Pedagogical Change in Undergraduate Mathematics Education

by
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If you read the early reports such as *Toward a Lean and Lively Calculus* (Douglas, 1976) or *Calculus for a New Century* (Steen, 1989), it is clear what the goal of calculus reform was. The original intention was to revise the *content* of calculus — to make it lean and lively, to make it more relevant to applications, to include some contemporary mathematics, and to reflect the new technology. All of the early talk was about changing the topics that were covered in a calculus course.

This is not what happened. The calculus reform syllabi are beginning to appear and their lists of topics invariably include polynomial, rational, exponential and trigonometric functions; differentiating by using product, quotient and chain rules; applying derivatives to motion, curve graphing and economics; integrating by techniques and by numerical approximations; applying integration to physics, economics, and statistics; and, of course, the fundamental theorem of calculus. The textbooks are still hefty and many of them are still in a format of discussion sprinkled with worked examples followed by exercises to practice.

It is my opinion that after a fairly short period of relatively superficial manipulations, the set of topics in the calculus will settle down to pretty much what it was. Indeed, it is hard to see how it could be otherwise. There is certainly some evidence that the topics have not changed for a very long time (Dubinsky and Ralston, unpublished report). This could be due to inertia, but it could also be that it is not the content that we want to change.

Today, many people are coming around to the point of view that what needs to be changed is not the content, but the pedagogy — not what we teach, but how we teach it. This is true, not only for alculus, but throughout undergraduate mathematics.

True, this is not yet a majority view. Many people appear to believe that effective teaching is actually quite easy to achieve, if only you care enough to give it a certain amount of attention and energy. The suggestions offered by proponents of this view are, in my opinion, little more than common sense, well understood by a very high percentage of members of our profession. The analysis ignores the fact that a really large number of mathematicians *are* conscientious and dedicated in their teaching. Very many of us *have* used these suggestions in our teaching and have been doing so for many years. The important point is that, in spite of all this, our students are still not learning mathematics. For me, the inescapable conclusion is that much more is needed than common sense suggestions gleaned informally from experience. I am convinced that we need to reconsider and revise our pedagogy — and we need to do it in conjunction with research into what it means for a student to learn a mathematical concept.

I would like to touch on some questions about this pedagogy issue. What are the new pedagogies that people are talking about? Why do their proponents expect them to lead to improvement? To what extent is pedagogy actually changing? And, do we see any results that suggest things are getting any better?

I think there may be something to say about the first two questions, but it is much to soon to expect anything definitive regarding the last two.

What are some pedagogical innovations in collegiate mathematics?

There are three major teaching changes to which a growing number of college faculty are giving at least lip service. The first is fundamental in our conception of the very nature of teaching and learning. We are beginning to move away from a feeling that mathematical knowledge is a kind of mental commodity that can be delivered by a teacher to a collection of students using various media such as speaking, writing, and even demonstrating on computers. We are beginning to understand that each individual must construct knowledge for her- or himself and the role of the teacher is not to explain mathematics in a classroom, but to induce students to construct it in their minds.

The second change we are talking about has to do with using computers to support the first change — that is, to foster students' constructing mathematical knowledge. This can be done in many ways. A small but growing number of people, including this writer, are convinced that the best way, by a long shot, is to have students implement mathematical ideas on computers — that is, write appropriate programs that implement mathematical processes and objects.

The third change that is fairly sweeping the country — if not in our classrooms, at least in our conference reports — is called cooperative learning. Students work in teams to learn mathematics cooperatively, not competitively and not as isolated individuals.

There are many other teaching innovations that are discussed in UME Trends, in the publication *A Source Book for College Mathematics Teaching* (Schoenfeld, 1991) and many other places. They include such methods as writing, self-paced instruction, student projects, and the Moore Method. But I believe that the three I mentioned are the most important today and I would like to concentrate on them for the remainder of this paper.

Replacing lectures

So what *do* you do if you are not standing in front of the class explaining mathematics? Some people talk for a while, ask a question, try to get a discussion going and go on from there. Some people send the students to the blackboard to work on problems.

Here is what we do and it is all part of a general scheme for getting students to construct mathematical concepts for themselves. The students work cooperatively in teams (see below) on tasks that, from their point of view, are related to work they have done in the computer lab (see below). From my point of view, the tasks are related to the mathematics that I would like them to construct but broken down according to the particular mental constructions that my research has suggested they can make in order to learn the mathematics.

More specifically, I will give the students a (paper and pencil) task to work on in their teams and then try to get them to discuss what they have done. At the end I will summarize and emphasize the mathematical ideas that I think they have constructed.

Suppose, for example, I would like them to understand the mathematical construction of a function E which assigns to each real number M the sequence $E(M)$ whose j^{th} term is $Mj(j+1), \quad j = 1, 2, \ldots$.

This requires that they understand a sequence of numbers as a function which consists of a process of transforming positive integers (domain) into real numbers (range) Then they must understand that a sequence is itself a mathematical object, a notion that permits the construction of a set of sequences or a parameterized family of sequences, or a function which maps numbers to sequences. I will discuss in the next section a precursor to the following program which I would have the students work with in the computer lab.

```
E := func(M);
        return func(j);
                    return M*j*(j+1);
                end;
     end;
```

That is, they might write it, modify it, try to graph it answer questions about it, or solve problems with it. (Dubinsky and Schwingendorf, 1992).

Now, the task in class might be to explain what the computer does if you enter `E(32.78)(6)`. (It returns the value at 6 of the function `E(32.78)`.) Or I might ask for an explanation of the computer's response if you say `E(32.78)` (it returns an internal designation for this function) and compare that with the previous response. Or, I might ask them to explain why `E(32.78,6)` would make no sense to the computer. It turns out that most of the conceptual issues — such as those concerning a function whose value is a function (sequence), a function as a process vs. a function as an object, the meaning of the graph of a family of functions and so on — will tend to arise in such a situation. Not only does it arise, but you can actually see the students constructing appropriate and powerful mathematical ideas as a result of their experiences with the computer and the opportunity for reflection that is provided by this kind of classroom activity.

Using Computers

As with trying alternatives to lectures, there are a multitude of ways in which people are using computers in college level mathematics courses. The role of the computer in these approaches ranges from the easy to implement but fairly barren "electronic blackboard" to a tool for investigation and discovery or an environment in which it is natural and helpful to construct sophisticated mathematical notions (such as function as a total entity).

The Source Book mentioned above (Schoenfeld, 1991) delineates some of these computer uses. I would like here to only expand slightly on the comments I made previously regarding the use of computers in our project.

Students can use the computer to perform calculations, discern regularities and make discoveries about patterns. They can use it as a powerful tool to solve problems and to apply various mathematical concepts to problems in other fields such as management, engineering, and the physical sciences. Details about such uses in our project and others for Calculus courses appear in (Tucker, 1990, Leinbach, 1991).

In our project, we use computers in an unusual way. Our theoretical and empirical investigations of how people learn help us to lay out a series of mental constructions that students can make in order to understand a mathematical concept. Often, however, they do not find it easy to make such constructions. For example, it turns out that constructing the idea of a sequence as a process which converts a positive integer to a real number is both important and difficult for students. We help them perform this mental construction by asking them to write and work with code such as the following (which explicitly constructs the function `E(32.78)` discussed above and evaluates it at three points.)

```
M := 32.78;
s := func(j);
        M*j*(j+1);
     end;
s(3); s(17); s(23167);
```

Working with sequences requires the student to see them not only as processes, but also as objects which can be manipulated. For example, the concept of sequence of partial sums is an operation on a sequence that transforms it to a different sequence. We help students construct this extremely difficult (for them) notion by asking them to write a program such as

```
PS := func(s);
        return func(j);
                  return s(j);
               end;
  end;
```

Working with this code helps students understand that `PS(s)` represents the sequence of partial sums of the sequence `s` and `P(s)(17)` represents its 17^{th} term. It is not that programming helps them directly with notation but rather that it helps them understand the underlying concepts and *that* makes the notation less problematic.

Working cooperatively

As I have indicated in discussing alternatives to lecturing and the use of computers in our project, we have students working cooperatively in teams. This has recently become a very popular idea in undergraduate mathematics teaching and, once again, there is a multitude of ways in which students can work cooperatively. These range from the informal suggestion, made from time to time by the instructor, that students should get together to work on some problems, to a more formal, permanent arrangement.

In our project, we work at the latter extreme. Students are assigned at the very beginning of the course to teams of three or four. Unless special difficulties arise, the assignments remain intact throughout the course and students do all of their work in these teams. This includes lab assignments, homework, classroom interaction, and even some examinations. We work hard to help the students develop a team spirit and a commitment to the progress of all members of their team.

Many questions arise when you begin to think about cooperative learning. There are practical questions. How are the teams selected? What are productive and non-productive ways in which a team can function and how can the teacher influence this? How do you translate the work of a team into an individual grade? There are also more foundational questions. What is the relation between learning and problem solving in teams? How are evaluation and assessment affected? What role does language play? How can technology be incorporated into cooperative learning?

Although a great deal has been written about cooperative learning in grade school, almost nothing exists about using it in collegiate mathematics classes. Based on our project, we are writing a book which will be a summary of our experiences and a practical guide for using this approach. It will also include an annotated bibliography. Hopefully, it will be available by the time this article appears.

Why do we think these new pedagogies are worth trying?

I would like to try to explain my contention that research into how people can learn mathematical concepts is, or should be, the main answer to this question. That is, the interaction among our theoretical analyses, implementations, and observations provides specific pedagogical approaches that are based on more solid ground than just the personal impression that come from informal reflections on experience which, I submit, have proven inadequate for the educational problems our profession is facing. I have not always been commited to this point of view and it may be useful to share my view of my own development.

I come to the question of how to find effective instructional strategies as someone who has been teaching college mathematics courses for almost 40 years, since 1956. For all of that time I have felt unsatisfied with the results — in terms of student learning. Most of the time I was trying one or another kind of teaching innovation, either my own idea or a technique suggested by someone else. Where did these ideas come from, and why did I think they were worth trying? They came, for the most part, from honest, sincere and dedicated people trying to use their experience to think of the best ways they could of helping their students learn. I believe that always, especially today, there are hundreds if not thousands of people who are doing this and until recently I considered myself to be one of them. I think we should applaud their intentions but at the same time face the hard reality that it is not working. That is, it is in the face of all this experience, sincerity, dedication, and hard, creative thinking about specific innovative teaching methods, that we are still having the same results. Our students are not learning mathematics as well as we would like them to — as well as we believe it is possible for them to do.

If we are to succeed in materially improving undergraduate mathematics education, we have to find a better way of coming up with new and different things to do. Moreover, after we try our new methods, we also have to determine how well they worked. This is the assessment issue and that is a matter for another discussion. Here, I am discussing sources for innovative pedagogy — how we come up with it and what might be useful criteria for deciding what to try. I think that one lesson we must accept from our experiences is that it is not enough to just think of ideas in any way that we can and try everything that occurs to us. We must look for guides to which pedagogical approaches are promising and worth the tremendous investment of time and money that it takes to try them.

I would like to suggest that one guide should be the theoretical analyses that are part of research in how people learn. Indeed, the three pedagogical innovations that I layed out in the previous section arose as applications of the theoretical perspective with which I am working. I would like to spend the rest of this section indicating very briefly how our ways of using computers, what we do in the classroom, and our approach to cooperative learning are all guided by this theoretical perspective.

What is our theoretical perspective?

Let me begin this very brief sketch of my theoretical perspective with the following general statement which expresses my constructivist point of view.

> An individual's mathematical knowledge consists of a tendency to respond, in a social context, to a perceived problem situation by constructing, re-constructing, and organizing in her or his mind, processes, objects, and schemas with which to deal with the situation.

To this I would add two things. First, constructing and organizing takes place by performing mathematical procedures in these problem situations and *reflecting* on what you are doing. Second, the statement alone is fairly barren and only comes to life when the theoretical analysis proceeds to specifications of the nature of specific constructions that are to be made. This latter activity is the heart of our project's research. The present short essay is not the place to go into details, but the interested reader can consult our papers in the literature.

What I *can* do here is begin to describe how these points relate to the three pedagogical innovations of the previous section.

How does our theoretical perspective guide computer activities?

The theoretical perspective provides a framework for what follows, and that has to do, in terms of the constructivist point of view expressed above, with the construction of specific processes and objects, organizing them into schema and reflecting on all of this mental activity.

Here is how the theoretical perspective guides the design of computer activities. The theoretical analysis (buttressed with the results of empirical investigations) tells us, roughly, what processes and objects need to be constructed. We then design computer activities to make these constructions on the computer. We have found that this very often results in the student making analogous constructions in her or his mind.

We have given some examples above of how this works in calculus and we have described elsewhere some examples of how it can work in abstract algebra (Leron and Dubinsky, to appear.)

How does our theoretical perspective guide classroom work?

Classroom work consists mainly of students working in teams to perform tasks using only paper, pencil, textual material, and their minds. A task might ask the students to solve a short mathematical problem, or explain some phenomena that ocurred in their computer work, or look for an example/counter-example (often without knowing beforehand which is the case). A good example is connected with the computer program for the functions E and PS together with the classroom tasks and questions that I discussed earlier in this paper.

After the teams have worked on the task, the instructor tries to get a discussion going. Sometimes, when the instructor is convinced that a fair number of students have made appropriate mental constructions, he or she will summarize the mathematics, taking the opportunity to point out standard terminology and notation.

There are a number of pedagogical goals for this kind of classroom work that relate to our theoretical perspective. In some cases, the tasks are chosen as additional stimuli to have students make the same kind of mental constructions that were expected from the computer work. Using recently constructed objects and processes to solve an unfamiliar problem can induce students to form these mental constructs into coherent schemas that can be used for solving similar problems. Also, this work can get students to reflect on their mental constructions, and how they may be used.

How does our theoretical perspective guide our approach to cooperative learning?

The most important theoretical support for our use of teams lies in the need, according to our theoretical perspective, for students to reflect on the calculations that they perform. Working in a group, it is natural for students to ask each other questions, explain what they are doing and, in general, to discuss the mathematics they are working on. We see this in the classrooms, the labs, and during informal interaction among students. This sharing of ideas gets students used to reflecting on what they are doing. It helps mathematics become, for them, not so much a mindless "repertoire of imitative behavior patterns" (Moise, 1984), but a form of thinking in which they are consciously engaged.

There are several other, mainly affective, contributions that we see coming out of cooperative work. The support that students give each other, both emotionally and just in terms of getting the work done is invaluable. Setting up a permanent (for the semester) team and suggesting to students that each member of their team is responsible for the learning of all members of the team, tends to make their experience with mathematics more joyful. If nothing else in our approach works, the more positive attitude towards the course that students seem to get from working in teams is a big gain that alone can justify our efforts.

Conclusion

In conclusion, let me summarize what I have tried to say here.

My main point is that real improvement in undergraduate mathematics education cannot and will not occur in the absence of substantial pedagogical change. Moreover, one ingredient that can make a major contribution to this change is research in how people learn mathematical concepts.

I have tried to describe the major pedagogical changes that people are thinking about: alternatives to lecturing, technology, and cooperative learning. I have discussed the way in which these three instructional strategies play a role in the curriculum development project with which I am involved. In particular, I have pointed out the relationship between our use of these strategies and our research in learning, especially from the point of view of theoretical analyses.

In closing, I would suggest that major pedagogical changes are today more honored in the conference report than in the classroom and the jury will still be out for a long time on deciding about their long term value. But it would be a mistake to end this note on such a negative tone. As I tell my students struggling to understand mathematics, one should not always concentrate on far there is to go, but sometimes it is helpful to look back and see how far one has come. In the case of pedagogical change in undergraduate mathematics education, it is possible to hope thast our dismay at the daunting length of the former, may be overcome by the awe inspired by the substance of the latter.

References

Douglas, R. G. (Ed.) **Toward a Lean and Lively Calculus**. Report of the Conference/Workshop to develop Curriculum and Teaching Methods, MAA Notes, no.6, Math. Assoc. Amer., 1986.

Dubinsky, E. and A. Ralston **Undergraduate Mathematics — Now and Then**, unpublished report.

Dubinsky, E. & K. Schwingendorf, **Calculus, Concepts, and Computers, Preliminary Version**, West: St. Paul, 1992.

Leinbach, C. (Ed.) **The Laboratory Approach to Teaching calculus**, MAA Notes No. 20, Mathematical Association of America:Washington, 1991.

Leron, U. and E. Dubinsky, *An Abstract algebra Story*, American Mathematical monthly, to appear.

Moise, E.E. *Mathematics, computation, and psychic intelligence* in "V.P. Hansen & M.J. Zweng (Eds.), **Computers in mathematics education**," (1984 Yearbook of the National Council of Teachers of Mathematics, pp. 35-42. Reston, VA : NCTM (1984).

Schoenfeld, A.H. (Ed.) **A Source Book for College Mathematics teaching** MAA Reports, no. 2, Mathematical Association of America:Washington, 1991.

L. Steen (Ed.), *Calculus for a new century* MAA Notes, no. 8, Mathematical Association of America: Washington 1989.

Tucker, A. (Ed.) **Priming the Calculus Pump: Innovations and Resources**, MAA Notes No. 17, Mathematical Association of America:Washington, 1990.