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COMPUTER EXPERIENCES IN LEARNING COMPOSITION OF FUNCTIONS

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Computer instantiations of abstract concepts could assist mathematics education by providing concrete experiences and by inducing the mental process of construction that Piaget called reflective abstraction. Two classes of college students ($n = 13$) given 6 weeks of computer experiences to help induce reflective abstraction scored higher on a test of their understanding of functions and compositions than one class of students ($n = 17$) who were taught according to traditional methods. The comparison was based on questions intended to indicate whether reflective abstraction had taken place.

It is acknowledged by educators that students have great difficulty in learning many of the standard topics in the undergraduate curriculum—concepts such as composition of functions, limits, and induction (Moise, 1984). Efforts to facilitate learning generally take the form of revising curricula, utilizing new technology in teaching, or trying to find ways in which teachers can better convey course material through classroom lectures (Howson & Kahane, 1986). What is usually lacking is an attempt to orient instruction around the learning process (see, however, Campbell, 1979; Fuller, 1976). Instead of hoping that students will learn new material by imitating the teacher's behavior or listening to a lecture, teachers may find it more fruitful to consider the mental processes by which new abstract concepts are acquired. This latter approach requires an understanding of, or at least a hypothesis about, the psychology of concept acquisition.

Piaget's epistemology provides a theoretical foundation for understanding how knowledge is constructed. In this paper we use his theory to suggest how and why computer experiences could stimulate the mental processes that lead to the acquisition of mathematical concepts—in this case, mathematical functions and their composition. After explaining how Piaget's theory relates to the concepts, we describe an instructional intervention in which students' experiences with the computer operating system Unix were intended to induce specific cognitive activities necessary for understanding function and composition. This instruction is discussed in the context of an

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experiment in which a second group of students was taught according to traditional methods. Performance of the two groups on questions relating to these concepts was compared to see if the computer treatment appeared to increase the likelihood that the students would acquire the desired concepts.

THEORETICAL BACKGROUND

Piaget proposed the process of *reflective abstraction* as the key to the cognitive construction of logico-mathematical concepts (Beth & Piaget, 1966). Reflective abstraction refers to the cognitive process by which a physical or mental action is reconstructed and reorganized on a higher plane of thought and so comes to be understood by the knower. Piaget's theory is uniquely appropriate for our work because of its clear articulation of the connection between activity and representation and of the desirability of concrete activity for developing adequate representations of abstract concepts.

In particular, we believe that four types of reflective abstraction that Piaget identifies—*generalization*, *interiorization*, *encapsulation*, and *coordination*—are necessary to construct the concepts of function and composition. We briefly describe them.

Most students, even in college, are restricted in their understanding of function to the idea of an algebraic formula. Many functions, however, in mathematics, physics, and computer science are presented by more complicated descriptions that do not involve formulas. The student must *generalize* her or his notion of function to include these new objects. The key feature of function at this more general level is that it involves some sort of process that transforms elements in one set (domain) into elements in some other set (range).

Understanding the concept of function includes the ability to *interiorize* this process, that is, to form a mental representation of the (possibly mental) action of the function.

To perform operations on functions, the student must consider functions as mental objects. Thus the process that has been mentally represented for a particular function must be consciously understood or *encapsulated* into a single, total entity. As a result, the student becomes able to think of a function both dynamically as a process and statically as a mental object.

Finally, suppose that one begins with two functions, F and G, understood as individual mental objects that involve certain specific processes. In order to form the composition $F \circ G$ of these two functions, one must call forth the processes one at a time. The results of these separate operations are to be *coordinated* (if possible) to form a new mental process that consists of first performing the process of G, generating a result, and then performing the process of F on that result. The new process is interiorized, and the resulting mental representation is encapsulated as the new function $F \circ G$.

A student who has acquired the concepts of function and composition as described above will find them to be powerful tools for understanding many ideas in the science and mathematics curriculum. Such an understanding is to be compared with the restricted notion mentioned above in which composition can only be performed by substituting a formula for one of the symbols in another formula.

Several difficulties confront the student in constructing these concepts. In the first place, unless the student has developed the ability to think formally (as described by Piaget), he or she will be able to form mental representations of only those actions that have been experienced concretely. Recent research (Adi & Pulos, 1980; Chiappetta, 1976) suggests that many college students will have this difficulty. But even for students who can think formally, reflective abstraction as described above does not happen automatically. Despite the importance of this process for education, little seems to be known about how to induce reflective abstraction in the classroom (Gallagher, 1978; Pascual-Leone, Goodman, Armon, & Subelman, 1978).

Computer activities may be an answer. Papert (1980) argued that they could facilitate learning mathematics for children, and Abelson and diSessa (1980) suggested similar approaches for older students. Computers have certain special features that suggest that experiences with them might help students construct the concepts of function and composition as described above. For example, converting a process to an object, by making it a procedure or operating system command, is done very often in the course of normal programming activities. Also, a structure that exists on a computer is concrete in that it is an electronic object, but at the same time it is abstract in that what exists electronically is very different in form from the construct that the user is thinking of. Thus computer constructions are intermediate between concrete objects and abstract entities.

In the next section we describe computer experiences that were designed to take advantage of such features.

INSTRUCTIONAL INTERVENTION

The intervention consisted of a combination of directed and nondirected experiences with various features of the Unix operating system (custom-tailored for this purpose) followed by classroom explanations of the concepts based on these experiences.

Two Unix features were used: *shell scripts* and *pipes*. In Unix a large number of commands can be applied to data entered at the terminal. These include both standard arithmetic operations on numbers and less familiar text-processing functions (count the number of lines containing the string 'and', capitalize all letters, etc.). A meaningful sequence of commands can be saved in a file and given a name, after which the sequence can be performed as a single command by invoking its name. This file is called a shell script. For example, the commands necessary to calculate $2x + 3$ could be saved as a shell script under the name POLY. Then the command POLY followed by the input 5 would give the result 13.

Two shell scripts can be piped together. For example, if another shell script to compute $3x - 7$ were constructed under the name POLY1, then the command POLY1@POLY (the symbol @ is the pipe) followed by the input 5 would produce the result 32.

Finally, a shell script can have a parameter. Thus one can construct a command COUNT 'WORD' where 'WORD' is a parameter, and COUNT 'the' followed by an input text would return the number of occurrences of the string 'the'.

Our hypothesis was that experiences with shell scripts and pipes could directly stimulate the types of reflective abstraction that, according to our theory, are required in the construction of the concepts of function and composition.

For example, thinking about what the computer is doing when it is performing a sequence of commands can lead to interiorization and can stimulate the construction of a mental representation of this process—even when no formulas are available. Making a shell script out of this sequence is precisely the process of encapsulation. Indeed, a shell script is an "object" in the physical (or at least electronic) sense. The formation of a pipe is a coordination of two shell scripts. Thus the student can begin with two shell scripts, think about their processes, coordinate these two processes in a pipeline, think about the resulting process, and encapsulate it in a new shell script. This is quite similar to our analysis of the composition of functions and may well stimulate the mental constructions we have discussed. Finally, working with text-processing functions (and creating new ones) provides students with a broader repertoire of examples that may help them to generalize their overall concept of function from purely algebraic representations to a richer set of examples.

METHOD

Subjects

The students were enrolled at a small university in upstate New York that emphasizes engineering, management, and science. Every student is required to own (and use in virtually all classes) a personal computer. Three sections of an optional first-year college mathematics lab were used for the study. All of the students were concurrently taking an introductory calculus course, in which composition of functions was dealt with briefly. The lab course was designated pass-fail, and the students were not graded for work done as part of this study.

The students had been divided into three sections by the registrar in an action unrelated to this study. Two of the groups were designated Computer 1 (10 students, Instructor B) and Computer 2 (14 students, Instructor A).

The third group was designated Paper-and-Pencil (20 students, Instructor A). The data analysis was restricted to 30 students because 8 did not take a pretest covering the concepts of function and composition, and 6 scored so high that we thought they had already acquired the concepts we were attempting to teach.

The elimination of these 6 students was done on the basis of their average on the pretest. Any student who scored 71.2% or more was eliminated. The highest score of the remaining students was 67.5%. After the elimination the average was 37.1% for the Computer 1 group, 49.0% for the Computer 2 group, and 38.7% for the Paper-and-Pencil group. The number of students in each group was 7 in Computer 1, 6 in Computer 2, and 17 in Paper-and-Pencil.

Treatment

About 50 Unix shell scripts (with and without parameters) corresponding to various functions including both standard mathematical functions and less familiar text-processing operations were prepared to be used in computer labs by the students in the Computer groups. For the Paper-and-Pencil group, a similar set of functions was used.

All students in the course came for six weekly 2-hour sessions. In the first session they were given the pretest.

During the following two 2-hour class sessions (one per week) the students in the Paper-and-Pencil group practiced with written exercises involving functions and compositions. The functions were of the type used in first-year calculus courses. The concepts were not explicitly emphasized but rather were expressed in terms of formulas and substitutions. While working the problems, students had the opportunity to seek assistance from their instructor. In the meantime, the Computer groups devoted the same amount of time to performing computer tasks that consisted of applying shell scripts to particular data and making their own shell scripts, either by specifying the value of a parameter or by piping two shell scripts together. Their instructors were available for assistance. Again, the concepts were not made explicit. The instructors suggested to the students in all groups that they think about the calculation processes as they were being performed (by the students or by the computer).

In the fourth session all students were given a 2-hour lecture on functions and composition. The concepts were dealt with as in traditional undergraduate courses, and no attempts were made to relate the material to any of the preceding exercises. To increase the likelihood that all students were dealing on the conceptual level with the same notion of function and composition, Session 4 was a joint session taught by Instructor A.

For the fifth session, each section met separately with its original instructor. Connections were drawn between the lab activities (written exercises for the Paper-and-Pencil group, computer exercises for the Computer groups) and the lecture, further written exercises were given, and the students were again urged to form mental images to represent functions.

During the sixth session all students took a written posttest. While scoring exams, the instructors did not know to which group the students belonged. Copies of the tests were scored independently, and the few discrepancies were resolved by conference.

The main differences in treatment between the Paper-and-Pencil and Computer groups occurred in Sessions 2, 3, and 5. These differences are illustrated in Figures 1, 2, and 3.

Paper and Pencil	Computer
Q. Evaluate the formula $F(x) = x + \sqrt{x^2 - 1},\ x \geq 1$ for the value $x = 2$.	Q. Evaluate the command pow2 for the value 4.
A. The student is expected to write $F(2) = 2 + \sqrt{2^2 - 1} = 2 + \sqrt{3}$	A. The student is expected to enter the following at the terminal: pow2 (return) 4 (return) control-D At this point the computer prints to the terminal: 16
Q. Evaluate the formula $F(x) = a^x$ when the parameter a has the value 3 and $x = 5$.	Q. Evaluate the command with parameter text.str when the parameter str has the value 'the' and the input is the given text (several lines of English text are provided for the student to type in).
A. The student is expected to write a^x when $a = 3$ gives 3^x $3^5 = 243$	A. The student is expected to enter the following lines at the terminal: text.the (return) the given text control-D At this point the computer prints those lines of the text which contain the string 'the'.

Figure 1. Comparison of paper-and-pencil and computer treatments for the evaluation of a function at some value in its domain.

Paper and Pencil

Q. Given that

$$F(x) = \frac{x}{\sqrt{1-x^2}}, \quad -1 \le x \le 1$$

$$G(x) = \sin x, \qquad \text{all real } x,$$

evaluate $F(G(x))$ when $x = \frac{\pi}{4}$

A. The student is expected to write

$$F(G(x)) = \frac{\sin x}{\sqrt{1-\sin^2 x}}$$

$$= \frac{\sin x}{\cos x} = \tan x$$

Therefore, the value at $\frac{\pi}{4}$ is

$$\tan\left(\frac{\pi}{4}\right) = 1.$$

Computer

Q. Given the commands pol3 and pol5 (these represent the polynomials $x^2 - 2x$ and $x^2 + 1$, respectively, which the students know and have worked with), pipe them together and evaluate the resulting command at 4.

A. The student is expected to enter the following lines at the terminal:

```
pol3 @ pol5     ⟨return⟩
4               ⟨return⟩
control-D
```

At this point the computer prints to the terminal:

255

Figure 2. Comparison of paper-and-pencil and computer treatments for the composition of two functions.

Paper and Pencil

Q. Each of the following functions is considered to act on some text written in English characters:

F interchanges the letters 'o' and 'w'.

G counts all occurrences of the string 'two'.

What is the effect of applying these two functions in sequence, first *F* and then *G*?

A. The student is expected to answer that the effect is to count all occurrences of the string 'tow'.

Computer

Q. Create a new command by piping the command #lin (which the student knows to count the number of lines in a text) with the command text.the (which the student knows to return those lines of a text containing the string 'the') so that the latter is performed first. Describe the new command.

A. The student is expected to type the following at the terminal:

```
makfun name        ⟨return⟩
#lin @ text.the    ⟨return⟩
control-D
```

The student is then expected to answer that the new command, name, counts the number of lines in a text that contain the string 'the'.

Figure 3. Comparison of paper-and-pencil and computer treatments for the construction of new functions.

Generally speaking, the Paper-and-Pencil group evaluated functions by paper-and-pencil calculations and composed them by substituting in formulas. They were advised to think carefully about the meaning of the processes they were performing. Parameters were handled by replacing a letter with a value. The formation or creation of a new function was discussed but was not related to any specific action other than writing phrases like, "Let the function F be defined by. . . ."

The Computer groups, on the other hand, evaluated a function by typing its name, the particular input, and a single go key (∧D). Composition was explained as identical to piping (the symbol @), and there was an explicit facility (*makfun*) that was used to create a new function and attach a name to it. The students were advised to develop the habit of predicting the result of a calculation before performing it on the computer and then to compare the actual result to their prediction.

The treatments for the Computer and Paper-and-Pencil groups were similar in several ways. Each was given the same tasks, with one group expected to use paper and pencil while the others used the computer. Also, all groups worked with lists of functions that were mixes of algebraic and text-processing functions. For the Paper-and-Pencil group, the majority of the functions in the list were algebraic, whereas the reverse was true for the Computer groups. An example of a task that was similar for all groups but used different functions is shown in Figure 3. The main tasks that were identical for all students were as follows:

1. Evaluate a single function or the composition of two functions.
2. Look at a list of functions and find all triples F, G, H that satisfy the relation $H = F \circ G$.
3. Given two functions G, H, construct F such that $H = F \circ G$.
4. Given two functions F, H, construct G such that $H = F \circ G$.
5. Given two functions, determine if it is possible to compose them.
6. Play around with comparisons of $F \circ G$, $G \circ H$, $F \circ (G \circ H)$, $(F \circ G) \circ H$.

Instruments

The pretest consisted of eight questions. Five of them were the standard sort of questions that might appear on an examination covering functions and composition in a typical advanced high school or elementary college course, and the remaining three were similar to questions on the posttest (described below). One was similar to Question 4 on the posttest, one was similar to Question 5, and one was an easier version of Question 6.

The posttest is given in Figure 4. The questions were designed to indicate whether or not the reflective abstractions involved in the construction of the concepts of function and composition actually took place. For example, in Question 1, explicit mention of $F \circ G$, $G \circ H$ as new functions that had been formed would suggest that encapsulation may have taken place. In Question

2, an understanding that $G(1)$ has no meaning before the selection and is equal to 0 afterward indicates that the student may be using encapsulation to construct a mental object corresponding to a particular function. The functions mentioned in Questions 3c, 5c, 6c, and 7c were unfamiliar to all the students. Success with them suggests that the students might be generalizing their concept of function. In Parts b and c of Questions 3, 5, 6, and 7, the functions given and searched for are such that substitution may not be helpful and, if used at all, would require the use of formulas derived on the spot. A more likely explanation of any success with these questions would be that the students are interiorizing the definitions, forming mental representations of the processes, and coordinating these processes. This explanation seems particularly appropriate for Question 5 and even more so for Question 6, where reversals of the processes seem to be required.

1. Let F, G, and H be three functions and assume that all compositions in the following expressions are possible.

$$(F \circ G) \circ H, \; F \circ (G \circ H)$$

   Describe the meaning of these two expressions and explain why they are equal.

2. Denote by F_t (where t is an integer) the class of functions whose domain and range are both the integers and whose rule of association is:

$$F_t(x) = tx^3 + 3x^2 - (t + 1)x - 2$$

   Respond to the following questions.

   a. Why do we speak of a "class" of functions rather than a function?
   b. Explain what is meant by the following instruction.
      "Select a function from this class by taking $t = 2$ and call it G."
   c. What is the difference between the meaning of the expression $G(1)$ before performing the instruction in b and after performing it?

3. Find all possible ways that the functions in parts a, b, c satisfy the relation $F = G \circ H$.

   a. The domain of all functions is $x > 0$. The association rules are given by:

$$F_1(x) = \frac{1}{x^2}$$
$$F_2(x) = \ln(1 + x)$$
$$F_3(x) = \ln(x^2 + 1) - \ln(x)$$

   b. The domain of all three functions is English text. The actions of the functions are given by:

      F_1 changes the case of capital letters only
      F_2 changes the case of all letters
      F_3 changes the case of small letters only

Figure 4. Posttest.

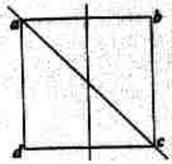
c. The domain is squares of the type indicated.

The action of the function is given by:

F_1 rotates 90 degrees clockwise about the center

F_2 reflects in the diagonal *ac*

F_3 reflects in the vertical axis



4. From the following list of 6 functions, find all possible triples *F, G, H* such that the composition formula $H = F \circ G$ holds. For example if *a* is the composition of *d* and *e* (IT ISN'T) then write $a = d \circ e$. Functions *F, G, H* should be distinct.

The domain is any set of English text. The statement which appears describes the result of applying the function to a particular value in the domain.

a. The first four lines of the text that contain the word "hat"

b. Those of the first four lines that contain the word "hat"

c. All lines in the text which contain the word "hat"

d. The constant 4

e. The number of lines in the text

f. The first four lines of the text

5. Solve for *F* in the relation $H = F \circ G$ where

a. Domain is all real *x* and rules of association are

$$H : \cos^2 x$$
$$G : \sin x$$

b. Domain is any English text and the result of applying the function to a particular sample of text is:

H: all words in the text containing the string 'TH'

G: all words in the text containing the string 'H'

c. Domain—triples of integers—*a, b, c*. The result of applying the function to a particular triple is

H: the sum of the second and third integers

G: the original triple with the second integer replaced by the sum of the original three integers

6. Solve for *G* in the relation $H = F \circ G$ where

a. The domain is all real *x* and the rules of association are

$$H(x) = \cos x$$
$$F(x) = \sin x$$

b. The domain is any English text. The action of the function when applied to any particular value is

H: counts all words containing the string 'TH'

F: counts all words containing the string 'OW'

Figure 4. Posttest—*Continued.*

c. The domain is the set of triples of integers—*a*, *b*, *c*. The action of the function when applied to a particular triple is

H: computes the sum of all three integers

F: computes the sum of the last two integers

7. Describe the composition $F \circ G$.

a. Domain is $x \geq 0$ with the rules of association:

$$F(x) = 1 + \frac{2}{x + 1}$$
$$G(x) = 2 + \frac{3}{x + 1}$$

b. Domain is any English text and the result of applying the function to a particular sample of text is given by:

F: all words that contain the string 'MA'

G: all words that do not begin with the string 'MA'

c. Domain is any set of 4 digit integers. The result of applying the functions to a particular set of 4 digit integers is given by:

F: those 4 digit integers where the sum of the two digits on the right is 8

G: those 4 digit integers where the second from the right is 6

Figure 4. Posttest—*Continued.*

RESULTS

The mean scores of the students in each section on each question are presented in Table 1. The scores of the Computer groups are generally higher than those of the Paper-and-Pencil group. The only exceptions are Question 5a, which involved functions more familiar to the Paper-and-Pencil group; 7b, on which the Paper-and-Pencil group did better than one of the Computer groups but not the other; and 3c, where the uniformly low scores may have been due to a misunderstanding reported by students when trying to perform F_2 after F_1 or F_3 (see Figure 4). In the wording that was used, it was not clear whether "*ac*" refers to the diagonal before or after it is rotated or reflected. This was the only ambiguity on the test reported by the students.

DISCUSSION

The scores on the posttest are consistent with our hypothesis that the computer experiences given to the students in the Computer groups were more effective in inducing the reflective abstractions involved in constructing the concepts of function and composition than was the traditional treatment given to the Paper-and-Pencil group.

A student who is thinking of a function only as a process is less likely to refer to the function itself as opposed to its action or its value at a particular

Table 1
Average Percent Correct on Each Posttest Question by Group

Question	Paper-and-pencil	Computer 1	Computer 2
	Designed to favor neither group		
1	25	46	43
2	30	75	59
3c	26	24	29
4	16	35	36
5c	45	53	80
6c	45	74	57
7c	68	90	90
	Designed to favor computer groups		
3b	25	40	50
5b	80	88	87
6b	24	57	36
7b	52	62	42
	Designed to favor paper-and-pencil group		
3a	40	70	86
5a	86	70	69
6a	29	46	80
7a	87	89	94

point than is a student for whom function is a mental object. Thus, although the argument is far from conclusive, the scores on the questions requiring explicit mention of a function that has been constructed, either through composition or by selection from a class, suggest that the students in the Computer groups might have been more likely to encapsulate a function as a cognitive object.

It is hard to imagine how a student could construct a process that fits in for an unknown in an equation involving processes unless he or she is able to interiorize the process of a function and manipulate it mentally. The results on questions requiring the students to solve for one of the functions F, G, H given the other two in the equation $H = F \circ G$ suggest that the computer treatment led to students being better at interiorizing the processes of functions and forming mental representations of them.

The superior performance by the students in the Computer groups on questions involving functions that were new to them might have been due to a successful generalization of the concept of function beyond algebraic formulas.

Finally, the results on questions involving functions more familiar to the students in the Paper-and-Pencil group do not suggest that the computer treatment was in any way detrimental to the ability of the students in the Computer groups to perform the standard (and extremely important) manipulations with functions given by algebraic formulas.

Of course there are many questions that must be answered before these indications can be sharpened to conclusions. For instance, the better scores attained by the Computer groups could be due partly or entirely to the general effect of computers as opposed to the particular tasks used and the attempt to induce the students to perform reflective abstractions. On the one hand, computers may simply be more motivating than paper-and-pencil sessions. On the other, there is evidence that they are not invariably helpful in postsecondary education (Kulik, Kulik, & Cohen, 1980). It should be noted also that since the students in this study attend a university that requires every student to own and use extensively a personal computer, the novelty of the computer is minimal for these students.

The size of the groups was very small. It would be useful to see how such a treatment works with larger groups of students in other kinds of schools. It may be possible to improve the questions in their ability to indicate whether or not specific reflective abstractions have taken place. It would be better to interview the students in depth as they are studying these concepts. All of these issues will be the objects of future research.

Nevertheless, the results of this study seem to us to provide some evidence in support of the effectiveness of computer experiences for helping students construct the concepts of function and composition. We believe that this is the case because such experiences help induce the reflective abstractions necessary to acquire these concepts. Future reports will describe our considerations of other topics for which such an approach may be useful and our attempts to develop appropriate computer experiences, based on this theory, that will implement instructional treatments for these topics.

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