Determining Linearity:
The Interplay between Visualization and Analysis

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1. Introduction

One of the long debates in mathematics education research is over the value of visual thinking in mathematics. Most mathematicians agree that analysis lies at the heart of mathematics, but visual exploration and reasoning can also be essential tools and sources of inspiration in mathematical work. Despite a reluctance to accept visual approaches as finished work, most mathematicians do use visual exploration and reasoning as essential analytic tools and also as sources of inspiration in their daily work (Eisenberg and Dreyfus, 1991). Recent research suggests that students too often use a combination of both visual and analytic approaches when dealing with mathematics (Zaskis et al., 1996; Clements, 1982). For some students, the visual approach can be absolutely essential (Mayer, 1989; Krutetskii, 1976), and can provide an entryway to understanding advanced concepts in mathematics.

In this paper, we examine the interplay between visualisation and analysis in the thinking of undergraduate mathematics students, when dealing with problems involving linear thinking (Cuoco et. al, in prep.). In particular, we are interested in looking at the ways the students use analytic and visual thinking to determine the linearity of a transformation.

Our research indicated that students use a variety of different ways and techniques for investigating linear algebra problems combining both analytical and visual thinking that are not usually revealed in the regular classroom environment. Contrary to the expectations of other mathematics educators who state that students have a strong tendency to think algebraically rather that visually, choosing, whenever possible, a symbolic framework to process mathematical information rather than a visual one, (Eisenberg and Dreyfus, 1991) the students we interviewed repeatedly constructed visual images of the problems we gave them and took from there on several different paths for investigating the situations they were given based exactly on these visual images they constructed. This use of the various combinations of analysis and visualization it the students' work when determining the linearity of a transformation is what drew our attention and motivated our attempt to examine the students' thinking.

2. Research Background

The work described in this paper is part of a broader research on undergraduate students' learning and understanding of Linear Algebra concepts. Of the six questions, designed to provide insights into students' ability to reason about calculations and to use linear thinking, two gave us information on how students determine the linearity of a transformation geometrically:

1. Consider the functions given by the following expressions:
   (i) S: R3’R3 is the function that reflects a point through the xy-plane
   (ii) L: R2’R2 is the fuction that moves a point over 3 units to the left and then 4 units up.
   (iii) F: R3’R3 is given by F[X]=Proj(1,3,2)X.
   If A and B are vectors and c is a scalar, which of these functions satisfy the following properties:
   ù f[A+B]=f[A]+f[B]
   ù f[cA]=cf[A]
2. Suppose that S: R3’R2 is a linear mapping. Suppose that A and B are vectors in R3 and you know that S(A)=(1,0) and S(B)=(-3,-1). Find S(3A-2A).

The linearity of a transformation can be determined both analytically and visually and we chose the above problems to allow for both approaches. The problems can be solved using either an analytic approach, a visual approach, or some combination of a visual together with an analytic approach.

The analytic approach that is the one most commonly used in traditional Linear Algebra curricula, involves a purely algebraic modelling. Students may describe each of the above functions algebraically and then move on to check whether the two properties of linearity (additivity and scalar multiplication) hold (or fail) in the usual algebraic manner. For example, a student may choose to solve problem 1(i) above by ......

Alternatively, the approach that we believe involves visual thinking is not so common in the existing curricula. Students may choose to first construct a visual image of the action of a transformation and then check how the two properties of linearity are preserved (or not) by this action. For such a geometric understanding of linearity we may interpret the scalar multiplication as a stretching or shrinking of a vector that passes through the origin, and the addition as the sum of the images of two vectors as being the same as the image of their sum. A student who chooses to use this approach may solve problem 1(i) by ....

We used the above questions to interview fifteen undergraduate students majoring in mathematics who had just completed a linear algebra course. The analysis of our data was done partly based on a modified version of the theoretical framework of the Visualization/ Analysis (V/A) model suggested by Zaskis et al. (1996). This model fits reasonably smoothly with our data and hence provides a useful language for discussing student thinking.

Our work in particular did not indicate that the students choose to go through visualization and analysis as discrete, separate actions. What we observed is that the students who were able to connect their visual to their analytic thinking were the ones who did a mature, rich investigation of each of the problems they were given, and this investigation brought them to a better understanding and, in most cases, to a complete solution.

Hence, two major questions were raised while analyzing our data: First, we were interested in understanding the reasons that motivated these students to investigate the linearity of a transformation using a combination approach of analytic and visual thinking. Second, we looked at the ways the students used this interplay to determine the linearity of a transformation and made a first step towards understanding how this approach helped them succeed in solving the problems. What follows is a discussion of our understanding of the students' thinking regarding these issues. Throughout our discussion we are using exerpts from students' interviews to illustrate our findings.

3. Choosing to work combining a visual with an analytic approach

Eisenberg and Dreyfus (1991) indicate that students are to use visual frameworks to process mathematical information. However, the first reaction of most of the students we interviewed using the above problems was to make a visualization act either by drawing a picture or doing some kind of physical motion that described a picture. Thus, the first issue we looked at was the reasons that motivated the students to make these visualization acts.

Our wording of the questions was probably one reason that caused this reaction conveying visual information to the students. The students were presented with functions describing each a geometric action (reflection, translation, and projection) and were not given the explicit algebraic form of each of them. In order to determine the linearity of each of these functions in the usual algebraic manner (check whether $F(a+b) = F(a) + F(b)$) the students had to first give the algebraic description of the functions, which is not always trivial work. On the other hand, making a visual interpretation only required to transfer into a drawing on paper the visual information that was part of the given information of the problem. Thus, the way we presented and worded these problems motivated the students to think in terms of the geometry instead of the algebra:

Dean: All right, so a geometric interpretation would be good for (i) I suppose, which reflects a point through the xy-plane.....

Ben: Okay, reflects a point through the xy-plane. R3 to R3. So we have to draw a three dimensional, so if I have a point down here reflected into a point up here.....

However, a closer look at the students' responses shows that our wording of the problems was not the only motivation for the students to start of with a visual interpretation. Some students resorted to the drawing and use of a picture in order to reduce the level of complexity and abstraction of the question. Students often need something concrete to work with and view the picture as a less abstract way of dealing with the problem that will help them reach a solution.

Ben drew a picture in order to avoid the "theoretical" proceedure. Becky, too, hoped that a drawing would help her produce an answer to the probelm she was given because she "can't think of it in her head". Finally, Tom explained how the difficulty and complexity of linear algebra problems can be reduced with the use of visual images:

Ben: Okay, this is really... I remember when my professor did this he said it was really theoretical. I remember it being really theoretical also. So pictorially the easiest way to represent it.....

Becky: Mmm... This one I'm gonna have to draw a picture too because I can't think of 'em in my head. You have, a point here, 1, 2, 3, ... 1, 2, 3, 4 ... I don't know about this one. About moving the point 3 units to the left and then 4 units up...

Tom: What I do when I see a problem in a subject like linear algebra in which you are dealing with things that you can kind of understand in a physical geometrical sense, I mean I can write a vector on that page and know exactly what that means, and so if I'm just talking about aby arbitrary vectors I don't understand that, but I can write in three dimensions, I can write all the vectors down and I can see exactly what is going on...

So, as the students were given these interview problems they found themselves in territories unfamiliar that they had never before explored in one of their classes. As recent research (Dorier, 1991) suggests, many problems given to students in a traditional linear algebra course can often be solved by direct manipulation techniques that do not require that tools of the general theory. Giving students unfamiliar problems to work with (like the ones given in these interviews) causes a state of disequilibrium in the students' minds, and thus the students have to make some action that will bring them in a more familiar situation (Zaskis et al. 1996). Some of our students clearly stated that since these problems were completely new to them they had to resort to some familiar ways of dealing with them and use tools, visualization being one of them, that would help them get, at least, some intuition for the problems:

Tom: This is something I've never really seen before... I guess it's gonna be... Let's go and draw another picture. I don't really have an intuition for this at all. [...] I'm just drawing pictures, random vectors and doing some addition... [...] This is kind of a slow way to go about this problem, but it never hurts to get some intuition...

This is neither a surprise nor new. As Polya (1973) argues "[f]igures are not only the object of geometric problems but also an important help for all sorts of problems in which there is nothing geometric at the outset".

4. How do students determine geometrically the linearity of a transformation?

Tom: So... I now have a picture and then... you know, just by reasoning things up..[...] as I drew the picture I said, "Oh, this is... they seem to be... you know, maybe [...] Oh, I see... there is a relationship here and in fact we ought to be able to get that... So this is..." You want it straight out?

Most of the students we interviewed originally reacted to our posing of the problems with an act of visualization (drawing a picture or perfoming some physical action that describes an image) in order to determine the linearity of a transformation. For some students, this was as far as they could get; they could not see how the picture they drew could add anything to their understanding of the problem they were faced with and either turned to a different technique, or abandoned the effort altogether. For some other students this visualization act provided them with a better insight into the problem and they used that visual image as a tool in further analysis.

Let us now take a closer look at the students' work in determining the linearity of a transformation through the interplay of geometry and analysis. We first looked at the original visualization action of the students and then moved on to see how they combined it with an analytic act or they dropped it.

One of the two ingredients in a geometric understanding of linearity is the interpretation of scalar multiplication as the stretching or shrinking of a vector that passes through the origin. This presented no difficulty for the students we interviewed. Most could draw or perfom physically some action that described an image, a vector, say A, becoming longer after being multiplied by a scalar. Relating the effect of scalar multiplication to what happens to the coordinates was a little more of a problem. We see Becky struggling with it when solving problem 2:

Becky: Um, if it's 2A, A is going to be a vector, and if this is 1A, 2A is going to be a point twice as far away, in the same direction. [...] Or perhaps... Well, I'm questioning my logic. Trying to decide if what I just said is correct. If 2A is really just doubling all the components. And I think it is, but I wouldn't swear by it.....

In thinking geometrically about the linearity of a reflection through the xy-plane in R3 (problem 1(i)), some students do not get very far. Jessica and Mia too seem to see the reflection geometrically but are unable to connect their first visual acts (here described with the physical actions of reflecting a point) with symbolic representations. This prevents them from making any further progress in their investigation of linearity. This lack of ability to connect a diagram with its symbolic representation was not only seen in our interviews. It is refered to also by Krutetskii (1976) as the most harmful but common difficulty with visualization.

Jessica: I think they should both hold, but from like thinking about points in my head, just cause there both going to be vectors there's not going to be any nonlinear combinations going on. So just, both properties should just hold, should hold. I don' really know how to check it on paper.

Mia: R3 to R3 is the function that reflects a point through - the xy - plane. So, are you just saying, that you've got - this, and then you've just got this plane, and so if you had a point up here it would just go down here?

Mia and Jessica failed to connect their visual image with its symbolic representation and thus could not move on in proving the linearity of the given function. However, a closer look at their work indicates that their inability goes beyond that point. Showing geometrically that the two properties of linearity hold for a specific function requires first some meaningful understanding of the nature of these two properties and this does not have much to do with the visualization itself. The two students used either the wording of the particular problem or some past problem solving experience in making a picture of the situation involved in problem 1(i) but their visual image could be of no use unless they had some understanding of linearity and its properties. Apart from that, even though their visual images did contain all the information that was probably needed to proceed in solving the problem, this information was not necessarily obvious to unexperienced and uninitiated students in interpreting information from visual sources and did not know how to take advantage of their own visual constructions. Thus, failing to proceed with this problem is not necessarily a problem related to the use of visualization but rather a broader problem related to their understanding of the properties of linearity.

Becky seems to have a better understanding of, at least, one of the two properties of linearity and this understanding enables her to establish vaguely the preservation of scalar multiplication for problem 1(i).

Becky: Hmm. This function... [pause] Well, I have a point here, and I reflect it, through the xy-plane, and then we get a point here, and this is x, y, and z. And if I take, say, a half, then I'm gonna get half, so I guess that has to be true...

Assuming that the students feel that the visual image they constructed for themselves is helpful and enlightening in the process of solving the problem, their next step is an act of analysis which consists of some kind of coordination of the objects and processes constructed visually. As Tom commented, the visual image was only a first step towards the investigation of the problem that provided him with the necessary insights and intuitions. The visual image by itself would not consist of a solution without the complex intermingling of it with analysis.

Tom: The way I solved it, I did the geometry and got some intuition because I just looked at this and I've never seen anything like that before, and so my first reaction is to get some intuition as to what you are looking for [...] With doing the geometric solution I got a fairly broad base of knowledge of work with and than I got to the question and it just asked something trivial and specific. And if I were on a timed test I would be kicking myself....

This interplay between analysis and visualization can happen in a variety of complex ways. Our interviews showed that students use different ways and techniques for investigating linear algebra problems that are not usually used or revealed to us in the regular classroom environment. After the students constructed a visual image they took on several different paths for analyzing the situations they were given. As the excerpts from the interviews show, the analysis that different students made reached different levels. From here on we will present the various techniques that these students used while trying to understand the students' thinking behind them.

One simple way to think about a problem is to pick a simple specific case and examine it. The next thing would be to examine the general case. There is a danger associated with the use of these specific diagrams in mathematical investigations such as this ones. This danger stems from the possibility of appealing to accidental features of the specific diagram in the investigation which may lead to errors (Barwise and Etchemendy, 1991). So, students may sometimes just convince themselves that the first step of examining the specific case is enough by completely overlooking the need for looking at the general case. There is though, another case that the students are being able to see that there is nothing special to this example and thus it may as well be a general one (Goldenberg et al. in preparation). Becky made progress because she picked very simple cases for scalar multiplication and addition and did appear to see geometrically that these are preserved. However, she left the problem there without attempting to solve for the general case:

Tom: I believe the reflections are linear transformations and I think if you are in 3 dimensions you are reflecting through the xy-plane, then that takes the vector (x,y,z) to the vector (x,y,-z) so... Is that linear? Here, here, here, yeah, that's linear. In the three components. What I'd like to do is write a matrix that does this. That matrix:

I'm pretty sure that would be. It definitely seems that it should be linear and satisfy both of these properties.

Dean, while solving the problem, commented on linearity in a way that showed very nicely how his visualization helped his analysis.

Dean: If you're reflecting through the plane, you're just changing the z coordinate to negative. So we'll call f(x,y,z)=(x,y,-z), so you'll have a matrix here which would be (1,0,0) which would keep your x in the top, and then (0,1,0) which would preserve the y in the middle, and then (0,0,-1) which would reverse your z coordinate. So its linear because it takes lines to lines and preserves the origin, and its linear because you can write this matrix and so it satisfies these properties.

Tom used the theorem that Dean previously used about the preservation of the origin (0->0) to show that the translation function (problem 1(ii)) is not linear but this did not convince him, so he calculated to show that the property of addition was not preserved.

Tom: L: R2’R2 is a function that moves a point over 3 units to the left and then 4 units up. That's a translation. Maybe it's not linear. (x,y)’(x-3,y+4). No, it's not linear because it doesn't fix the origin. (cx,cy)’(cx-3,cy+4). Definitely can't define a matrix that's gonna get this done by multiplication... Obviously this is not.... If you had (z,w)’(x+2-3,y+w+4) we have . If you add these together you get.... that's not linear. Doesn't satisfy the constant. I know I didn't write this out very clearly but what I'm taking here is... If you add vectors (x,y) and (z,w) this is the image of the vector (x+z, y+w), but this is the image of the vector (x,y) and this is the image of the vector (z,w) and if you add those, this plus this doesn't equal that. So, the translation is not linear and I believe the ultimate reason why is it doesn't fix the origin.

The projection (problem 1(iii)) can be explored in a variety of ways. We saw Tom taking a totally different path to explain both the linearity and the properties of the projection. This student chose to fully explore the geometry of the problem so he used his knowledge about similar triangles to show that the scaling property of linearity is preserved. However, he had trouble keeping his geometric method for proving the additivity so he chose to work completely analytically for that part of the problem:

Tom: O.K. we have the vector (1, 3, 2) and some other vector way up there, so to find the projection, the image under the map, I'm just gonna try to think of what happens geometrically. So, we take c times this vector, magnitude here, there, this is similar triangles so f(cA)=cf(A). Adding vectors... It's gonna be a little harder geometrically. [...] projection is not a real complicated equation so let's write the damn thing out. Let's work with this one. Certainly could be. Sketching is tough, so let me actually write out the projection. It seems like it should be. My intuition is that it should be linear because it's just a map. It does fix the origin.... How is the equation? The magnitude of this vector, there's an angle, we get... [...] So the magnitude of this vector is |x|cosqäv/|v|.

Chad tries to determine geometrically that "over 3 and up 4" does not preserve scalar multiplication, but he is uncertain:

Chad: Well when you multiply by a scalar, it just basically stretches the vector by a set amount. Say our, in this case it's just on the xy plane cause z=0. So we'll say x and y and say, you know we're going over (a) amount and we're going down the same amount. So if we go, our vector gonna be a 45 degree angle pointing downwards. And to the right. If we multiply by 3 it's just gonna be extended 3. If, lets say you had a vector, I'm gonna draw the same, geometrically I'll draw xy, lets say we go over (a) amount and then we go down a-1. It should be something like that. When we multiply this vector by 3. Okay. We're gonna be going over 3a amount right here. And then we're gonna be going down 3a (a-1), (3a-3). Is not gonna take us back to the same, it's not gonna have the same slope. It gonna be a, it gonna be different by a factor of 3, whatever, whatever (a) is. Our slope is gonna differ. Which means it's not closed under scalar multiplication because it didn't just extend the vector, it actually changed the vector.[...] scalar multiplication will just increase the length of a vector, it won't change the direction of it.

In considering a general linear transformation, Matt is unable to give a general formulation and Chad does only a little better. Tom seems to combine geometric and analytic thinking to investigate linearity:

Interviewer: Those two properties taken together, do they have --- some kinda name?
Matt: Uh, linear, subspace, yeah.
Int.: Where is the subspace here?
Matt: It describes, the two equations, hmm. R^2. Because, if R, this function here [function R], sends [...]anything from R^2 to R^2, yeah.
Int.: So now let's go back to these two conditions. What do they have to do with the subspace? Where is the subspace?
Matt: R^2, in R^2. And it, it spans all of R^2.
Int.: If you see a subspace there then you gotta tell me where it is.
Matt: Um, R takes any vector in, R^2, and it, sends it to a new vector. Which is, just that, just, that m-, matrix times x, and y.

Chad: I would think that you'd be allowed to use the distributive property

because we're not, we wouldn't be changing anything. I mean 3A-2B, it doesn't really, it doesn't matter whether or not we do the transformation before or after you do the addition because it's just stretching it or shrinking it, doing whatever. So
Interviewer: So you can stretch or shrink right after or before, it doesn't make a difference.

Interviewer: Just it expands in a couple of directions?
Tom: Geometrically, now I'm thinking geometrically what it does. It definitely fixes the origin and then all it does it stretches in the y-direction, flips and stretches in the x direction, flips and stretches in the z direction. This one I solved by actually writing down what is the equation of the determinant because in three dimensions we have a nice simple equation of the determinant, so you have an actual definition of this map which is a lot easier to look at than that. Number (vii). F:R^3’R^3 is given by...

Becky: [problem 1(i)] This is the reflection --- hmm. Okay, suppose that X is this vector here, and Y is this vector, then --- where am I? Then S acting upon X is going to bring it --- over here, and S acting upon Y will leave it alone. And so the sum of that is here, and if we add these two together [pause] --- and it holds for that case. I would probably want to do another one, without such nice angles, to make sure, but it suggests that that's the same --- I'm trying to reason it out and think.

Becky: [problem 1(iii)] Um, what kind of projection? Hmm. I might have projected in two dimensions, just um, if we had a point here, it projects --- here, so suppose we take this point, which is about half -- it projects --- here, and it's twice... works out really nice with right angles. I wonder if it's not the same with right angles. I think that would be true as well.

While Becky was not able to make a full algebraic investigation to determine the linearity of any of the two functions given in problems 1(i) and 1(iii), her visual representations enabled her use her understanding of the two properties of linearity and move one step further to make the short investigation that is demonstrated here. Her visual image guided her investigation in some special cases but she never reached a general solution.

Matt's thinking helps him move further and prove the linearity of the functions of both problems 1(i) and 1(ii)., More than in any of the rest of the students we interviewed, in this student's thinking we can see the analytic and visual elements successing each other and intermingling to produce a solution. It is this kind of thinking that can be modelled using the Visualization/Analysis model (Zaskis et al. 1996). He chose to investigate and prove the linearity of the function given in problem 1(i) in a geometric way, so he used a sheet of paper and his hands to physically describe the reflection. At first, he made no progress, but then, with the prompt of the interviewer, he started reasoning analytically. This consists of his first act of analysis, which led him to a subsequent act of vusualization. He eventually visually showed fairly clearly that the addititve property of linearity holds using symbols.

Matt: Here's your point. And uh, you have a point here, it reflects it --- through, just through it. So, [...] to the other side. Right underneath it, okay. Simple enough. Um huh. Yes I believe so. Um, how would I check that?
Int.: Well, so far you haven't written anything. But, you made a lot of motions with your hands.
Matt: Every point that would be, above the xy-plane, if, well, say there were two of `em, and their sum, if that were reflected, down to the bot-, underneath, then, that function that reflects them would also reflect their sum. Which could be, then calculated, to be the sum of them. I see. Yeah, it's hard to prove it. So my answer is yes for both...
Int.: Okay, and you've, given us sort of a geometric-intuitive, uh, explanation [...] but, how would you go about proving it if you were going to?
Matt: Mmm. [...] I'd set up two vectors, U and V, say. And sum them together and then show that well, let's see. Show that their k component it would be negative k, and that's what the function will do to them. And then, in that process, that would show that, the sum, the function of the sum is, the function of each of `em. The sum of the functions. [...] Uh, the [k component is the] unit vector, in the k direction.

Unlike Becky who looked at a specific case to check whether the two properties of lineartiy hold, Matt chose two generic vectors and reasoned that what is affected by this transformation is the "k component" and thus the "function that, reflects them, would, also reflect their sum".The same student looked at the projection (problem 1ii) geometrically and first decided it is not linear. Then he considered the issue a little further (not clear if geometric or analytic) and decided that it actually is a linear function. This shows nicely an interplay between geometric and other kinds of thinking:

Matt: A projection on (1, 3, 2). Hmm. [pause]. I would say no. Because, [...] hmm, do it like this: If this were A, and this were B, and we're projecting that to a line 1,3,2)], uh I assume that's a line that's projecting, [...] So, if you just geometrically look at it, this would be the length of, the, the projection of B, call it B'. And, if you're adding, say in the, in the first example, f[X + Y] = f[X] + f[Y], and, this would be your, your projection of A --- huh, this is B', this is A'. Aha. And, so if you, add them together --- hmm, throwing it out maybe it looks like it will. Oh, wait yeah. Huh. It, looks like it --- how would I prove it...

A similar description regarding scalar multiplication was given by Ben, who constructed a visual image as a base and used it to produce a good arguement about this property of linearity for the function given in problem 1(i). Ben, like Becky earlier, used a specific pair of vectors A and B for his demonstration. However Ben is able to see that the vectors he chose could actually be any arbitrary given vectors so he used the specific example to generalize his findings.

Ben: I'm thinking what geometric representation of this would be let's say that this point is a+b. Ya, I think it would be true. Let's say my function is the same. Let's say A is this and say B is this.[...] Since this is a reflection A this whole line B will also reflect up to here A and this whole line B will reflect up to here and they got the same point. I would say the first property is true. And this should be the same thing. Because, this is basically the same idea.
Int.: This meaning the second [scaling] property [of linearity]?
Ben: Ya, multipling by a scalar basically takes a line and stretches it so I think ya, this is fine.

Two other students, Tom and Dean used their geometric interpretation to investigate linearity in a different manner, closer probably to what they were taught in their classes. Both Dean and Tom looked for the matrix of the transformation that was described in problem 1(i) and used their knowledge of matrix multiplication to assert linearity. This is a more standard way to determine linearity, in the sense that these students tried to give themselves what the problem (had it come out of a traditional textbook) would have provided them with: the standard algebraic description of the function. Both succeded in doing it, but both did it by first looking at the visual image that guided them to the matrix of the transformation.

Tom, sketched the three-dimensional system and after drawing a random vector pointed with his finger how each coordinate would be transformed and saying "Here, here, here, yeah, that's linear, in the three components". The writing of the matrix of tranformation served more as the formal and final solution of the problem.