, realized they were used in the proof of Lagrange's theorem and showed solid action, process and object understandings with an ability to coordinate them through interiorization, encapsulation and de-encapsulation. In the following excerpt Eli displays much of this and we see also an example of his going back to an action description to support his process understanding.

Eli: OK, it's ah, let me see. Every element of G is one, so the order of the subgroup is the same as the order of H and the order of the coset.

And what that means. Hmm... (pause)... oh, OK. Since every element belongs to one of the cosets and only one of the cosets and the order of the cosets is H; well, they can only go on one, and that would be the order of all the cosets is the same. Then, if you have to divide it's gonna be the factor; the order of these cosets would be, well the number of the cosets would be the order of the cosets divided by the order of starting group.

I: So, that would be the idea. Can you write that down?

Eli: I'll try to write down more like, I'll give another example like Z_{12}, I should probably pick smaller example. Let see, H is ah; what is the bigger one? Has to contain the identity, H will be a subgroup zero, four, eight; and there will be four subgroups like Lagrange's theorem proves that; like I said the set of all left cosets: zero, four, eight; one, five, nine; two, six, ten; three, seven eleven, and that's it. Now, since I... every element belongs to one coset, just the one coset and the order of all the cosets is the same, here is three, then you have... it has to evenly to divide. So the total number of elements in those x number of cosets is gonna be equal to the number of... Oh, OK, the total number of elements is gonna be the same as the order of G. Since the order of all these is the same is gonna be a factor. Well, I think that was right.

4.3.9 Second interview, Question 4. Center of a group.

Some students were unable to say very much about their understanding of cosets when responding to the task of showing that the center is normal, because they were unable to think about normality, even when prompted to varying degrees, including the definition provided by the interviewer. For those with some understanding of normality, the responses indicated an object conception of cosets and showed how they dealt with the problem of de-encapsulating the object to a useful process. Those students who were able to do this, even when prompting was required, generally succeeded in completing the proof.

We begin with a student who recalls the definition of normality in terms of the equality of two sets (object conception) but does not succeed in de-encapsulation. This suggests that the definition may have been learned mainly as a string of symbols.

Karen: OK. Hmm. I think... I remember from this semester was doing something with inverses, like, uh... oh, I remember showing that $a^{-1}Na = N$?

I: OK, yeah, that's one definition of a normal, what it means for a normal subgroup. Means that that holds.

Karen: All right, so I have to show that that is normal.

I: M-hmm.

Karen: OK... Oh, maybe I start with g, and say that's $a^{-1}ga$, or do I use ga first?

I: Um, I would use just g, the element of the center.

Karen: OK. OK, then, um... then since g is in the center of G,... all right.

I: And then that's gonna be true for the whole set? The whole subgroup?

Karen: Right. So, $a^{-1}C(G)$...

This student was unable to say anything more on the subject.

In the following excerpt, Lorrie recalls her difficulty with encapsulation. She begins with the relation between objects, $gH = Hg$ and indicates some de-encapsulation to the process. When pressed, she appears to be going all the way back to an action where she can "... think of it by trying out some things." Then she points out how the statement $gH = Hg$ confused her and it may be that she was confused because it was, for her at that time, an equality that did not deal with objects. She then goes on to show what definitely appears to be a process understanding and, with prompting, eventually explains the proof.

I: Can you show me that it's actually a normal subgroup?

Lorrie: OK, um, let me think of, there's like, four different definitions for a normal subgroup.

I: Uh-huh. Which one do you usually think of?

Lorrie: Uh, the first one that comes to mind would be this one.

I: OK, and you wrote $gH = Hg$. That's the first definition that comes to your mind. Can you explain what that says?

Lorrie: It means, um... g is an element of the group...

I: Uh-huh.

Lorrie: ... and H is the subgroup.

I: OK.

Lorrie: So every element of... the group acting on the subgroup would equal... the opposite of that.

I: How do you actually do that, when you say acting on the subgroup, what do you really do?

Lorrie: Um, the cosets.

I: Oh, okay. All right, so how would we use this definition maybe to show that this is a normal subgroup?

Lorrie: Um... I always think of it by trying out some, some things, but...

I: OK, what could we try out?

Lorrie: Um... (pause)... I just remember I got kind of confused in the class when we could just say $gH = Hg$, I didn't, I kind of forgot how... you need to try it out, like you just take a, um, element of G...

I: Uh-huh.

Lorrie: ... and then multiply it by the whole subgroup.

I: OK.

Lorrie: And then, you just, I think you just have to take an element out of the subgroup... and see if those two together equals... um, they'll probably, I mean operating on them this way, see if they're equal.

I: OK, if we were doing that here, what would we do then? I take a little g out of the group and then I have to take something out of the subgroup, so what would that look like?

Lorrie: Um... I don't, okay g is in G, if you take x in the...

I: Uh-huh.

Lorrie: ... subgroup, g operation x equal x operation g.

I: Is that true?

Lorrie: Um... yeah, I think it's true.

I: Why?

Lorrie: Um, what a subgroup is made of, it just says any element from the group commuting with an element from the subgroup equals this.

I: OK. So if we agree that it's true in this one case, is that enough or do we have to say anything else to prove your statement $gH = Hg$? You picked one g and you picked one element out of H.

Lorrie: You have to make sure that every element of G, and any element of G...

I: Uh-huh.

Lorrie: ... acting on, you know, every element of H would equal this.

I: Would that be true, in every case?

Lorrie: Um, yeah I think it would be true.

I: Because?

Lorrie: Because that's just like the definition that we started with.

I: OK.

Lorrie: That every element of the subgroup has to, that commutes with any element from the group.

I: Every element from the group.

The next excerpt is interesting because the student appears to have all of the required knowledge. She begins with a relation between objects, de-encapsulates it to deal with processes, even going back to what may be an action because of her need to label the elements of H so explicitly, and completes the proof. Her knowledge, however, must be elicited at each step with prompts that do not give any information, but just make up for the fact that this student does not seem able to function autonomously.

I: All right. Now, can you show me that the center is a normal subgroup?

Kathy: Normal subgroup, the center. Well, we have... are we, are we assuming that it is a subgroup.

I: Yes, we are. We said that's assumed...

Kathy: OK, like we proved it. OK, we know it's a subgroup, and H is a commute, that's what H is, the center. If we took, some ele... it's kind of like, like the coset thing where we took an element of big G and multiplied it by H.

I: Um-hmm.

Kathy: And we got something. Then we had to multiply H times that little element again. And if we came up with the same thing, they're normal.

I: So another way of doing that, since we don't actually know what H looks like to be able to compare the way you said. What else could we do?

Kathy: I have no clue. That's how, um, how else could we decide, let see what else. I have the conjugate here that I can use. If I did it...no. Is there any way you could...

I: What's the definition of this g operation capital H, what do you actually do to find that?

Kathy: I multiply.

I: Multiply what?

Kathy: Multiply, uh, I multiply um an element of G, little g, times the subgroup, every element of the subgroup H.

I: OK. Every element, so for example, write one of those little multiplications down here.

Kathy: You mean like gH equals g times h_0, g times h_1, dot dot dot.

I: OK, we could do that.

Kathy: g times h_i, is that what you mean?

I: All right, so what would capital H little g look like?

Kathy: Capital H little g would look like h_0g, h_1g, dot, dot, dot, h_ig.

I: OK, now, why would they be equal?

Kathy: Would they be equal?

I: This particular subgroup.

Kathy: Because they're commutative!

I: Can you explain that to me? Because what's commutative? How are you using it?

Kathy: Bec... up here we have ab where a operation b equals b operation a, a such that a and b are in G and for all a in G dot dot dot.

I: OK, so.

Kathy: So here we have, since g operation h up here in our definition of center, it's the same thing as h operation g. Or h_0 operation g.

I: OK. So you know...

Kathy: So, it's a commutative, since we're dealing with commutative, the center is commutative, we know that it doesn't matter which way.

Finally, we give an example of a student who works autonomously to give a correct proof of the normality of the center, clearly displaying a coordination between process and object conceptions.

I: OK. Can you show that the center of a group is a normal group?

Eli: Normal? ...(mumbling)...it's $g^{-1}Hg$ is in H for all g in G. Actually this is just in the definition of the center. So, to be a member of H is gonna commute with everything, so $aH = Ha$. ...(pause)...ah sort of. I just picked some arbitrary elements in the center...hmm...those are all the cosets, all the cosets commute, left coset and the right coset. So if I start with left coset and left coset just gonna...ah...operation through the subgroup, well then, hmm all these elements are commutative by definition. So, it just carries through the right subgroup and that's equal.

4.4 Performance

4.4.1 Test 1, Question 5.

All seven teams chose an appropriate subgroup and gave all of its left cosets by listing the elements in each case. Four teams preceded each listing of a coset by $aH =$ where a was an appropriate element of S_4.

4.4.2 Test 1, Question 7.

Two teams stated the theorem completely correctly, while the other five only neglected to mention that the group must be finite.

All seven teams also gave essentially correct overviews of the proof of Lagrange's theorem, the only errors being minor wording inaccuracies. The average score on the last part, giving the details of the proof, was 76%, and the lowest score was 50%. The errors that did occur mainly consisted of not going far enough in providing details or leaving out a point.

4.4.3 Test 2, Question 4(a).

Regarding the four kinds of responses listed above for this question, 5 students indicated little understanding of cosets, 5 students indicated an action conception, 3 indicated a process conception and 18 indicated an object conception.

The average score on this question was 68%. Much loss of score was due to students proving only an implication between two conditions and not an equivalence. In many cases it is not clear if this really was an error because what was written was a sequence of relations that was clearly reversible.

4.4.4 Test 2, Question 6.

Of the 31 students, 24 took A_3 to be the subgroup and two took the trivial subgroup. All of these students computed cosets correctly. The other five students did not appear to understand very much about cosets.

4.4.5 Test 2, Question 7(a).

Of the 31 students, 28 were successful in computing the cosets. The average score on this question was 94%.

4.4.6 Proofs on the Final.

The one student who selected this statement gave a correct proof.

4.4.7 First interview, Questions 2 and 5.

As we indicated above, most of the students appeared to have a strong conception of coset as a process and as an object. From the responses on these interview questions, we would estimate that no more than three students did not have solid process conceptions and all three appeared to be in transition from action to process. Another four students demonstrated a process conception, but may not have progressed to an object conception. The remaining 17 students did appear to have an object conception, either in talking about the meaning of normality or in listing the elements of S_4 mod K. Of these 17 students, 10 explicitly demonstrated an ability to de-encapsulate their objects to cosets as sets of elements formed by multiplying a single element by every element of the subgroup.

4.4.8 Second interview, Question 3.

Data presented in the form $a(b, c)$ represents a responses, b from students who took a lecture-based class, and c from students who took the experimental class.

A total of 11 (4 from a lecture-based class, 7 from the experimental class) gave a correct statement of the theorem. An additional 9 (3 from lecture-based,6 from experimental) students were able to give a correct proof after some prompting. Another 2 (0,2) students gave the lemma — that the order of an element of a group divides the order of the group and one of them subsequently gave the correct statement after prompting. Thus, 21 (7,14) of the 37 students had some knowledge of Lagrange's theorem.

One or more applications were given by 15 (4,11) students. Of the 4 students from the regular course, 3 required prompting before any applications were mentioned.

In particular, 9 (1,8) students gave as an application the general problem of finding the subgroups of a given group. Of these students, 2 (0,2) spoke of specific groups ($\mathcal{Z}_4$, and S_6). The fact that a group of prime order had no proper, non-trivial subgroups was given as an application of Lagrange's theorem by 4 students from each of the classes. The fact that the order of an element of a group divides the order of the group was given as an application by 3 (1,2) students.

With one exception, all of the students who were able to recall applications of Lagrange's theorem were among those who gave (in some cases after prompting) a correct statement of the theorem. The exception, a student from the experimental course, said that he knew the statement of the theorem, but could not recall the specifics. This student did say, without prompting, that the theorem was used to see if a group was a subgroup of another group.

Of the 5 students who took the lecture-based course at the same time as the students who took the experimental course, 4 gave a correct statement of Lagrange's theorem and 3 did so without prompting. Of these 5 students, 3 recalled some applications, 2 without any prompting.

Regarding the question of remembering and/or reconstructing the concept of cosets, there were 11 (8,3) interviews in which there was no mention of cosets. In an additional 8 (5,3) interviews, the interviewer brought up the idea of cosets, but it did not become part of the discussion. In many cases, the lack of discussion of cosets was due to the fact that the student appeared to have little or no understanding of the entire set of ideas connected with Lagrange's theorem.

In 6 (4,2) interviews, the interviewer mentioned cosets and it became part of the discussion. In 12 (3,9) interviews, the student brought up cosets and they were discussed in the remainder of this portion of the interview.

Taking into account the time elapsed between the classes and interviews, 2 of the 5 students who took the regular course at the same time as the students who took the experimental course brought up the idea of coset without prompting. For the experimental course, 9 of 17 brought it up on their own.

Of the 18 students who were able to discuss cosets in this part of the interview, there were 6 (3,3) who were unable to talk about cosets other than as objects and who may not have reconstructed any knowledge; there were 4 (1,3) students who appeared to be struggling to reconstruct their knowledge and de-encapsulate their object conception; and finally, there were 8 (3,5) students who appeared to have succeeded in this de-encapsulation and seemed to coordinate their action, process and object conceptions of cosets.

4.4.9 Second interview, Question 4.

Of the 37 students interviewed, 20 (7,13) were not stopped by an inability to think about normality and proceeded to the point of discussing cosets in the proof that the center of a group is normal.

Of these 20, 9 (3,6) students were able to give the proof with little or no prompting. Another 7 (2,5) completed the proof with some prompting, but indicating a substantial amount of knowledge, possibly requiring no more help than stimulating memory. The remaining 4 (2,2) did not get very far, even with extensive prompting.

5 Normality

We begin with an historical note. According to Nicholson (1993), Galois proved that if the group G of a polynomial has a normal subgroup H then the polynomial can be decomposed into two polynomials, one whose group is H and one whose group is what we recognize today as the quotient group G/H. Thus the property of normality appears to have arisen historically in a specific context rather than as a general conceptual insight. Today we might interpret normality as a generalization of commutativity which carries with it at least a part of the essence of the more restrictive property.

5.1 Epistemological analysis of normality

We may consider that normality consists of an action on a pair (G, H) of a group and one of its subgroups. The result of this action, as with any mathematical property, is the Boolean value *true* or *false* corresponding to whether the pair does or does not satisfy a certain property.

In Dubinsky et al. (1994) the action in the normality schema was analyzed. According to that analysis, the action consists of selecting an element a of G and coordinating two instances of coset formation, one to get a left and one to get a right coset. The process of coordination consists of checking the equality of these two cosets. This process is encapsulated by varying the value of a to obtain a proposition valued function whose domain is G and whose value for a given $a \in G$ is the truth value of the assertion $aH = Ha$. Finally, a quantification schema must be applied to this function to obtain the condition of normality — that the truth value is *true* for all $a \in G$.

The quantification schema is thematized to be an object consisting of an action which may be applied to any group and any subgroup.

5.2 Instructional strategy specific to normality

The concept of normal subgroup comes up approximately in the 9^{th} week of the experimental course. At this point, students have worked with left and right cosets using tools they have constructed on the computer and calculated by hand. About a week before the concept of normality is studied, students write the computer function `PR` (see Section 4.2) and use it to calculate products of the form gH, Hg, gHg^{-1}, HK where g is an element of a group and H, K are subgroups of it.

Normality is introduced by having the students consider a fairly large number (about 10) of examples of pairs consisting of a group and subgroup taken from modular groups and groups of symmetries. The students are given 5 relations which are, in fact, equivalent definitions of normality (although this is not stated right away) and they are asked to determine which pairs satisfy and which do not satisfy each of the five relations. The students are also asked to use coset multiplication to define a binary operation on the set of cosets in each of these examples and check whether this binary operation forms a group.

These computer activities evaluate a large number of examples which students can do quickly by using the computer function `PR`, which they have written. The idea is for them to notice that for each example, the relations as well as the group definition are either all satisfied or all violated.

At this point the students are given a problem in which they are asked to look at a long list of conditions (about 10, all of which are definitions of normality) and determine which are equivalent to each other. No advice is given the students on whether to work by hand or to use the computer.

After students have worked on these activities, the list of conditions is formulated as a set of equivalent definitions of normality. A few equivalences are proved, but most are left for the student.

Very little motivation is given for studying normality, but the students are given an opportunity to realize that one can think of it in terms of generalized commutativity.

5.3 Mental constructions

We next look at what mental constructions concerning normality we observed in the student test responses and during the interviews.

5.3.1 Test 2, Question 4(a). Conditions for normality.

The overwhelming majority of students (26 out of 31) understood that normality was a property that a subgroup might or might not have and this could be determined by applying a condition to a given subgroup. Their equivalent conditions were either correct or had errors that did not seem to indicate difficulty with the concept of normality. The remaining students either appeared to be totally lost or gave answers that were sufficiently far from correct that they might have been wild guesses.

For most students, work on the proof of equivalence showed the level of their understanding of cosets, but not of normality. There were some students who used an alternative condition for normality as an intermediate step. One student used $gH = Hg$ to prove the equivalence of the given condition with $HG = GH$. Another, in trying to prove the equivalence with $gHg^{-1} = H$ wrote: "$gH = Hg$, which is the definition of H being a normal subgroup of G". Another individual wrote: "$gx = xg$, definition of normal" in trying to establish the equivalence with $\forall x \in H,\ gxg^{-1} \subset H$. (Note: $\subset$ is the student's error) One person suggested $gh = h'g$ as a condition for normality. In the proof, the student wrote: "so take an $h' \in H$ such that $gh' = h'g$, we can say this since H is normal". Somewhere in the proof it was suggested: "since H is normal, $hg = gh$".

5.3.2 Test 2, Question 4(b). Normality of the kernel of a homomorphism.

Again, the responses told us whether or not the concept of normality was understood, but not much about the nature of this concept other than the action of applying it to a subgroup and checking one of the equivalent definitions. Some students did this correctly and others were correct as far as normality goes but made major errors in carrying out the proof. The most common of these errors was to assume that the kernel was equal to the identity and this is an issue that has to do with the concept of homomorphism and not normality.

One student gave an incorrect counterexample to the assertion that the kernel was a subgroup and so did not get to the issue of normality. The error was that the map given was not a homomorphism. Another student's paper was copied so poorly that nothing could be made of what was written on this question.

Finally, some students gave nothing in response to this question that indicated any understanding of normality.

5.3.3 Test 2, Question 6(a). Normal subgroup of S_3.

The students who got this correct selected either the alternating group A_3 or the trivial group $\{(1)\}$. They all appeared to understand that normality is a property of a subgroup and their responses were consistent with the notion of applying an action to a subgroup (i.e., determining that it has the property).

The incorrect responses included a subset that was not a subgroup and the two element subgroup (obtained by keeping 3 fixed), which is not normal.

In several cases, a student gave an example that was not a normal subgroup but otherwise seemed to work correctly with the notion. It is possible that this is a simple factual error that could arise out of focusing on the cycle structure idea and forgetting about the condition of being a subgroup. This might be an example of students having different pieces of knowledge but not coordinating them.

Although the question did not ask for explanations, some students gave them anyway. The correct explanations were either that it was closed under cycle structure (offered, not always correctly, by all of the students whose example was incorrect) or that one of the equivalent definitions of normality was satisfied. The incorrect reasons given were an incorrect statement of being closed under cycle structure or that the subgroup is isomorphic to $\mathcal{Z}_3$ which is normal, presumably because it is commutative.

5.3.4 Definitions on the Final. Definition of normality.

With the possible exception of one student whose copy was unreadable, every student tried to give one of the equivalent definitions, and no student suggested that the proposition that a subgroup H of S_n is normal if and only if it is closed with respect to cycle structure was a definition of normality. (We henceforth refer to this proposition as the *structure theorem*).

The most popular definition offered was that $gH = Hg$ for all $g \in G$. Two other correct definitions were that $abH = aHbH$ for all $a, b \in G$ and that for all $g \in G, h_1 \in H$ there exists $h_2 \in H$ such that $gh_1 = h_2g$.

The only incorrect definition offered was that $hg = gh$ for all $g \in G, h \in H$.

It appears that all of these students had constructed a conception of normality as a property that a subgroup might have and this was determined by a certain relationship among the elements of the group and the subgroup.

5.3.5 Proofs on the Final. Normality of image.

Only one student chose to give this proof, verifying that the image satisfied one of the equivalent definitions.

5.3.6 First interview, Question 2. Normality of subgroup of S_4.

There were some students who took the concept of normality to be identical to commutativity, or to be very similar to the notion of the center of the group. These are examples of applying a schema which appears on the surface to be appropriate but which is not. These particular misconceptions were not always held very strongly. In the following excerpt, Carla gives several explanations of normality under quite minimal prompting. Several of her explanations are incorrect, but at least one is correct. At one point, it appears as if she does know the difference. On the other hand, her understanding does not appear to be very robust.

At first, Carla talks about showing that K is a group. It is possible that she is acknowledging that a normal subgroup must first be a subgroup, but her later discussion suggests that, for her, the concept of normal as applied to K might mean something different depending on whether K is considered as a subgroup of S_4 or as a group in its own right. In any case, she goes on to indicate that, for her, normal is identified with commutative.

I: So, the first question about this subgroup is how would you go about deciding if K is normal?

Carla: Normal. Um first you would have to show, (pause) you know it's a group?

I: Yeah, let's say that you already checked if it is a subgroup. Right.

Carla: OK. You just have to check if when you multiplied two elements 2 elements together if it is the same as if you did them in reverse order. Like if you take $1, 2, 3, 4$ and compose it with $1, 4, 3, 2$ see if you get the same thing as if you take $1, 4, 3, 2$ and compose it with $1, 4, 3, 2$ see if you get the same thing as when you take $1, 4, 3, 2$ and compose it with $1, 2, 3, 4$.

I: So it's um

Carla: Commutative.

I: Commutative, uh, between two elements.

Carla: Yeah.

I: Now if it is not commutative,

Carla: Then it is not normal.

The interviewer then gives a prompt that appears to be quite minimal but sets Carla to thinking in a different direction. She indicates here her different ways of thinking about K. The new way amounts to the idea of K being (a subset of) the center of the group.

I: OK, does it have anything to do with sets?

Carla: What do you mean "with sets"?

I: Does

Carla: Like with this set?

I: Right.

Carla: Um, well it depends if you are talking about a normal group or a normal subgroup.

I: What would the difference be?

Carla: This is a subgroup of S_4, and in order for it to be normal you would have to be able to take every element of S_4 and have it be commutative with every element in K. Where if as you are talking about a normal group you just have to make sure that it is commutative within the group.

I: Is that a normal group? If it is commutative within the group, do you call that a normal group?

Carla: Um, I would. That doesn't mean it is right. (They laugh) The only time we talked about normal was with subgroups.

I: With subgroups right.

Carla: So…

I: What you were talking about, I think people generally call a commutative group, or a

Carla: Abelian group.

I: Right. Yeah. OK, so let's go back to this idea of a normal subgroup.

Carla: OK.

I: So that, now what is that again?

Carla: You take an element out of S_4

I: Mmm.

Carla: and multiply it times an element in K.

I: Mmm.

Carla: And if you get, if you take the element in S_4, like say the permutation (1 2), and multiply it times the element in K, (1 2)(3 4), if whatever you get for that is the same as when you take (1 2)(3 4) and multiply times the element in S_4. And you have to be able to do that for every element in S_4.

I: So this has to be done for every element in S_4.

Carla: Right. Times every element in K.

I: And each time element by element, they have to be the same.

Carla: Yes.

Now the interviewer gives a prompt that has considerable content and the result is that Carla gives a correct explanation of normality. She even realizes (perhaps with some embarrassment) that her previous definitions were wrong.

I: OK, now what about the whole subgroup, what about if you multiply this, your single element of S_4 by everything in the subgroup.

Carla: They all have to, it has to be another element in the subgroup, maybe. Um, well the same thing has to hold true, you multiply an element in S_4 times every element in the subgroup you'll get another set,

I: Right.

Carla: And then when you do the reverse, when you take the set K times every element in S_4 you should get the same set back.

I: OK. So you are saying that if you take an element in S_4, and multiply it by everything in K

Carla: Right.

I: That gives you one set.

Carla: Right.

I: And the other thing you might do is you take everything in K, and

Carla: Times that one element in S, S_4.

I: Right.

Carla: And then you should get the same set as you did when you took the element times the set.

I: OK. Uh, now is that the same thing as saying that each element, each time you multiply it by an element, you get the same thing as if you did it in the reverse order?

Carla: Uh, no.

I: So which one is normal?

Carla: The second one. (smiles and laughs)

These excerpts suggest that Carla has at least three competing notions for normality of subgroup H of G: H is commutative; every element of H commutes with every element of G; and $gH = Hg$ for every $g \in G$. She seems to be aware that normality is the concept expressed by the last notion, but she is not very strong in this knowledge. It is possible that the source of her difficulty is that her conception of coset is as a process but that she does not easily encapsulate this process as an object. The next excerpt appears to support this suggestion. It shows her difficulty in attaching a name to the process, and her response to a mention of the name by the interviewer is to describe the process. Her comment at the very end suggests that she may be developing the ability to encapsulate this process into an object.

I: The second one. Right. When you take an element of S_4,

Carla: Mmm.

I: OK? and you multiply it by everything in K.

Carla: Mmm.

I: You said you get a set.

Carla: Right.

I: Does that set have a name?

Carla: Probably. Well it should be the same as K.

I: Well try it. Why don't you multiply, take your (1 2) and multiply it by everything in K.

Carla: OK.
(pause)

Carla: Guess you don't.
(pause)

Carla: You get that, unless I did something wrong.

I: And is that the same as K?

Carla: No.

I: No. It's different from K.

Carla: Right .

I: Does it have a name?

Carla: I don't know. I don't think so.

I: Could you write a notation for using that single element from S_4 and K?

Carla: Well it'd be like $(1\ 2)K$.

I: OK, and does $(1\ 2)K$, does that have a name?

Carla: It just tells me that you take (1 2) times every element in K. And you get that subset.

I: What is a coset?

Carla: I don't know, that's what it is, huh? (They laugh) It's when you take an element and you multiply it times every element in the subgroup. That would be a coset.

I: So you sort of knew that.

Carla: Oh yeah.

I: Once I reminded you.

Carla: Sort of yeah. And then to get another one you take another element in S_4 that is not in K and then multiply it out and you get another one.

Now when the interviewer returns the discussion to normality, Carla explains her two different ways of thinking about K, and it may be that confusion between the role of a subgroup as a subset and as a group in its own right may be another source of difficulty. In any case, we see that Carla's problem is not in any lack of knowledge. All of the necessary pieces seem to be there, but she has not succeeded in organizing her various bits of knowledge and cannot be clear about these ideas without a fair amount of prompting.

I: OK. Let's go back to this definition of normality that we talked about, that you said...

Carla: OK.

I: We finally agreed that normal means that if you take one element, any element from S_4, and multiply it by everything in K, that you get the same set as you do if you do it in the reverse order.

Carla: Right.

I: And originally you were saying that you had to get the same answer every time you did it for a single element.

Carla: Mmm.

I: Which you agree is different.

Carla: Right.

I: Why do you think you had that confusion? Between those two...

Carla: Because I was thinking of, I was thinking as this as a group.

I: K_4. You are pointing to K_4.

Carla: Right.

I: You were thinking of that as a group.

Carla: And then I was thinking that it had to be commutative.

I: Right.

Carla: But I wasn't thinking that if you had a set, that you could multiply two elements together and get one element and then, one element in the set, and then reverse the order and get another element in the set, I was thinking it had to be the same element, but if you are talking about the sets, since order doesn't matter it just has to be another element in the set.

In the final portion of this excerpt, we see the interviewer trying to get Carla to state her definition of normality, which is correct, in the succinct form of *Each left coset is equal to the corresponding right coset*. Carla does not accomplish this and the discussion suggests that her conception of $gH = Hg, g \in G$ is a collection of separate statements (one for each coset) and she may not be able to think of them as a single statement about left and right cosets.

I: Mmm, OK, good. All right, Now let's go back to this thing that you've written down there, this $(1\ 2)K$?

Carla: Mmm.

I: OK, and we agreed that that is called a what?

Carla: Coset.

I: A coset. OK. What kind of a coset?

Carla: Uh, left.

I: Left. OK, what other kind of coset is there?

Carla: A right.

I: OK, good. (They laugh) What would the right coset be?

Carla: It would be K times (1 2).

I: OK, Now going back to this subgroup K, and try to decide whether it is normal. So you would have to simply check that.

Carla: You would just have to figure out what that is.

I: Right, and you would have to do that, would it be enough to do that (1 2)?

Carla: No. You would have to do it for all of them.

I: For all of them.

Carla: All the elements of S_4. There should be six cosets so you would have to check all six of them.

The main response to this question (or to a request for an alternative after the structure theorem was given by the student) was for the student to state one of the definitions of normality and apply it to this example. Again, this shows a conception of normality as a property that subgroups may have and that is checked by applying an action to the subgroup.

Kristen's response on page 19, in which she responds to the question by stating the definition and then, at the request of the interviewer shows how she would check it, is a typical example.

5.3.7 Second interview, Question 4. Normality of center.

For this question we could discern nine different kinds of responses that can be ordered linearly.

1. Essentially nothing that indicates an understanding of normality. In some cases the student suggested that he or she could not think of any meaning for this term other than as orthogonality or some meaning outside of mathematics.

2. A beginning of an understanding in the sense of being able to work with a definition of normality once it is essentially supplied by the interviewer. This could amount to being able to work with cosets but still not having any understanding of normality.

   In the following excerpt, James requires significant prompting but then seems to be able to work with the concept.

   I: Now we want to show that it's a normal subgroup.

   James: Normal, normal I normal. The only thing I can think of when I hear normal is 90 degrees.

   I: OK, that's normal in the sense of vectors, right?

   James: (Mumbling)I, I can't remember what normal means.

   I: OK, it has to do with the cosets, again. Remember at the beginning we formed left cosets and right cosets.

   James: Normal means the left coset equals the right coset.

   I: Right.

   James: Oh jeez I so I have to show that I given a, uh, an a in C that the, uh, a. . . something like, um, C over a, is that how it's formed?

   I: Not the, not the quotient group, but just the cosets.

   James: OK, just the cosets.

   I: Just the cosets.

   James: I get it now.

I: OK, so it might be denoted aC or Ca.
James: $aC = Ca$.
I: Yeah.
James: That's what I have to show.

3. A response that suggests the student thinks of normality as an action applied to H, but does not appear to have any conception of what that action is, or the conception is not very close to a definition of normality.

   In the following excerpt, Carla speaks of *it* which suggests that there is an action to be applied to the subgroup, but her action confuses normality with commutativity. It may be that she still has the misconception of identifying normality with commutativity that she displayed on the first interview. It is also possible, however, that her phrase "...to be normal it has to be commutative" is for her a loose way of expressing the fact that commutative implies normal.

   I: OK, how would you show that the center is a normal subgroup.
   Carla: OK, in order for to be normal it has to be commutative and from the definition of how you put the elements into the center you want them to be commutative, so it would have to be commutative and we just proved that it's a subgroup.

   The interviewer did not pursue the issue.

   All of the remaining types of responses are from students who appear to think of normality as an action applied to H. The response is categorized according to what the nature of that action appears to be for the student.

4. The responses are confusing and it is not easy to see what kind of action the student is thinking about.

5. A condition such as $ag = ga$ where both g and a are elements of H.

This kind of response may simply represent the student identifying normality with commutativity. This may be the case with Kristen, who would then be an example of a student whose understanding deteriorated after the course ended (since she gave a much better response to the question in the first interview.)

> **I:** OK. How would you show that the center is a normal subgroup?
>
> **Kristen:** C be the normal of G. So we have to prove that g operation a equals a operation g for all g in C.
>
> **I:** What does it mean for a subgroup to be normal?
>
> **Kristen:** That every element commutes.
>
> **I:** OK. How would you show then that the center is normal?
>
> **Kristen:** I have to prove that for every element in the subgroup C is commutative. We have to prove that g operation a equals a operation g for all a and g in the subgroup. I know that a and g in C we know that a times, or a operation g and g operation a are both in C. We have to prove that a operation g equals g operation a.

6. One of the equivalent conditions defining normality such as $gH = Hg$ or $gHg^{-1} \subset H$ with g clearly an element of G but no apparent universal quantification over G.

(Note: The meanings students seem to be attributing to such conditions was discussed in the section on cosets.)

In the interview from which the following excerpt is taken, Jeff never tells us anything about g.

> **I:** Can you show me that it is a normal subgroup? What would it mean for it to be normal?...
>
> **Jeff:** So, OK. Let's call it the center, since we know it's a group, call that C.
>
> **I:** OK.
>
> **Jeff:** So, to show that it's a normal subgroup, that means that gC equals Cg.

7. The same as the previous one with the addition of some mention of the role of g in a way that may or may not come from thinking about a universal quantification of g over G. In the following excerpt, when Jocelyn says, "...where g is in your group" we cannot be sure if she is thinking about g statically as a single element of G or as a variable running over the domain G. This might not be a simple omission since in the subsequent discussion, Jocelyn indicates that she is quite able to perform universal quantifications over G and does so in talking about the center. Thus, it is not clear why she is not equally specific with normality.

   I: How would you show that this is a normal subgroup?

   Jocelyn: OK, a subgroup is normal if for all h in the center ghg^{-1} is in the center where g is in your group. OK. So, ghg^{-1} being in the center means that, there is an easier way to do that. Hmm, OK, h is in the center means that gh equals hg for all g. So if we take the inverses g, if we operate the inverses g on the right side we get ghg^{-1} equals hgg^{-1} and gg^{-1} cancels so you get ghg^{-1} equals h and that's another way that the something is normal subgroup.

8. The same as the previous ones except that in subsequent discussion the student does seem to be thinking about this condition holding for all $g \in G$. In the following excerpt, Lynette wrote the condition $gH = Hg$ and close to it she had $g \in G$ but at first, it was not clear whether she was thinking of a single element of G or all elements in G. From the subsequent discussion it would seem that she had the latter in mind.

   I: OK. All right, then let me give you a harder question. Can you show me, if, if you can work all this out, you would come to the conclusion that it is a subgroup, okay? Can you show me that it's actually a normal subgroup?

   Lynette: OK, um, let me think of, there's like, four different definitions for a normal subgroup.

   I: Uh-huh. Which one do you usually think of?

   Lynette: Uh, the first one that comes to mind would be this one.

I: OK, and you wrote little g capital H equals capital H little g. That's the first definition that comes to your mind. Can you explain what that says?

Lynette: It means, um... g is an element of the group...

I: Uh-huh.

Lynette: ...and H is the subgroup.

I: OK.

Lynette: So every element of... the group acting on the subgroup would equal... the opposite of that.

9. A definition that is correct in that one of the equivalent conditions is stated as being required to hold for all $g \in G$.

In the following excerpt, Diane wrote the following three lines after being asked to focus on normal.

$$\{g \in G, s \in S \qquad g \circ s \in S\}$$

$$\{g \in G, S : gS = Sg\}$$

$$g \circ S = S \circ g$$

I: OK. Normal. Can you focus in on that?

Diane: Normal... normal.

I: OK. Tell me what you wrote down.

Diane: That if you take any element of G, in the group G...

I: Uh-huh.

Diane: And you have the subgroup S...

I: Uh-huh.

Diane: That little g times the subgroup is going to give you the subgroup times little g.

5.4 Performance

Next, we look at the performance of the students on both the exams and on the interview questions.

5.4.1 Test 2, Question 4(a).

Of the 31 students who took this examination, 16 gave a correct equivalent definition of normality and 10 wrote,

$$ghg^{-1} \in H, \quad \forall h \in H$$

which might have come from interpreting the question as asking for an equivalent formulation of the statement $gHg^{-1} \subset H$.

Of the remaining 5 students, 3 wrote

$$ghg^{-1} \in G, \quad \forall h \in H$$

which could have been a typo (writing G for H). One student wrote $GH = HG$ and the other seemed to be totally confused.

The responses of five students to the equivalence part of the question suggested that some of them may have a fixed definition of normality in their minds. In the case of others, it might be that they got stuck. For example, they needed Hg instead of gH somewhere in their proofs, and they chose this way to get the result they wanted.

5.4.2 Test 2, Question 4(b).

Of the 30 students (omitting the one whose paper did not copy), 16 applied a definition of normality correctly and another 7 gave a correct definition, but made a serious error in applying it. Two students "proved" commutativity and incorrectly concluded normality from that and one student gave an incorrect example of a kernel of a homomorphism that was not a subgroup. The remaining 4 students gave nothing useful in response to this question.

5.4.3 Test 2, Question 6(a).

A total of 26 students gave correct examples of normal subgroups; 24 were the alternating group and 2 were the trivial group. Of the incorrect examples, 4 were subsets that were not groups and one was S_2.

Including the 5 who gave incorrect examples, 24 students offered reasons for their example being a normal subgroup. All of the group whose examples were incorrect gave the structure theorem as the reason and 3 of these examples were, in fact, closed under cycle structure (but were not subgroups). Of the 19 students who gave correct examples and offered reasons, 12 applied the structure theorem, 3 checked one of the definitions of normality and 4 gave incorrect reasons.

5.4.4 Definitions on the Final.

Of the 30 students whose copy was readable, 27 gave the condition that $gH = Hg$ for all $g \in G$ as a definition of normality. The other two correct definitions and the one incorrect definition were offered by one student each.

5.4.5 Proofs on the Final.

The one student who tried this proof gave a correct proof.

5.4.6 First interview, Question 2.

Of the 24 students interviewed, 15 responded to the question with a correct definition of normality. An additional 3 gave responses that might have come from understanding the concept, but they were so hesitant and fumbling that it was not possible to be certain. There were 4 students who confused normal with commutative and 2 students whose response suggested very little understanding of the concept of normality.

5.4.7 Second interview, Question 4.

We made the following counts for the nine kinds of responses to this question described in Section 5.3 on Mental Constructions above, using the notation introduced in Section 4.4.8. Eight students, all from the standard course, gave responses which did not indicate any understanding of normality. Once the definition of normality was given by the interviewer, 5 (4, 1) students began to exhibit some sort of understanding of the concept. The beginnings of an action conception were exhibited by 5 (3, 2) students. One student from the experimental class appeared to think of normality as an action applied to H, but it was unclear to see what kind of action the student was envisioning. Another student from the experimental course conceived normality as a condition which involved only elements from H, while 4 (3, 1) students involved at least one element of G. It was unclear whether the responses of 7 (1, 6) students indicated some understanding of a universal quantification over G due to statements like, "...where g is in your group." The responses in the subsequent discussion with the interviewer of 3 (1, 2) students indicated more clearly that the condition held for all g in G, and three students, all from the experimental group, gave a correct definition of normality.

6 Quotient Groups

In this section we will first look at the epistemological analysis of quotient groups and how that was used to formulate the instructional strategy. We then move to reporting the mental constructions observed in the student exams and interviews followed by a summary of their performance on both the exam and interview questions.

6.1 Epistemological analysis of quotient groups

The quotient group concept is a coordination of three schemas: the coset schema, which was considered in section 4, and the binary operation and group schemas, which were considered in Brown, DeVries, Dubinsky and Thomas (1996). This coordination consists in selecting specific constructions from these schemas and applying them to the quotient group situation. Thus, from the coset schema we have the formation of a set of cosets. We have the construction of coset product from both the binary operation schema and coset schema. ¿From the group schema we have the property of normality and that the coset product together with the set of all cosets formed using a normal subgroup of the given group satisfies the group axioms. Finally, the group schema may also help to identify the resulting quotient group with some familiar group.

With an object conception of coset, it is possible to apply the binary operation schema. To actually define the operation on cosets one can use representatives or apply coset multiplication. In either case this requires a de-encapsulation of the cosets to the process of coset formation, and the results of either method must be coordinated with the group schema. With the representative method, it is necessary to check that the definition is independent of the choice of representatives and with coset multiplication, it is necessary to determine that one has a binary operation in the sense that the product is again a coset. In other words, the notion of normality contained in the individual's group schema must be applied in order to justify using either method.

6.2 Instructional strategy specific to quotient groups

Most of the work in helping students construct the concept of quotient group is done in the process of constructing the three prerequisite schemas. In particular, as part of the coset schema, students are asked to write computer functions which define a binary operation on a set of cosets, to compute Cayley tables for these binary operations, to check the group properties and to identify the resultant groups.

If the development of the three schemas has been satisfactory, all that remains for a formal consideration of quotient groups is a coordination of them. A standard mathematical treatment was used to develop this coordination.

6.3 Mental constructions

We next look at evidence of mental constructions concerning quotient groups we observed in the student test responses and during the interviews.

6.3.1 Test 2, Question 6(b). Identify the quotient group S_3/N.

The richness of the requisite object conception of coset can be revealed in students' thinking about the binary operation on the set of cosets that is used in forming the quotient group. Unfortunately, this particular question did not reveal much about this thinking. The only students who gave any details about how they computed coset product used the representative method. This is a reasonable choice in terms of convenience but it does not tell us much about the students' understanding of cosets.

Most of the students who formed the set of cosets and were aware of the operation (in that they mentioned it even if they gave no details) were able to identify the quotient group as being isomorphic to $\mathcal{Z}_2$ with the operation of addition mod 2 if their normal subgroup was A_3, or as being isomorphic to S_3 if their normal subgroup was the trivial group.

A similar response was to form the set of cosets and then assert that it was the same as $\mathcal{Z}_2$. It is not clear if these students were aware of the operation and possibly also the fact that there is only one group of order 2 (so that details are unnecessary) or if they were thinking of a group only in terms of its underlying set. An indication of the latter is the fact that each of these students referred only to $\mathcal{Z}_2$ and made no mention of the operation. An indication of the former might be obtained from their responses to the next question on the exam.

A response that was more likely to indicate that a student was thinking about a group only in terms of its underlying set (a difficulty which is discussed in Brown et al., 1996) was one in which the student formed the set of cosets and gave no more on the problem, apparently considering that this sufficed as an identification of the quotient group.

There were several responses that were erroneous (as opposed to being incomplete) and some of them seemed to be due to errors in the schemas on which the quotient group concept is based. One error was to use only the elements of the normal subgroup N in forming the set of cosets gN rather than allowing g to range over S_3. We saw this in Section 5. Another arose from selecting as a "normal" subgroup a set which was closed under coset multiplication but was not a subgroup, indicating a weakness in the group schema. The responses in this situation were similar to the ones in which the student was working with what was actually a normal subgroup. This leads, of course, to contradictory information, but none of the students who made this error noticed any contradictions.

6.3.2 Test 2, Question 7(a). Describe the ring $2\mathcal{Z}/6\mathcal{Z}$.

Most students listed the cosets correctly, either mentioned or used the group operation and computed the Cayley table for addition correctly. Judging from their calculations and, in some cases by their explicit statements, their method of calculating with cosets was to use representatives. The students seemed quite comfortable with cosets as objects but, since they were using representatives, we cannot tell from this question anything about whether these are objects that come from encapsulating the process of coset formation.

In some cases, the student only listed the set of cosets and did not mention the operation until responding to Question 7(b), which asked the student to prove or disprove that the ring is isomorphic to $\mathcal{Z}_3$. This may be a transition from thinking of a group as set to thinking of it as a pair consisting of a set and an operation. That is, these students mention only the set in describing the quotient, but in their work they show that they are quite aware of and can compute successfully with the operation. This could be an example of success coming before understanding (Piaget, 1978). One other response was to mention the set and operation but to compute the Cayley table incorrectly.

Some students listed the elements of the quotient incorrectly. Two simply listed the representatives without the subgroup. The others demonstrated serious flaws in their coset schemas, in one case "canceling" a two from $2\mathcal{Z}/6\mathcal{Z}$ to get $\mathcal{Z}/3\mathcal{Z}$.

6.3.3 First interview, Question 5. Operation table for S_4/K.

The most common response to the request to compute the operation table for S_4/K was to calculate the set of cosets of K by elements of S_4 and begin to work out some of the products of these cosets. These students seemed to have schemas for groups, binary operations and cosets (as processes and objects) which they could draw from and coordinate answer this particular question. In many cases the student began with a clear overview statement such as the following.

I: . . . We want to look at the operation table for the quotient group.

Nathan: OK. The operation table, you'd form the group table for that.

I: Yeah.

Nathan: With uh, our operation would be. Do you want me to write it out?

I: At least get started on it and tell me about how you'd go about doing it.

Nathan: OK. Well, what this is, is just the set of cosets of, uh, K with respect to S_4, and with the operation coset multiplication in quotes there, basically. So you'd have your operation.

Nathan then goes on to compute cosets and coset products.

When the schemas of coset, group and binary operation are well developed, their coordination can be used to locate and resolve problems. In the following excerpt, Randy computes the set of cosets and is asked to compute the operation table. He makes a mistake, as the interviewer observes, in that he only obtains 4 cosets. In computing the coset product he gets an answer which is not one of his four and he concludes that there must be five.

Randy: The operation table. Well you'll have, how many cosets will you have? So you'll have an identity and uh, three other cosets, yeah. (pause)
OK. (pause)
It's right here, whatever you need for this. The whole problem here, OK.

I: So what are you trying to do now?

Randy: Figure out the four cosets. One will have (1 2) in it, One will have (1 3) in it, One will have (1 4) in it, I think.

I: What are those?

Randy: This is the first element of the four cosets. These are the, they go on these are the four cosets and I am going to multiply them together to find out what other coset I'll get for up here.

I: OK.

Randy: And I'll just go e. . .

I: OK.

Randy: OK? (pause-writing)
I'll get one sooner or later.
(pause)
It's obviously b or c.
(pause)

I: Tell me what you are doing.

Randy: Um, I just found out what um, an element in a...

I: Mmm.

Randy: With another element in a is, and that was, this is what it is, and now you have to find out what coset it is in, b or c, I just have to figure out which one it's in and then that'll go there, from then on you do the same.

I: So you are doing a operation a.

Randy: Yeah and I got this. And I am going to do b operation, ( ), you know this just to find the rest of this I have to see if this in it and if it's not I know it's in c. (1 3)...

I: Can you relate this coset of a, the whole coset a?

Randy: The whole coset?

I: Yeah.

Randy: Yeah. It's a bear.
(pause)

I: OK, and then you do the rest.

Randy: Oh, OK, I see.

I: Uh-huh.

Randy: There is 5.

The difficulty recurs and Randy eventually figures out that there are 6 cosets.

Some students had difficulty incorporating the binary operation schema into their thinking on this problem. Jocelyn began to work out the cosets and did not move to the product of cosets until the interviewer suggested, "Why don't you write out one or two elements and I have a list over here and I will then give you the rest of them." After this small prompt, she goes on to compute products and is able to explain what she is doing.

Other students had considerable difficulty in thinking about anything beyond the set of cosets when considering S_4/K. In the following excerpt, Lynette has succeeded in considering the set of cosets, but she shows considerable resistance to thinking about the operation.

I: Cosets, OK. Now get back to the point of the question, compute the operation table for S_4.
(pause)

Lynette: Um, I don't really know what the operation table is, a place that you, OK, I thought it was just these set of cosets.

I: Well the set of cosets is S_4 mod K.

Lynette: OK, can compute the operation table.

I: Right.
(pause)
What do you want to do with these on account of notation?

Lynette: 1,2

I: ... but you, you did in the first problem you had things in the beginning, you said how did you get these things, you used uh

Lynette: Permutations, I mean

I: No but you told me you computed. You even wrote it down.

Lynette: (laughs) um

I: Do you know what it means to get the operation table?

Lynette: I forget what we did.

I: Well what are you doing when you do these, when you do these from here, do you do the multiplication, taking the operator of this element on that element, you're talking about this element, right so how else do you do the operation for this, S_n mod K is the set of...

Lynette: All cosets.

I: ... all the cosets.

Lynette: All right.

I: And so how would you operate on the cosets?

Lynette: By multiplying them together, your total

I: That's what I want for you to do to make it to make it...

Lynette: Like $(1\ 2)K$ or something?

I: However you want to do it.

For some students, being put in a situation which requires the coordination of schemas was rather disequilibriating. In the excerpt below, Mitch is at first unable to coordinate his coset and binary operation schemas, so he falls back on his group schema, and considers K on its own, rather than the cosets.

I: OK. OK, let's move, then, to this question. Compute the operation table for S_4 mod K. If K is that given subgroup. Now, how would you go about computing the operation table?

Mitch: The operation table…um, that's a…I would take…I would take an element minus four, S_4 mod K…(long pause, some mumbling). What do you mean exactly by operation table? Table, what do you mean by table? You want a, okay.

I: …for S_4. So what would you need in order to make the operation table?

Mitch: List the elements…

I: Tell me what are you doing there?

Mitch: Huh?

I: What are you doing there?

Mitch: I'm listing the K.

I: OK, listing all the elements?

Mitch: Yeah. Of K and then I was going to, and then list them across the top too and then you can, I think that's what you want, is, uh…

I: And then?

Mitch: And then you'll be able to tell what all the products are. I'm not sure…what I'm doing here. But that's what I'm thinking. Um…S_4 mod K, that's a s-, those are sets, I mean, isn't it? Um, I'm stuck, I don't know, I'm just not thinking today. Yeah, I can't think, yeah. Um…I'm not sure.

It is perhaps not surprising that most students preferred the method of representatives for calculating coset products because it is much easier than the definition via the set of all products of elements taking the first from one coset and the second from the other. But how developed is the level of the student's understanding of the representative method? Students who are able to give a reasonable explanation of it appear to have more than an action conception of the quotient group's binary operation, for it seems they have interiorized the action. Following is a particularly detailed and complete explanation that was not too unusual. Notice that Thomas's last computation includes one example in which the product of two representatives was a member of the resulting coset other than the representative that he had chosen. He also includes the fact that normality justifies his use of this method.

I: OK. Now I'm not going to make you do all of them, here are all the cosets. Um now as we construct this table what I am really interested in is how you take the product of 2 cosets.

Thomas: OK.

I: So, let's just write out say the first portion of the table.

Thomas: OK. So that would be K. (1 2)K, (1 3)K... Um, knowing that it is normal makes it real easy to just take 2 representatives from each of these cosets and multiply them together and then you locate which coset it is in and find, you know and then just write which representative coset it would be, but K is going to act as an identity because it has the identity in it so I can just write out that's K that's going to be (1 2)K, (1 3)K,... do you want me to finish all this?

I: That's fine.

Thomas: Do the same down like that, then if I was multiplying (1 2)K with (1 2)K I know since it's they've both got (1 2) in it (1 2) and (1 2) is it's own inverse so that is going to be K. If I was multiplying (1 2)K with (1 3)K that's just going to be (1 2 3) which is I'll call it (1 2 3)K. And (1 3) with (1 2) then is going to be 1, (1 3 2) which is I'm going to call (1 2 4)K.

Not all explanations of the representative method were a clear as Thomas's. For example, in the following excerpt, Travis seems focused on the representative and never really moves beyond an action conception to relate the product to the entire coset.

I: Good. OK. Now what I would like you to do, and you won't have to finish this but just make a start on writing out the, uh, operation table.

Travis: Oh, okay. And with this what we do is have to work with representatives because we've worked with that before.

I: OK.

Travis: And it's a lot simpler to do than to worry about other things. (Pause). Now, obviously, those two rows will be real obvious.

I: OK. You don't have to write them, just... (long pause). OK, could you explain to me how you got the one that you just got?

Travis: Oh, that's... yeah, I, I did it the hard way. Actually a two cycle multiplied by itself is going to be giving you, it's going to be its own inverse.

I: OK, so...

Travis: Therefore it's going to undo what it's already done.

I: OK, so can you explain just what exactly you did, you were...

Travis: Well, what I originally did was I just said one goes to two, two goes to one...

I: Right.

Travis: ...therefore one goes to one.

I: OK.

Travis: Then two goes to one, one goes to two, therefore two goes to two.

I: OK, what cosets were you performing the operation on?

Travis: I was performing the operation on the cosets of, um, $K(1\ 2)$ and $K(1\ 2)$.

I: OK, and then, and what did you do to get that answer?

Travis: I multiplied the, I mean I composed the two representatives of those sets.

I: So you used representatives.

Travis: Right.

A few students chose the set product definition as their first method of multiplying cosets, but, in general, the students were quick to mention the other of the two methods when asked. Interestingly, one student, who had chosen set product, gave as her "other way of doing it" a mixed calculation (also correct for normal subgroups) in which a representative of one coset was multiplied by every member of the other coset. This "half-way" method might have come from the important formula,

$$(aH)(bH) = abH$$

which the student might have interpreted as,

$$(aH)(bH) = a(bH)$$

6.3.4 First interview, Question 6. Justify the method of representatives.

Asking students to justify the use of the method of representatives in computing coset products revealed a lack of coordination of a number of schemas they had constructed. Some students responded by talking about the partition of a group into its cosets by a given subgroup. Others referred to the property of coset product (defined as the element-by-element product of two sets) being closed and some brought up closure with respect to cycle structure as the reason, valid only for permutations, that one was allowed to use the method of representatives. All of these schemas, together with the definition of normality, had, for the most part, been constructed by the students (see the corresponding sections on these topics), but their confusion about the relationships suggests that the students had not thematized these schemas into objects that could be related to each other with each finding its proper role in a consideration such as the formation of quotient groups.

In many cases, the students gave a short response to the question which was accepted by the interviewer who did not always probe further. Hence, we cannot get any comprehensive view of these students' understanding of the relation between normality and the use of the method of representatives. However, we could distinguish between a variety of levels of coordination between the concept of normality and the method of representatives.

Some students mentioned normality as the justification for use of the representative method, but said so little about it that it could have been only a memory. This is particularly likely since a great deal of the discussion at this point was about K being normal. For example, in the following excerpt, Aimee talks around the problem a great deal and mentions many things, including the fact that K is normal, but it is not clear that she has any understanding of this particular point.

I: How do you know that?

Aimee: Because I've worked with these so... for so many years, and in my wise old age I... no, um... (interviewer laughs)

I: Another way to phrase the question is, what gives you the right...

Aimee: The audacity! To say such a thing!

I: ... the audacity to do it this nice, convenient, simple way when it's actually defined another way.

Aimee: Um, well, is starts off by you do actually write them all out and start doing them all. Um, if you want me to prove that it's that, it will take me a few minutes...

I: That's not really what I'm asking, necessarily.

Aimee: What are you asking? Say, Danny, [interviewer] what are you asking?

I: Well, you said you could prove it, someone could prove it...

Aimee: It can be proven.

I: You might prove it if we sat here for awhile, you say. Um, I guess, what is it in this situation that assures you of that, if anything?

Aimee: Because K ... K is a normal subgroup?... I don't know. It fits nicely into our little mode of the way we do things. Um, I've worked with it before. So that, you know, with my incredible brain power I know that it just fits in there.

I: OK.

Aimee: No, I mean, it's just... because this is a subgroup, this... this... S_4 mod K is going to be a group. These six elements form a group. With coset multiplication...

I: OK. So, some combination of experience and stuff.

Aimee: Well, and in working with... um, yeah, mostly, mostly the experience of working with so many different groups and subgroups. And because K is a subgroup, not just a subset, then I know that S_4 mod K is gonna be also a group.

We see in the preceding excerpt the vague idea that something must be proven. In the following, Nathan appears to connect the use of the representative method with normality and the need to prove something — which he believes he can do.

I: And what allows you to do it this other way?

Nathan: Um... By multiplying the representatives because we know that it's normal, because we have K is a normal subgroup of S_4.

I: OK. And somehow that told you you can...

Nathan: Uh, yeah there's... Well, I believe I can...

I: I guess you're saying there's a theorem or something.

Nathan: Yeah, there's a theorem, and I could possibly prove it if I sat down.

I: All right.

Nathan: Using the fact that it's normal, that would be pretty easy, yeah. Actually, I know I could prove it, yeah.

I: OK. Well, unless you really want to...

Nathan: OK. I don't really want to.

In addition, we could distinguish a number of specific issues which gave trouble to particular students.

1. Most students selected the method of representatives while working on the operation table for S_4/K, but some did not seem to have much understanding of it. They appeared to be stuck with an action conception of the method. For example, in the following excerpt, Arlene describes the calculation but can give no more justification than "it makes sense".

   I: OK, so can you see what or guess what gives you the right to do it this way that you've done it here?

   Arlene: Because we are taking every element and doing every combination and by um... Hmm...

   I: Guess what I'm saying, for example you showed me how to operate these 2 things together (1 3), (1 4)

   Arlene: Yeah.

   I: And I'm wondering how is this related to the actual definition of taking these 2 cosets and doing coset multiplication, or maybe not the relationship but what lets you do it this nice slick and easy way?

   Arlene: We found this coset by multiplying this times K.

   I: Uh-huh.

   Arlene: And we found this coset by multiplying this times K.

   I: Yeah.

Arlene: So it would make sense that if you can so these 2 together then do it all on K, you would get the same answer.

I: OK. I see what you are saying. Yeah.

Arlene: It makes sense to me here hard to explain. Like if, like just in general algebra if you have, kind of like distributive property in a way, you know this is going to be the same as if you do it this way.

I: So you, this little relationship you've written down which,

Arlene: We're not really adding or multiplying but...

I: Right, this is an analogy.

Arlene: Yeah.

2. Many of the students who used the method of representatives were aware that it was not the same as the definition, but some were not. For example, Jocelyn seems to have forgotten the set product method. A prompt by the interviewer leads her to recall the definition of coset product, but not much else about it.

I: $K(1\ 2)$ is right. You did $K(1\ 2)$, times K, how did you get that answer?

Jocelyn: I picked representatives out of each coset. So out of $K(1\ 3)$ I picked the cycle $(1\ 3)$ and out of $K(1\ 2)$ I picked the cycle $(1\ 2)$ and multiplied them.

I: OK. Is that how a coset product is defined?

Jocelyn: It's uh, for subgroups one you can pick representatives and just multiply them and then your answer will be the coset that contains the product.

I: Right, but that is not the original definition of this.

Jocelyn: Why not?

I: Do you remember what the original definition is?

Jocelyn: Uh, I think we had to go through and multiply every single element in the first coset by every single element in the second coset.

I: OK.. The two ways of doing it are the same as you just said. And you have to prove it. Right? Somebody has to prove that, but again there is no immediate reason why that is true I guess. It's just if somebody said to you why is that true?

Jocelyn: You know I don't recall off hand right now. I know it made sense before.

3. Once the student was aware of two different methods, one of which is the definition, the question arises as to whether the alternate method will always work. In the fol-

lowing excerpt, Jeff has just explained the method of representatives (correctly) and the interviewer asks him if this requires normality. At first, possibly influenced by the fact that the question is asked, he indicates that it does. On further probing, he does not use the fact that K is normal (only that it is a subgroup) to justify his belief, and again, perhaps influenced by the interviewer's questions, he decides that normality is not required to ensure that the method of representatives works.

I: OK. How come you can do that? How can you take one of these out of each of these cosets and just take the product of those two instead of doing the longer form, how come you get that choice?

Jeff: Because, um, it has to do with the fact that K is normal, and if, if it's like, you're taking an element from that quotient group, and multiplying it by, if you're, if you're going to choose just the, if you're going to make it, if you're going to pick a representation of that certain set and multiply it by K you are always going to get that set back. So rather than multiply (1 2) times K, then this times that, you could just take a representative that gives you the, another, that same, in the case of the identity, it'll give you back that same set or would give you some different set by multiplying your cycles together.

I: So how are you using the idea that K is normal or do you really need to have a normal subgroup to do this?

Jeff: Commutativity for example holds and all kinds of things like that that makes this necessary in order to compute cycles. I would think.

I: I'm not — I just don't understand what you said. Could you try to explain that for me?

Jeff: Since K is normal you are able to pick a representation of a certain set because if you multiply that times K again you are going to get back the original, you're going to get back the set that you were represented it by, with (1 2).

I: And that doesn't hold true for a subgroup in general? It only holds true a normal subgroup? If I take subgroup and I pull an element out, and I multiply it. . . or any coset, maybe. . .

Jeff: Well, it would hold for any coset. I think. Yeah it would hold for any coset because the definition, hmm, cosets,. . . is if it's an element of G times the entire coset would return that coset again.

I: OK, so does it have to be normal for that, or. . . ?

Jeff: No.

I: OK. So let's go back to the original question, I've gotten a little lost now. So the reason you are able to pick just one representative element out of each coset and multiply those to get an answer is why again?

Jeff: Because if, if you take an element from the coset, and multiply it by a group, an element in G , you are going to get back that same coset, so instead of using (to self) I think that's what... (aloud) Instead of using the whole entire coset you only need to choose one because, the definition of cosets state that an element, a coset, an element in a coset times a group, an element of G, will return that same coset.

I: OK, so when you started by saying it was because K was normal, does that, is that in the picture or not?

Jeff: No. Says it works for any coset in general.

4. In addition to realizing that the method of representatives does not always work, there is the question of how it could fail, that is, the issue of independence of the representative. The only time this came up was in the interview of Kristen. She is working with the "half-way" definition of coset multiplication in which you take a representative of the first coset and multiply it by everything in the second (in spite of the interviewer's prompt). The issue, however, is the same as in the method of representatives.

Kristen: So when you take an element out of a coset and you multiply it by a coset, you just get another coset.

I: Right. So if I took a different element out of this coset and a different element out of this, would I get a different coset?

Kristen: Well, you shouldn't.

I: You shouldn't? Why not?

Kristen: Let's say I took the second element of $(1\ 2)K$ and multiply it by $(1\ 3)K$. OK. Now you're saying take the third element out of $(1\ 2)K$ and multiply it by $(1\ 3)K$. Will the second element times $(1\ 3)K$ equals the third element times $(1\ 3)K$. Is that what you're asking me?

I: I was asking you about being unique, because you kept using the word 'unique'.

Kristen: Um

I: What do you expect?

Kristen: I expect that they would be equal.

I: No matter which one I take here and which one I take here?

Kristen: Well if you take, no if you take one element out of (1 2)K multiply it by the whole set and that set would equal any of the elements in (1 2)K times the set.

I: OK. Why is that true? Well even before I ask why is that true, I think you just said something different than what you said a minute ago. Before you said take one out of here and one out of here.

Kristen: No

I: Then I thought I just heard you say take one out of here and all of this one

Kristen: Right

I: What are you saying. Clarify that for us.

Kristen: If you take one element out of this coset (1 2)K multiply it by the WHOLE coset of (1 3)K, that should equal any other element of (1 2)K multiplied by the whole coset of (1 3)K.

Finally, there were students who were able to coordinate the three schemas, and could go ahead and give a reasonable proof.

I: That's, yeah that shows me your method. Now why can you do this, as you'll remember we defined another way of multiplying the cosets at the beginning and you are using a shorter method here, why do we know that you can use the shorter method?

Thomas: Because we've shown that K is a normal subgroup.

I: OK. And why does normal allow us to do that?

Thomas: Um, back to this definition here, being normal that if you have an element aH it can also be written as Ha, so then when you are multiplying, like if you have aH, the coset aH with the coset bH, you can regroup through associativity you can write it as $a(Hb)$ grouping like that, H and then this here can be written since we know that we've got this property here um, you can write that as $a(bH)H$ and then again through associativity you can regroup it and you get $(ab)(HH)$, then with these two here having the identity in it, you are going to be multiplying it and get the same thing, so HH is just H, so you get it's abH.

6.4 Performance

We now turn to a summary of the performance of the student responses regarding quotient groups.

6.4.1 Test 2, Question 6(b).

Of the 31 students, 26 chose a normal subgroup correctly in part (a). Four chose a subset that was closed with respect to cycle structure but was not a subgroup. All 30 of these students formed a set of cosets and appeared to be thinking of cosets as objects in this construction. Of these, 22 chose a normal subgroup in part (a) and then listed the cosets correctly, although one of them did not eliminate duplicates. Two of the students who had chosen a set in part (a) that was not a normal subgroup, did seem to be thinking about the cosets reasonably. Another 2 of these 30 students merely said that S_3/N was the set of all cosets but did not compute anything. There were 4 students, including the other 2 who had not chosen a normal subgroup correctly, who made the error of computing the cosets $xN,\ x \in N$ rather than $x \in S_3$. The remaining student appeared to be lost on this question.

Of the 22 students who were working with a normal subgroup and had listed the cosets correctly, 9 identified it as $\mathcal{Z}_2$. Of these, 4 mentioned the operation explicitly and one other was clearly working with a Cayley table. The other 4 just referred to the set $\mathcal{Z}_2$ with no mention of an operation.

For 2 of these 22 students, the quotient group was identified in terms of the set of cosets and Cayley table for the operation, but it was not related to any familiar group.

For the remaining 11 students, the only identification given was the set of cosets.

6.4.2 Test 2, Question 7(a).

A total of 28 students listed the cosets, mentioned or used the operation and computed the Cayley table correctly. The other incomplete or incorrect responses were given by 3 students.

6.4.3 First interview, Question 5.

Of the 24 students interviewed, 15 responded to the request to compute the operation table for S_4/K by computing the set of cosets and calculating products. In addition, 1 student did so after a brief prompt. For the most part their calculations were correct although some struggled with the details and took a long time to complete a calculation. Another student made an error in computing the cosets but was able to correct it in the course of calculating products. There were 2 students who tended to think only about the set of cosets and not the operation and 1 student who appeared to have trouble with forming a set of sets. There were 2 students who wanted to calculate the cosets of S_4 by elements of K rather than the reverse, one of whom had little difficulty in recognizing and correcting his error. There was 1 student who appeared to be totally confused, and we consider the remaining response below.

The method of representatives for coset product was preferred by 18 of the 24 students. Another student needed some prompting for any method at all but then chose the representative method. Of these 19 students, 17 gave an acceptable explanation of the method. There were 4 students who chose the set product method and their explanations were, of course, reasonable (since it is hard to see how to refer to this method without explaining it). The remaining student never mentioned coset products.

One response deserves special consideration. In the excerpt below, Karoline is able to work her way through the problem calling on past experience, and demonstrates a decent understanding of coset representatives. However, in the process, she expresses many times her conviction of her own ignorance. We present the following excerpt in two ways. First we give it in the form in which the words were spoken. But then, we divide the original excerpt into two parts. The first part has all comments that do not relate to the content of the discussion removed and the second part is a list of what was removed — mostly Karoline's very pessimistic evaluation of her knowledge. It seems that hiding behind Karoline's low opinion of her skills is a fair amount of understanding.

Here is the original excerpt.

I: OK, here is next question.

Karoline: Compute the operation table for K_4, er, S_4 mod K? Do you use the same K in S that was there?

I: Yes.

Karoline: OK. K was one... one, two, three, four; one, three, two, four; and one, four, three, two. OK. Compute the operation table? Have we ever done an operation table? It doesn't sound familiar. I don't know what an operation table — I've never heard it like that before — 'an operation table'. I know this is equal to S_4 mod K. Are they talking about when we did it for our last test and we did it for multiplication and addition, those type. So, we did... how did we do that? I don't remember.

I: Do you remember using some representatives to find the product of two cosets?

Karoline: I remember doing it, but I can't remember! We did it... I remember substituting in ones and twos and threes and zeroes and things after to see if the tables were the same. OK, let's see. How did we do this? I don't know. I can't remember! I remember doing them, but I can't remember how to do them! Huh, break does this to me. I totally forgot. How did I do that? And that was easy part. I can't remember. I know when I checked it to see if it was isomorphic to other groups like this. I can't remember. I remember we'd have something like... I remember we had something in Z's, like Z_9 and I set up zero through eight on each side and modded it by nine. But I don't remember doing anything like this. S_4 with K... I don't know. Would you, you wouldn't write every element of S_4 out... Could use quotient groups? Quotient groups if we have. Hmm. Goodness gracious. Could take... set of four elements in K. So, I don't remember how to do this. This may have been the one on the test that I didn't do — didn't get. I don't remember. S... if quotient groups... I can't rethink of how to do it for this. I don't know. If you take, this one of K and you know... I can't remember. Would you find, I don't know, if you'd find cosets or something for them. That's the only thing I can think of, 'cause, well, all the cosets of S_4, I think they're six of them, aren't there? I think you have, you have your identity, and you have one, two; one, three; one, four, and then I remember we had something like two, th — two, three, four and two, four, three, I think are the other two. 'Cause I think when we did S_4 cosets we only, we had six, I believe. That'd be the cosets for S_4, or the representatives. Those are the representatives of the groups. Then, if we multiplied them... mod K? That's (??) hung up on. If we write these as a table, though. I'll never remember... this would have to be under multiplication, 'cause I don't remember this closed under addition. If we did this... this is what I had trouble on. It's the only thing I can think of how to do it. I know these would be the same. And then I'd have to go through and figure out all these cosets in order to find that, I'd put one-four times one-three. Let's try it: one-four times one-three would get one to four, four to four, so one to four, two to two, two to two, I can't do it like that, I have to do it... So, we, four-four-four, two-two-two, (mumble). So, it would be four cycle four. So, would it the cycle one-four-three? One goes to four, four goes three, three goes to one. So, it'd be the cycle one-four-three. Then I would look back in these cosets whichever they equal and find the sub, or the coset that has one-four-three in it and put it in there. Which I don't know, right of hand. And then, after doing that I'd have that table. Do you want me to do the whole thing, or...? The whole table?

I: Um, just a couple of them.

Karoline: Just a couple? So, all this would be one-four-three, the cycle, but I don't know what... one-four-three, so it would be, oh, one-two-four. My cycles are different: one-two-three, one-two-four. So, I don't need these. I would substitute them in for these. You don't have the identity on that? OK. One-two, one-three, one-four, one-two-three, one-two-four, one-two-three, one-two-four. So, one-four-three would be the same as one-two-four. And, then let's take one-two-three times one-four which would be four is not the same spot. Let's try this: one goes to two, two goes to two, so one goes to two. One goes to two, two goes to two, so one goes to two. Two goes to three, three goes to three, so two goes to three. Wait. Three? Four goes to four, four goes to one, can't do that. This would be four-two-three-one times two-three-one? I think. So, four-two-one-two, three-three-three-one, two-two-two-three... Wait. Four-four-four, two-two-two-three, three-three-three-one, four-one-one-two. So, one goes to four, four goes to two, two goes to three, three goes to one. So, it would be the cycle one-four-two-three, which is one-two. OK. Yeah, it's taking forever to do.

Now we have the excerpt with all non-content comments removed.

I: OK, here is next question.

Karoline: Compute the operation table for K_4, er, S_4 mod K? Do you use the same K in S that was there?

I: Yes.

Karoline: OK. K was one... one, two, three, four; one, three, two, four; and one, four, three, two. OK. I know this is equal to S_4 mod K.

I: Do you remember using some representatives to find the product of two cosets?

Karoline: S_4 with K... I don't know. Would you, you wouldn't write every element of S_4 out... Could use quotient groups? Quotient groups if we have. Could take... set of four elements in K. you'd find cosets or something for them. well, all the cosets of S_4, I think they're six of them, aren't there? I think you have, you have your identity, and you have one, two; one, three; one, four, and then I remember we had something like two, th — two, three, four and two, four, three, I think are the other two. 'Cause I think when we did S_4 cosets we only, we had six, I believe. That'd be the cosets for S_4, or the representatives. Those are the representatives of the groups. Then, if we multiplied them... mod K? this would have to be under multiplication, 'cause I don't remember this closed under addition. And then I'd have to go through and figure out all these cosets in order to find that, I'd put one-four times one-three. Let's try it: one-four times one-three would get one to four, four to four, so one to four, two to two, two to two, I can't do it like that, I have to do it... So, we, four-four-four, two-two-two, (mumble). So, it would be four cycle four. So, would it the cycle one-four-three? One goes to four, four goes three, three goes to one. So, it'd be the cycle one-four-three. Then I would look back in these cosets whichever they equal and find the sub, or the coset that has one-four-three in it and put it in there. Which I don't know, right of hand. And then, after doing that I'd have that table. Do you want me to do the whole thing, or...? The whole table?

I: Um, just a couple of them.

Karoline: Just a couple? So, all this would be one-four-three, the cycle... one-four-three, so it would be, oh, one-two-four. My cycles are different: one-two-three, one-two-four. So, I don't need these. I would substitute them in for these. You don't have the identity on that? OK. One-two, one-three, one-four, one-two-three, one-two-four, one-two-three, one-two-four. So, one-four-three would be the same as one-two-four. And, then let's take one-two-three times one-four which would be four is not the same spot. Let's try this: one goes to two, two goes to two, so one goes to two. One goes to two, two goes to two, so one goes to two. Two goes to three, three goes to three, so two goes to three. Wait. Three? Four goes to four, four goes to one, can't do that. This would be four-two-three-one times two-three-one? I think. So, four-two-one-two, three-three-three-one, two-two-two-three... Wait. Four-four-four, two-two-two-three, three-three-three-one, four-one-one-two. So, one goes to four, four goes to two, two goes to three, three goes to one. So, it would be the cycle one-four-two-three, which is one-two. OK.

And, for the record, we list those comments of Karoline that were removed.

Oh, goodness.

Compute the operation table? Have we ever done an operation table? It doesn't sound familiar. I don't know what an operation table — I've never heard it like that before — 'an operation table'.

Are they talking about when we did it for our last test and we did it for multiplication and addition, those type. So, we did... how did we do that? I don't remember.

I remember doing it, but I can't remember! We did it...

I remember substituting in ones and twos and threes and zeroes and things after to see if the tables were the same. OK, let's see. How did we do this? I don't know. I can't remember! I remember doing them, but I can't remember how to do them! Huh, break does this to me. I totally forgot. How did I do that? And that was easy part. I can't remember. I know when I checked it to see if it was isomorphic to other groups like this. I can't remember. I remember we'd have something like... I remember we had something in Z's, like Z_9 and I set up zero through eight on each side and modded it by nine. But I don't remember doing anything like this.

Hmm. Goodness gracious.

So, I don't remember how to do this. This may have been the one on the test that I didn't do — didn't get. I don't remember. S... if quotient groups... I can't rethink of how to do it for this. I don't know. If you take, this one of K and you know... I can't remember. Would you find, I don't know, if

That's the only thing I can think of, 'cause,

That's (??) hung up on. If we write these as a table, though. I'll never remember...

If we did this... this is what I had trouble on. It's the only thing I can think of how to do it. I know these would be the same.

but I don't know what.

Yeah, it's taking forever to do.

6.4.4 First interview, Question 6.

Of the 24 students interviewed, 10 mentioned normality as the justification and 3 of these said things that suggested that there was some understanding that normality was a condition which, if satisfied, guaranteed that the method of representatives would work, and that this guarantee had to be proved. There were 2 students who did not show much knowledge on this question and the remaining 12 students gave scattered responses which were not followed up satisfactorily by the interviewer.

7 Epistemological Analyses Reconsidered

7.1 Cosets

For the most part, the action/process/object analysis that came from Dubinsky et al. (1994) is consistent with much of our data. We may take the current study as supporting this particular epistemological analysis.

There are a few comments to make and points which can be added to the description offered by Dubinsky et al. (1994) as a result of this study.

1. There may be situations in which an action conception is very difficult, if not impossible to use. In another context, we pointed out that, almost by definition, it is necessary to have a process conception of a situation before one can think about anything having to do with infinity. This is for the simple reason that one cannot actually perform infinitely many steps.

   In the case of cosets, it is hard to see what could be an action conception of a generic coset. Here, the necessity to use an elision (i.e., representing the generic coset as gH instead of writing out actual elements like $\{gh_1, gh_2, ...gh_n\}$) seems to force a process conception. Another feature of an action conception is the requirement, not only of actually performing all manipulations that are involved, but doing so according to very explicit formulas or recipes. This works fine, for example in cosets of $\mathcal{Z}_n$, but students in this study did not have (or make up) formulas to use in working with S_n. Presumably, an action conception here would consist of writing out all of the elements in any given situation, but again, this seems difficult in case n is very large.

2. De-encapsulation appears to play a major role in working with cosets. While discussing

the proof of Lagrange's theorem, when checking the normality of the center of a group, and in thinking about and forming the product of two cosets, students very often appeared to be going back and forth between process and object conceptions of cosets and this seemed to be an essential part of their understanding.

7.2 Normality

The observations described here generally appear to support the epistemological analysis described in the beginning of this section, and they allow us to elucidate on it.

We may give a global description of understanding the concept of normality as the coordination of three general schemas: subgroup, coset and the general notion of an object having a property. A number of difficulties for students arise and it is not always clear whether they are caused by a weakness in one of these schemas or by the problem of coordination.

For the subgroup schema there appears to be a special importance in the distinction between a subgroup as a subset of a larger group and as a group in its own right. With the coset schema it appears to be necessary for students to be able to have both process and object conceptions of coset formation. The most serious difficulty students had was in confusing normality with commutativity. This appears to be related both to an inability to encapsulate the coset formation process and to think of normality as a property that a subgroup has as a group, independent of the containing group.

With these points in mind, we can offer the following more detailed description of a genetic decomposition of the concept of normality. In this description, we use the relation $gH = Hg$ which is the one most often chosen by students. There is little difference if one of the other relations leading to an equivalent definition is used, such as $gHg^{-1} \subset H$.

1. There is a stage which we might refer to as *pre-normal* in which the individual has no idea what the concept means and prompting does not help. Sometimes an individual at this level might be able to think of normality only in terms of its meaning in other contexts such as geometry or psychology.

2. The earliest notion of normality that we observed is that it is a property that the subgroup has, independent of the containing group. This is a manifestation of the subgroup confusion we just described.

3. The subject engages in thinking that leads to differentiating between actions in H and G and coordinating these to arrive at normality as an action applied to an object H *as a subgroup* of a group G. This action has its own genetic decomposition which we can describe as follows.

   (a) The first level is as an action on elements of H and G with little control. This might be expressed as $gh = hg$, $\forall g \in G, h \in H$. This can become another road to confusing normality with commutativity or it might just stay in this unstructured form.

   (b) The first step from this beginning to a construction of the concept of normality is to differentiate between the elements of G and H. A student at this level might refer to $gh = hg$, $\forall h \in H$ and consider g separately or not at all. The student's conception at this point is still as an action.

   (c) Once the elements of G and H are differentiated, the action relating to H can be interiorized into a process and encapsulated (as in coset formation) to reach a statement like $gH = Hg$. Often, although the statement is understood as a process involving the elements of H and a fixed element g, there can be a delay before the student is ready to iterate g over G. This may be the result of not encapsulating the equality $gH = Hg$.

(d) Finally, the relation $gH = Hg$ is encapsulated as a proposition which is an object that can depend on the parameter g. This parameter is iterated over G to form the process, $gH = Hg$, $\forall g \in G$. The process is encapsulated to the assertion that each left coset is equal to the corresponding right coset.

7.3 Quotient groups

The epistemological analysis described the quotient group concept as the coordination of three schemas — coset, binary operation, and group. This view seems to be useful for talking about a great deal of what was observed in this study and our observations permit us to expand on it. In our refinement of the epistemological analysis, we will include issues such as the role of normality and independence of representation .

We consider separately how several specific constructions are taken from the schemas of coset, group, and binary operation, and applied to quotient groups.

7.3.1 Set of cosets

In order to form a set of cosets, the student must be able to think about cosets as objects. Constructing individual elements of this set requires that these objects be de-encapsulated to the process of forming the cosets. Constructing the set of cosets itself requires first the encapsulation of the process of coset formation for each coset and then the process of iterating over the elements of the original group and forming the coset for each.

One error that can appear at this point is to reverse the process and consider the cosets of the group by elements of the subgroup. A related error is to use only the cosets of elements of the subgroup and not the entire group.

Another possible error is to go immediately to a representative for each coset and work with it to the exclusion of the coset. This is most likely in situations such as a coset of $\mathcal{Z}$ by a subgroup $n\mathcal{Z}$ where the representatives $0, 2, \ldots, n-1$ are used and the operation is not a product of cosets but addition mod n. If the coset is ignored in this case, then we have examples in which the student gets the right answers, but may not have much understanding of cosets.

7.3.2 Coset product

The schema for binary operation applied to cosets includes two processes: the method of representations and the method of set product. The quotient group schema will coordinate these processes through the concept of normality.

The set product method is a process to object to process construction very similar to other binary operations such as composition of functions. The cosets to be multiplied are considered to be objects which are de-encapsulated to their processes of formation. These are coordinated with the group operation to form the process of multiplying every element of the first coset by every element of the second. Finally, this process is encapsulated to form the new coset.

One aspect of encapsulating the set product process to an object is consideration of the question of closure. Is the set product of two cosets again a coset? This is an action on the set product interpreted as an object.

The other process is the method of representatives. This process was described explicitly by several students in words like the following.

> ¿*From each of the two cosets to be multiplied, select an element as a representative.* (To do this, the cosets must be considered both as objects and processes.) *Then apply the group operation to multiply these two elements (in the appropriate order) to get an element of the group. Iterate over each of the cosets and check the element for membership in each.* (Again, the cosets must be interpreted both as objects and as processes.) *The coset which contains the element is taken as the result of the product of the two original cosets.*

Again, with the method of representatives, there is the possible error of working only with the representatives and ignoring the cosets. Thus, after multiplying the two elements, the student might look, not for the coset which contains the result, but to see if the result is equal to one of the representatives. This will not work in general but, in the situation of $\mathcal{Z}/n\mathcal{Z}$ where addition mod n is introduced arbitrarily, the student will get correct answers.

7.3.3 Coordination through normality

Based on the different reactions of students that were observed in response to the First Interview, Question 6, we can postulate a developmental sequence for dealing with the formation of cosets and coset products. For each of the following steps we saw examples of students who had reached that point and examples of those who had not.

1. First there must be an awareness that there are two different methods for multiplying cosets and each may or may not "work". For the set product method the resulting set may not be a coset and for the representative method, the result may depend on which elements are chosen as representatives.

2. Second, there is the question of whether both methods give the same answer, and the student must realize that something must be proved in order to settle this question. Several students simply mentioned normality as sort of a mental button to push, indicating that if the subgroup is normal, then everything will work out fine. Other students advanced to the point of realizing that it is possible to prove various relations between normality and the product of cosets "working".

3. Finally, the proof for both methods giving the same result is a fairly simple coordination of processes involving associativity of the coset product and the equality of left and right cosets.

7.3.4 Group properties of the quotient

Once the set of cosets and coset product schemas are formed and thematized to objects, an action on them is to form the binary operation pair and check the group properties. The concept of normality reappears as the decisive criterion for the binary operation to satisfy these properties.

7.3.5 Identifying the quotient group

Once the student's quotient group schema has developed to the point of being able to construct a particular quotient group, then this can be coordinated with the group schema to form the action of identifying the quotient.

8 Learning Outcomes

In this section we describe the learning outcomes that seem to be reflected in the performances described for the various concepts in Section 4.

8.1 Cosets

The data suggests that a fairly high percentage of the students who took the experimental course had constructed a fairly strong understanding of cosets. They had both process and object conceptions and could transfer their thinking back and forth between the two interpretations. In most questions the number of students giving satisfactory responses was two-thirds or more. Two exceptions were the number of students who knew without prompting that the proof of Lagrange's theorem involved cosets (9 of 17) and the number of students who could actually give such a proof, possibly with some prompting (7 of 17).

Students from the regular course were only included in the second interview and they did not, on the whole, indicate a very strong understanding of cosets. The number of students who gave satisfactory responses was never more than one third. Very often, the student being interviewed was not able to say anything, even with prompting, that could be interpreted as indicating any knowledge of the point being considered. This rarely happened with the students from the experimental course.

Certainly this difference could be at least partially explained by the effect of time elapsed after taking the course. Overall, considerably more time elapsed for the students taking the regular course than for the students taking the experimental course. However, 5 of the students took the regular course at the same time as those who took the experimental course. They gave responses as good as those from the experimental course in some cases, but not in others and in no case did they do better as a group.

Another factor which we are unable to consider is the basic mathematical ability of the students. Some of the differences could have been the result of this basic ability rather than what was learned in the course. The only information we have on ability is the grades the students received in their course and since the various courses used entirely different grading schemes, this variable does not seem worth considering.

One might also compare the students in the experimental course with the participants in the study reported in Dubinsky et al. (1994). This latter group had great difficulty with thinking about cosets at all when the context moved from $\mathcal{Z}_n$ to S_n. The students in the present study from the experimental course seemed to be able to work with S_n as an example of a group and think about various cosets of subgroups of S_n.

8.2 Normality

The students who took the experimental course appeared to be developing reasonable understandings of normality at the time. One-half to two-thirds of them were able to deal with questions concerning normality while the course was going on. At the end, almost all of these students performed satisfactorily on final exam questions that involved normality. The question on the first interview, given towards the end of the course, led to reasonable and robust responses from about two-thirds of the students. The confusion with commutativity and difficulty in seeing normality as a property of a subgroup in its relation to the containing group were present in only a small number of students.

There is a side issue connected with the special case of normal subgroups of S_n in which the group of students as a whole tended to prefer the structure theorem rather than a formal definition of normality. It could be that their understanding of normality was entirely contained in the structure theorem. The performance on the final examination and the second interview, however, suggests that this was not the case as students were able to deal with normality in completely general contexts.

The second interview suggests that during the following semester, a fair number of these students still appeared to have a reasonable understanding of normality. Referring to the 17 students' responses to Question 4 on the second interview, no student was completely lacking in any relation to the concept of normality and only four were reduced to only a beginning of, or very partial understanding of this concept. Two students were very close, lacking only the last iteration of g over G, and another six students indicated some concern with controlling g. Five students showed an essentially correct understanding of normality in applying this concept to the center of a group.

This performance is in sharp contrast with that reported in Dubinsky et al. (1994) where similar questions were asked and the report concludes that "…normality…did not appear to be well understood by very many" of the participants.

The performance of the students in the experimental course is also very different from what was seen with the students from the standard course, although the latter group, on average, were more removed in time from when they took the course than were the students in the experimental course. Among the 20 students from the standard course who were interviewed, eight gave responses that did not show any understanding of normality at all and another seven showed only a beginning of a construction of this concept. Three students lacked only the iteration of g over G, one showed some concern with g and one gave an essentially correct definition. Referring to the numbered types of responses in Section 5.3, the distribution was:

Response	Standard	Experimental
1	8	0
2 - 4	7	4
5 - 6	3	2
7	1	6
8 - 9	1	5

8.3 Quotient groups

The quality of understanding of quotient groups by these students is different from what we have observed for cosets and normality as well as the concepts studied in related papers from this data (Asiala et al., 1996, Brown et al., 1996). For each of these we have generally seen that, roughly, two-thirds or more of the students appear to have constructed an understanding of the various components of the concept that is reasonably compatible with the understanding that mathematicians seem to share. This rate of success is maintained for the essential building blocks of the quotient group concept: formation of a set of cosets, product of cosets, and determination of normal subgroups. However, there are aspects of putting it all together on which the rate of success was closer to one-third than two thirds.

One such aspect is the idea that a group consists of *both* a set and a binary operation. In at least one question (Test 2, Question 6(b)), no more than 11 of the 31 students both constructed the set of cosets and considered a binary operation. Because of their performance on subsequent questions it seems that this omission may not be so much an inability to multiply cosets as it is simply not considering the binary operation as part of the object being constructed. On that same question, only 9 of the 31 students completed the problem by identifying the quotient group as $\mathcal{Z}_2$.

We note that on the question of identifying $2\mathcal{Z}/6\mathcal{Z}$ the students did much better than on the similar question concerning groups of permutations.

On the question of normality we see again that although a high percentage of students apparently understood this concept in isolation (as seen in the section on normality) very few students really succeeded in fully understanding the role of this concept in constructing quotient groups.

Finally, we note that none of the instruments used tried to determine explicitly student understanding of group properties as applied to the binary operation on the set of cosets. The only indication of this came from a few isolated examples in which students brought up the question of closure when coset product was defined in terms of multiplication of elements in two sets.

It is perhaps fair to summarize these students' understanding of the concept of quotient group by saying that a very high percentage of the students had completed the necessary preparations for constructing their concept of quotient group, but a much smaller number of the students were able to coordinate their knowledge to develop a fairly complete concept of quotient group.

We can say that, although these students have some distance to go in constructing their understanding of quotient groups, they have made progress and have gone beyond what might be expected from a class of this kind. They have certainly progressed much farther with this concept than did the participants reported in Dubinsky et al. (1994).

The topic of quotient groups has traditionally been a difficult one for students. The results of this study suggest that a constructivist approach could turn out to be the key to overcome this difficulty.

We end this section by referring to an aspect of student understanding that exists, no doubt, in other mathematical concepts as well. It is our observation that students may well understand something, even fairly strongly, and yet, in a given situation, tend to not display that understanding. We have provided one example in which a student's discussion of a mathematical issue is mixed (but not inextricably) with a great deal of self-deprecation. Obviously such an attitudinal issue is important in anyone's development of understanding of mathematical concepts, as well as being important to an instructor's pedagogy.

9 Pedagogical Suggestions and Open Questions

9.1 Cosets

The pedagogical conclusions supported by this study are that the strategy used in the experimental course seems promising and should be continued with further exploration of just what contributes to students learning the concept of cosets.

Although much more would need to be done in order to arrive at a decisive conclusion, the results of this study are consistent with the notion that writing the program `PR` (see Section 4.2) to construct a computer implementation of a generalized product that unifies the concepts of forming the product of two elements, an element and a coset, and two cosets is a major reason why students succeeded in developing their concept of coset. At the very least, the results of this study encourage us to reject the suggestion that the task of writing `PR` presents students with unhelpful obstacles.

In connection with the above discussion, the effect of `PR` in particular and the overall pedagogical approach in general on student understanding of cosets remains an open question. That is, from this study one can say that the students in the experimental class performed satisfactorily and it is possible that this is due to the pedagogical strategies that were employed. It will require further study to determine if this is a reasonable conclusion.

In particular the question of the effect of other factors remains open. For example, to what extent are results reported here affected by time elapsed after studying cosets? How much of the difference is due to variations in students' overall mathematical ability? Studies should be conducted to consider the effect of as many variables as possible.

9.2 Normality

The pedagogical treatment used in the experimental course appeared to be effective in helping a reasonable portion of the students construct a good concept of normality that was still available to many of them after several months. It seems that only refinements focusing on specific issues are necessary to bring along even more students. These improvements can probably be handled with minor changes in the computer activities with corresponding adjustments in the text and classroom discussions.

New activities could include examples of subsets of S_n which satisfy the structure theorem, but are not subgroups; examples of two subgroups that are isomorphic as groups but one is normal and one is not; and activities that distinguish between normality and commutativity.

The most important question is whether the results obtained with the experimental course can be replicated in situations with varying characteristics, or whether they were due to special circumstances in this particular situation.

Another important open question is the relationship between understanding normality and the time that has elapsed since the course was taken. We observed very strong differences in understanding normality some time after the course was completed between students who took the standard course and students who took the experimental course. The time elapsed was different for the two groups, however, and we do not know how much of the difference to attribute to the different pedagogy and how much to attribute to the differences in time elapsed.

Future studies may also attempt to find out more about the definition of normality that students carry in their minds. At least some students may think that there is a "real" definition of normality. Interview questions focusing on this aspect would be useful in revealing any misconceptions.

9.3 Quotient groups

Generally speaking, our conclusion from this study is that the pedagogical approach that was used for the experimental course should be kept but that a number of additions are called for.

Most important among these additions would be activities aimed at helping students coordinate all of the components of the construction of a quotient group. These activities could include having the students write a computer function which accepts a group (as a pair made up of a set and a binary operation) and a subset of that group. This function would check the subset for being a normal subgroup and if it were, then the function would construct the set of cosets and the operation of coset product and test this new pair for satisfying the group properties.

A variation of this function could be used to generate and study examples of what goes wrong when the subgroup is not normal. An auxiliary function could compare the results of doing coset products by representatives and by set product. Students could then check many examples and note how the results are related to the property of normality.

The course should put greater emphasis on the various proofs surrounding the property of normality, especially the fact that it implies that the method of representatives "works". Perhaps a normal subgroup could even be defined as a subgroup whose cosets form a group, and move to reconcile the two processes of forming the coset product from there.

Finally, a way must be found to increase the students' awareness that a group is not just a set but a set together with a binary operation. Perhaps this can be achieved through activities on which several different operations are used with the same set.

The most important open question is whether it will be possible to go beyond the progress reported here and help a greater number of students learn how to construct quotient groups and to understand what they have built.

Future studies should investigate students' understanding of the idea of multiplying cosets by representatives being independent of the choice of representatives. Also important is the question of students' understanding of what it means for the set of cosets with coset product to satisfy the group axioms. Finally, it would be interesting to look further into the "half-way" definition of coset product in which a representative of one coset is multiplied by every element of the other coset.

10 References

Asiala, Mark, Brown, Anne, DeVries, David J., Dubinsky, Ed, Mathews, David & Thomas, Karen (1996). A framework for research and development in undergraduate mathematics education. *Research in Collegiate Mathematics Education, 2*, 1–32.

Asiala, Mark, Brown, Anne, Kleiman, Jennifer & Mathews, David (1996). *The Development of students' understanding of permutations and symmetries.* Manuscript submitted for publication.

Brown, Anne (1990). Writing to learn and communicate mathematics: An assignment in abstract algebra. In A. Sterrett (Ed.), *Using Writing to Teach Mathematics* (pp. 131–133). Washington, DC: The Mathematical Association of America.

Brown, Anne, DeVries, David, Dubinsky, Ed & Thomas, Karen (1997). Binary operations, groups and subgroups. *Journal of Mathematical Behavior*, this issue.

Clark, Julie, Hemenway, Clare, St. John, Denny, Tolias, Georgia & Vakil, Roozbeh (In press). Student attitudes toward abstract algebra. *PRIMUS*.

Czerwinski, Ralph (1994). A writing assignment in abstract algebra. *PRIMUS, 4*, 117–124.

Dautermann, Jennie (1992) *Using ISETL 3.0: A language for learning mathematics.* St. Paul, MN: West Publishing Company.

Dubinsky, Ed (1995). ISETL: A programming language for learning mathematics. *Communications in Pure and Applied Mathematics, 48*, 1027–1051.

Dubinsky, Ed, Dautermann, Jennie, Leron, Uri & Zazkis, Rina (1994). On learning fundamental concepts of group theory. *Educational Studies in Mathematics, 27*, 267–305.

Dubinsky, Ed & Leron, Uri (1994). *Learning Abstract Algebra with ISETL.* New York: Springer-Verlag.

Freedman, Haya (1983). A way of teaching abstract algebra. *The American Mathematical Monthly, 90*, 641–644.

Gallian, Joseph (1976). Computers in group theory. *Mathematics Magazine, 49*, 69–73.

Geissinger, Ladnor (1989). *Exploring Small Groups* (Ver. 1.2B). San Diego: Harcourt Brace Jovanovich.

Hart, Eric (1994). Analysis of the proof-writing performances of expert and novice students in elementary group theory. In J. Kaput & E. Dubinsky (Eds.), *Research issues in undergraduate mathematics learning, MAA notes 33* (pp. 49–62). Washington, DC: Mathematical Association of America.

Hodgson, Ted (1995). Reflections on the use of technology in the mathematics classroom. *PRIMUS, 5*, 178–191.

Kiltinen, John & Mansfield, Lisa (1990). A writing fellows program meets an abstract algebra class: The instructor's and the fellow's perspectives. In A. Sterrett (Ed.), *Using Writing to Teach Mathematics* (pp. 93–98). Washington, DC: Mathematical Association of America.

Kleiner, Israel (1986). The evolution of group theory: A brief survey. *Mathematics Magazine, 59*, 195–215.

Leganza, Krystina (1995). Writing assignments in an abstract algebra course. *Humanistic Mathematics Network Journal, 11*, 29–32.

Leron, Uri & Dubinsky, Ed (1995). An abstract algebra story. *The American Mathematical Monthly, 102*, 227–242.

Leron, Uri, Hazzan, Orit & Zazkis, Rina (1995). Learning group isomorphism: a crossroads of many concepts. *Educational Studies in Mathematics, 29*, 153–174.

Makiw, George (1996). Computing in abstract algebra. *The College Mathematics Journal, 27*, 136–142.

Nicholson, Julia (1993). The development and understanding of the concept of quotient group. *Historia Mathematica, 20*, 68–88.

O'Bryan, John & Sherman, Gary (1992). Undergraduates, CAYLEY, and mathematics. *PRIMUS, 2*, 289–308.

Piaget, Jean (1978). *Success and understanding* (A. Pomerans, Trans.). Cambridge, MA: Harvard University Press

Selden, Annie & Selden, John (1987). Errors and misconceptions in college level theorem proving. *Proceedings of the Second International Seminar on Misconceptions and Educational Strategies in Science and Mathematics* (pp. 457-470). New York: Cornell University.

Zazkis, Rina & Dubinsky, Ed (1996). Dihedral Groups: A tale of two interpretations. *Research in Collegiate Mathematics Education, 2*, 61–82.

Zazkis, Rina, Dubinsky, Ed & Dautermann, Jennie (1996). Coordinating visual and analytic strategies: A study of students' understandings of the group D_4. *Journal for Research in Mathematics Education, 27*, 435–457.

A Instruments

Following is a list of the various instruments used in this study. Together with each question, we give a brief description of our expectations, before administering it, of what we would learn from the question.

Only the first of these questions is explicitly about cosets. The others involve cosets as they are used in Lagrange's theorem, the issue of normality and the construction of quotients. In some cases, students may only use properties of cosets and thus show something of what they know, but if this knowledge is very much developed, their responses might not tell us much about the construction of this knowledge. On the other hand, most of the details one would give in answering these questions fully would involve actions on cosets, so that we may learn something about students object conception of cosets, and de-encapsulation of these objects to compute specific cosets or the product of two cosets.

A.1 Test 1, Question 5.

Find a subgroup of S_4 that is the same as S_3. Calculate the left cosets of your subgroup.

Assuming that a subgroup is constructed, this item should tell us about the team's conception of coset formation. It will also tell us about the team's ability to work with cosets in the non-commutative case and where, unlike in $\mathcal{Z}$, $\mathcal{Z}_n$, no formulas are available.

A.2 Test 1, Question 7.

1. State Lagrange's Theorem on the order of a subgroup.

2. Give an overview of the proof of Lagrange's Theorem.

3. Give as detailed a proof of the theorem as you can.

In the overview of the proof we will see if the team can think of a coset as an object whose properties can be studied. In the proof we can see if the team can de-encapsulate this object and work with the process to perform calculations.

The team's ability to present a complete proof and deal with all of the details will tell us something about overall understanding of this part of group theory.

A.3 Test 2, Question 4(a).

There are many conditions that are equivalent to a subgroup H of a group G being normal. One is,

for all $g \in G$, it is the case that $gHg^{-1} \subset H$.

Give another condition for normality and show that it is equivalent to this statement.

In the student's dealing with equivalent conditions for normality, we may see something about how he or she thinks of a coset and also perhaps the way in which manipulations with cosets are performed.

It is not necessary that responses to this question will tell us much about student's understanding of normality beyond the idea that it is a property which a subgroup might have. One could answer the question, for example, by simply finding a condition equivalent to the given one, such as $\forall g \in G,\ h \in H,\ ghg^{-1} \in H$. If the student does write something about normality in the response, we could learn something.

Initially it was our thought that work on the proof of equivalence has to do with the student's understanding of coset and not of normality. Analysis of exams revealed that some students made statements about normality in their proofs that pointed to misconceptions that they might have about the subject. Therefore analysis of these responses have also been included in the results.

A.4 Test 2, Question 4(b).

Let $f : G_1 \longrightarrow G_2$ be a homomorphism. Prove or give a counterexample to the following statement.

The kernel of f is a normal subgroup of G_1.

Again if, as is not unlikely, the student uses one of the definitions given in the previous problem, we are likely to learn more about the student's understanding of coset and homomorphism rather than normality.

It is possible that this question and the previous one will at least identify any students who have no understanding of the notion that normality is a property that a subgroup may or may not possess and this can be checked in specific situations.

A.5 Test 2, Question 6.

Let S_3 be the group of permutations of three objects.

a. Find a normal subgroup N of S_3.

b. Identify the quotient group S_3/N.

Assuming that a normal subgroup is found and that the student chooses a proper subgroup, then the construction of the quotient group could show us something about the student's understanding of cosets. If the basic idea of coset is understood, the calculations of cosets for a particular example is straightforward, so it may not indicate very much beyond a first understanding of the concept, strong enough to be functional in the context of groups of permutations.

Both the selection of the subgroup and any justification for selecting it will tell us something about the student's understanding of normality.

This question is asking the students directly for their knowledge of quotient groups. An object conception of cosets is necessary to form the requisite set of cosets.

A.6 Test 2, Question 7(a).

Let $\mathcal{Z}$ be the ring of integers. Describe $2\mathcal{Z}/6\mathcal{Z}$.

Responses on this question will tell us about the student's ability to work with groups in the context of a ring.

A.7 Definitions on the Final.

What does it mean for a subgroup of a group to be normal?

This will give us the student's report on her or his understanding of the term.

A.8 Proofs on the Final.

Students were given 11 statements and asked to give a formal statement and proof for any two of them. The following question had to do with normal subgroups.

> The image under an onto homomorphism of a normal subgroup is again a normal subgroup.

The proof could require the student to work directly with individual group elements as well as cosets as objects and this would show us something about her or his understanding of cosets.

The proof requires the student to not only know at least one definition of normality, but also to be able to unpack it and hence we might learn something about what the concept means to the student.

A.9 First interview, Question 2.

In the group S_4, consider the subgroup

$$K = \{(1), (1\ 2)(3\ 4), (1\ 4)(3\ 2), (1\ 3)(2\ 4)\}$$

How would you go about deciding if K is normal?

In using one of the equivalent definitions of normality, the student may display the nature of her or his understanding of cosets.

It is expected that the student might respond by using one of the equivalent definitions of normality or by applying the structure theorem. In either case, the student is prompted to see if he or she is able to give the other response as well. Finally the student is asked to compare the two methods.

We don't learn much about the student's understanding of normality from the use of the structure theorem, but we do when another method is tried and when the student compares the two methods.

A.10 First interview, Question 5.

Compute the operation table for S_4/K (Here K is the same as in the previous question).

The construction of the operation table could show us something about the student's understanding of cosets.

Students are prompted to use both the method of representatives and coset multiplication, and are asked to explain their first choice.

This question is mainly about the binary operation on cosets, which was not considered in the section on cosets. Hence the students' responses will tell us about their understanding of the basic calculations in one very important category of examples of quotient groups — groups of permutations. Also, we can learn something about the students' understanding of quotient groups from their reaction to the question as phrased. Do they react to S_3/K by thinking about a set of cosets and the binary operation of coset product, or do they require some prompting to go from the general idea of quotient group to this particular computation?

A.11 First interview, Question 6.

What gives you the right to use the more convenient method of representatives?

This question is an attempt to see if the student can coordinate the method of representatives, coset multiplication and normality in the context of constructing a quotient group.

A.12 Second interview, Question 3.

1. What do you remember about Lagrange's theorem?

2. What is the statement of the theorem? If the student did not remember, then the interviewer prompted, first by suggesting that it had to do with subgroups and groups and then by adding that it involved orders.

3. How does one prove it?

This is essentially a repetition of Test 1, Question 7, but this time for an individual and in the setting of an interview.

For those who can reconstruct their knowledge enough to work with cosets, the discussion of the proof of Lagrange's theorem should indicate something about the student's ability to coordinate object and process conceptions of cosets.

A.13 Second interview, Question 4.

The student was given the definition of the center of a group. (This idea had been considered only very briefly in the course.) The student was asked to show that the center is a subgroup, and then to show that it is normal.

Once again, the proof could require the student to work directly with cosets and this would show us something about her or his understanding of cosets.

Also, the student is required to know what normality means and to understand this meaning enough to explain why it is so in this situation.