Applying a Piagetian Perspective to Post-Secondary Mathematics Education

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Introduction

Piaget did not write much about how his theories related to pedagogical practice. This was not because of any lack of interest in the topic (the little that he did write makes it clear that he was intensely concerned about education). His reticence seemed, rather, to stem from a certain distrust of pedagogy, at least as it is practiced in traditional schools. In an article ([23]) in which he relates his work to the ideas of Vygotsky (and finds much more agreement than is generally thought) he expressed the view that schools tend to be unaware of the use that could be made of the child's spontaneous development and that education often inhibits this development. Piaget even felt that existing teaching practice could be harmful to intellectual development.

> "In some cases ... the gifts of instruction are presented too soon or too late, or in a manner that precludes assimilation because it does not fit in with the child's spontaneous constructions. Then the child's *development is impeded, or even deflected into barrenness* (emphasis added) ...." (op. cit., p. 246.)

Although he rejected much of traditional education, Piaget did have things to say about how teaching *could* help a child to learn and it is these ideas on which I have based my own work on curriculum development at the post-secondary level, which is the topic of this report.

Because Piaget's writings about education do not seem to be as well known as his other work, I will begin with a brief introduction to his ideas in this area and try to show how they form the theoretical foundation for my own curriculum development activities. The development of this theory proceeds through the analysis of data from students trying to understand mathematical concepts. It is a constructivist theory and it focuses on pedagogy aimed at getting students to make certain very specific mental constructions. I will explain the overall structure of this pedagogical strategy and give examples of how computer activities are used to get students to make these constructions.

I will not report in this paper on the results of this research and curriculum development but rather refer, where appropriate, to relevant published papers which report these results.

1 Piaget on Education

Compared to the total body of his work (about 50 books and over 300 papers), Piaget only wrote about 6 papers and two books on education. Nevertheless, this work represents a substantial contribution to thinking about pedagogy and I cannot, in this article, give more than a few representative samples of what Piaget had to say on this subject. I hope that in the future someone will produce a comprehensive study of Piaget's thinking on education. To help in such an effort, I include in the references at the end of this paper the publications that I think represent Piaget's work on education.

1.1 The limits of pedagogy

Perhaps one reason why Piaget wrote so little about education was his awareness of the limits of pedagogy. As indicated above, he felt that traditional school learning could even be a negative influence on intellectual development. For example, it could be that the way in which formal procedures are taught in school tends to block the child from constructing her or his understanding. He was particularly opposed to the use of examinations which he considered to be a "veritable plague on education" ([22], p. 73). He was similarly concerned with teaching by "verbal transmission" which he felt could only be effective if it was prepared by preliminary activity on the part of the student (op. cit. 129.)

Piaget was also bothered by how little we know regarding the long-term effects of education ([17], pp. 696-697). He felt that what children learned in school did not stay with them and that their knowledge could even regress with time ([22], p. 93.) It is important that education not try to accomplish what is impossible and that it focus on what it can do. There is, according to Piaget, a fixed order of development of the major mental structures ([21], p. 179) and he was convinced that pedagogy can not change this. It can accelerate various aspects of this development, but Piaget also suspected that there was an optimal rate of development ([19], p. 21) and that rather than trying only to make everything happen as fast as possible, teachers should look for, and resonate with, rates that were most natural for their students. Perhaps Piaget was most opposed to what he is said to have referred to as the "American Problem" which is a futile attempt to enhance and accelerate intellectual development through direct training in the tasks Piaget used to locate where children were in their developmental progress ([14], p. 173).

Finally, Piaget considered that the emphasis, in traditional schooling, on learning by association, rather than through assimilation ([16], p. 185) was one reason why use was not made of the child's spontaneous developments ([23], pp. 244-346).

1.2 Intellectual Development

One of the major themes of the seminal work of Piaget and Garcia ([24]) is that the story of intellectual development is not about acquisition of specific bits of knowledge but rather has to do with the emergence of powerful mechanisms by which an individual grows in her or his ability to make sense out of complex situations. These mechanisms include reflective abstraction, the dichotomies of assimilation/accommodation and disequilibration/reequilibration, and the trichotomy of intra, inter and trans (op. cit.) More specifically, conceptual understanding of a given mathematical phenomenon passes through action, process, and object conceptions and these understandings are coordinated into schemas that are used to deal with the phenomenon ([22], pp. 95, 103, [17], p. 704, [18], p. 81.)

The development of these mechanisms takes place to some extent spontaneously as the result of maturation, but only in the presence of appropriate experiences and, as Piaget pointed out on numerous occasions, under the influence of social interaction ([22], pp. 98, 129, [23], p. 247, [16], p. 178.)

1.3 What pedagogy can do

What then, according to Piaget, is the role of pedagogy in this intellectual development and the learning that derives from it? In general terms it is to cooperate with the mechanisms of learning and to help the student develop, become aware of, and consciously invoke them. Indeed, although the mechanisms may develop in the mind of a particular student, he or she may not be aware of them and so may not invoke them in a given situation ([17], p. 71, [18], p. 86.) This is an area in which education can make a contribution.

In Piaget's view, the teacher must begin with the structures that the student has already constructed spontaneously as a result of factors we have described and help the child relate them to mathematical structures as the teacher understands them ([17], p. 72, [19], p. 19, [21], p. 179.) There are, at least, two ways in which the teacher can become able to do this. One is to involve mathematicians. Indeed, Piaget saw "... a great future for cooperation between psychologists and mathematicians in working out a truly modern method for teaching ... mathematics." Another is to really listen to children and pay attention both to what they are saying and what thinking may lie behind their words ([20], p. 24.)

Piaget offered a general program for education from pre-school into high school:

> "Mathematical training should be prepared, starting at nursery school, by a series of exercises related to logic and numbers, lengths and surfaces, etc., and this type of concrete activity must be developed and enriched constantly in a very systematic way during the entire elementary education, to change little by little at the beginnings of secondary education into physical and elementary mechanical experiments. On these terms, strictly mathematical education is grounded in its natural surroundings of equivalency to objects, and will give full scope to the intelligence which would have remained purely verbal or graphic." ([22], p. 104.)

The main strategy for doing this, according to Piaget ([18], p. 85, [23], p. 246, [14], p. 174) is for the teacher to create situations that facilitate the child's discovery or invention of mathematical ideas and to present disequilibrating examples so that the child will develop new ideas in order to re-equilibrate.

Finally, Piaget provided a number of pedagogical recommendations which he had inserted into the recommendations of the International Bureau of Education and UNESCO ([17], p. 703.)

- Guide the student into forming his (sic) own ideas and discovering mathematical relations and properties himself, rather than imposing ready-made adult thought upon him;

- make sure that he acquires operational processes and ideas before introducing him to formalism;

- do not entrust to automatism any operations that are not already assimilated;
- make sure that the student first acquires experience of mathematical entities and relations and is only then initiated into deductive reasoning;
- extend the deductive construction of mathematics progressively;
- teach the student to pose problems, to establish data, to exploit them, and to weigh the results;
- give preference to the heuristic investigation of questions rather than to the doctrinal exposition of theorems;
- study the mistakes made by students and see them as a means of understanding their mathematical thought;
- train students in the practice of personal checking and auto-correction;
- instill in students a sense of approximation;
- give priority to reflection and to reasoning.

2 From Piaget to post secondary situations

Rather than try to extend everything Piaget had to say about the development of intelligence and education, my approach has been to look amongst Piaget's work for those ideas which my experience and, subsequently, investigations suggest are applicable, possibly after some revision, to post secondary situations. In the case of intellectual development, I have tried to do essentially this in two earlier works ([7, 8]). In this article, I will focus on reformulating Piaget's ideas on education for application at the post secondary level.

One serious difficulty in making the transition is that in Piaget's theory, conceptual understanding has its source in the manipulation of physical objects. As the mathematical level of the concepts increase, it is necessary, according to Piaget, to construct new objects, no longer physical but mental, and manipulate these in order to construct mathematical ideas (see, for example, Beth and Piaget [2] and Dubinsky [10].) A major problem in mathematics education is to find appropriate substitutes for physical objects. In my work, the computer has been used for this purpose.

Another obstacle to a Piagetian approach at higher levels of mathematics is that much of Piaget's thinking is related to spontaneous development. Again as the level of sophistication increases, there is less and less of this. Consequently, the role of the teacher in creating situations that will foster the developments that need to take place becomes even more important than it is at the earlier levels on which Piaget concentrated his attention.

Following is a list of those of Piaget's ideas on education, taken from the preceding that I try to implement in the research and development work that I do.

- Focus on the mechanisms by which intellectual development takes place. These include reflective abstraction and the dichotomy of disequilibration/reequilibration.
- Help students construct actions, interiorize them to processes and encapsulate processes to objects.
- Help students to become conscious of the structures they have built, to connect them to mathematical concepts and to make additional constructions to deal with new situations.
- Change the role of the teacher from the disseminator of information to a guide and assistant.
- Pay attention to the voices of the students, their errors and their successes and try to understand their thinking.
- Create situations that directly foster students making mental constructions to deal with mathematical problem situations.
- Let students build experiential bases for concepts before they confront the formalism that structures the concepts.
- Give students an opportunity to discover mathematical concepts before they are explained either by other students or the teacher.
- Establish an environment in which students have an opportunity for rich social interaction with other students as well as the teacher.

There are three major components of my approach to pedagogy that are used in achieving these goals: research in learning, the ACE Teaching Cycle, and cooperative learning.

Research in learning. The studies are mainly qualitative, involving in-depth interviews of students about how they are thinking as they struggle to make sense out of a mathematical situation.

ACE Teaching Cycle. The structure of the course is a cycle with three components: activities on the computer in a laboratory, working in class on problems related to the computer activities and discussing these problems and their solutions; and exercises to reinforce what has been learned and point to future work.

The computer activities consist mainly of implementing various mathematical concepts on the computer and using them to solve problems. They are given to the students in the form of tasks which lead to disequilibration and provide them with an opportunity to build an experiential base for mathematical concepts and discover specific mathematical ideas. Succeeding in these tasks leads to reequilibration.

These activities are designed so that, as a result of doing them, or even trying to do them, the students make the reflective abstractions by which appropriate mental constructions of actions, processes and objects are made.

The problem solving work in the classroom continues all of this and the discussions help students reflect on the computer and classroom activities so that they can become conscious of the structures they are building.

Cooperative Learning. The students do all of this work including their homework and some their examinations in permanent (for the duration of a course) cooperative groups. This provides an environment for social interaction that can enhance the growth of their understandings (and there is specific research attesting to this effect [27].)

In this pedagogical system, the teacher's role begins to move from the central figure through which everything happens to one component of the total learning environment. He or she has a role as guide, facilitator, creator of situations, and assessor, but the basic responsibility for learning and mental constructions required for it become the responsibility of the student.

I turn now, in the remainder of this article to a more detailed description of this pedagogical approach and the research on which it is based. I will begin with some examples of student responses, in an interview, to a question about the order of elements in a group. I will try to show how this motivates the development of a theoretical perspective to use in making sense of the responses. After a somewhat general discussion of the characteristics of a theory of learning, I will explain the theoretical perspective, based on an interpretation of Piaget's ideas, that I work with. Finally, I will consider the ACE cycle again and give some examples of the computer tasks that occur in the activities.

3 Voices of students

The pedagogy described in this paper was used in an abstract algebra course at a large state university in the midwest of the United States. The students were not exceptionally strong and, although many of them were preparing to be high school mathematics teachers, quite a few appeared to be very frightened of subject and almost all were totally lacking in confidence in their ability to do anything more than closely follow the steps of an algorithm.

Just after the course ended, the students were interviewed in depth on a number of questions including the following.

> Suppose that G is a commutative group with an element of order 2 and an element of order 3. Must G have an element of order 6? Give a proof or counterexample. What would happen if 2 and 3 were replaced by other numbers?

Although the class as a whole did rather well on this question, there was a spread of performance and the following four excerpts are representative of the responses that were seen.

The first, Nathan[1], was not very successful. He wrote a collection of symbols such as a, bbb, a^2b, without any apparent coherence. The interviewer asks him to explain what he is doing and he responds as follows.

> Um. I wasn't really sure. Well, you know, if you have an element 2 times 3 is 6, right, okay, so you're gonna have... You see how I can make that connection, right? How can I make this connection 2 times 3 is 6, right. So you got a times a, you got bbb.

[1] The names of students have been changed to provide confidentiality

And if you multiply, you get an element that's in there, but not necessarily in order. Well, no, okay. Okay, e time e is e. So a times a times b times b times b, that's gotta be e. That's an element of order 5 isn't it? And this isn't the same element though. So…

Nathan makes no further progress on the problem.

The next student responds similarly, but begins to organize his activities by writing expressions such as $a, a^2 = e$, $b, b^2, b^3 = e$, and $ab, a^2b^2, ab^3 = a, a^2b = b, ab^2, a^2b^3 = e$.

He recognizes that this means there is an element whose 6^{th} power is the identity.

Ted: ab times ab squared, because it's commutative, a squared b cubed which is e… Because a squared is a and b cubed is e.

I: Okay, I'm with you so far. So, the question is, must it have an element of order six?

Ted: Well, we already know the orders of these. b squared, b squared times b squared is b to the fourth times, which is b cubed times, 2 b's cubed times b, which is e times b, and times b squared, so you multiply that by another b squared … and that's going to be b cubed, which is e. So then that order is three.

Ted: Um, ab times ab is equal to a squared b squared and we know that a squared is e so it would be b squared times ab… is … times ab is ab cubed, which is the same as a, multiplied by ab is a squared b which is b.

I: So your conclusion is what?

Ted: It does have to have an element of order six.

Ted does not show that 6 is the smallest power of ab that gives the identity, and so he has only succeeded to solve half the problem.

Next we see a student, Mitch, who solves the problem for orders 2,3.

Obviously it going to be, the reason why it works is because, um, … they're going to have to be, you're going to have to go that far, I guess, like, so that they're both equal to e and you don't have either a or b left over. And that's why it has to exist, or that, yeah, that is has to exist because otherwise you won't, a times b, um, and that won't

> be equal to the identity until, until they're, until a, um, is equal to the identity and b, or not a itself, but, um a times itself is equal to the identity.

Although he recognizes that the solution is the least common multiple of 2,3, he is unable to use this fact to solve the problem for two elements of arbitrary order!

Finally, we have a student, Jocelyn, who solves the problem in the general case as well and gives the following as her explanation, which is a fairly mature and complete mathematical description of the situation.

> OK so we've got a having a certain order meaning a certain number of times you have to multiply it by itself to get back to the identity. And b having a certain order, the number of times you have to multiply it by itself to get back to the other, So if you have ab together and you take powers of ab ... It's like has a certain no, a has a certain order and b has a certain order, so you are taking ab times itself. So you are going to get to a point where a is going to cycle back to the identity and, you know, b might have a bigger order so it is going to keep getting a bigger order until it cycles back to the identity, while it's doing that a is like going through it's cycle and... so there is like going to be a point when they are going to match up. And that will be the least common multiple.

4 Making sense of the responses

These responses raise two, related, questions.

> How can we explain these differences in performance?
>
> What can be done to eliminate the differences?

There are many possible answers to the first question. The students might simply vary in their ability so that some are more successful than others. It has been suggested that students don't do well because the explanations given in class are not good enough. It is also thought by many that the fact that the question is de-contextualized explains why some students have difficulty with it.

These answers to the first question imply possible solutions: students should be better prepared by the courses they take before abstract algebra; better, more inspiring lectures should be given; and problems should be couched in real-world situations.

We don't have a great deal of research data about such explanations or the corresponding solutions, but I am convinced, based on more than 40 years of college teaching that this approach is inadequate and it is necessary to look for explanations and solutions based on what might be going on in the minds of the students. In my view, the way to do this is through the development of a theory or theoretical perspective.

5 Characteristics of a learning theory

What characteristics might a good theory have? It should be coherent and parsimonious, generalizable and predictive, applicable and effective.

A theory should be coherent in that it provides a reasonable language for making sense out of data. Important concepts which the theory uses should have definitions that are operational in that it is possible for people other than the prime researcher to apply these terms. Nothing should be in the theory that is not really essential and where there are two competing explanations, preference should be given to the simpler one.

It is not helpful if a theoretical explanation is only valid for a given set of students or a particular topic. A good theory should provide tools that can be used in a wide variety of situations. These tools should also allow one to predict the difficulties which students might have. Such predictability is not only a valuable feature of a theory, but it provides a means of assessing its value.

It should be possible to apply a theory to design instructional treatments and these should be effective in improving student learning.

In the next sections I will present and describe one particular theoretical perspective and some pedagogical features that are derived from it.

6 A theoretical perspective

I begin the theoretical discussion with a statement about the nature of mathematical knowledge and its development in an individual.

> An individual's mathematical knowledge is her or his tendency to respond to mathematical problem situations by reflecting on them in a social context and constructing or reconstructing mathematical actions, processes and objects and organizing these in schemas to use in dealing with the situations.

There are many aspects of the education enterprise that are touched on by this statement. For example, the term tendency refers to the fact that very often a student will be asked a question, say on an exam, or in class, and he or she will not appear to know the answer. Yet, at a later time (with no discernible learning experience in-between), or even an earlier time, the student will give a perfectly reasonable response. How should we rate such a performance? Yes, it should not get "full marks" for this should be reserved for the case in which a good response is given every time. But the student should not get zero since this should be reserved for the case in which a reasonable response is never given. Somewhere in between, no doubt, but where? And what implications does this have for examinations. If a student will perform very differently on successive tries, and the exam only allows one try, how does the score reflect the fact that on a second try the student might do much better — or much worse?

Other items in this statement are more easily dealt with. The whole idea of "problem situations" relates to the disequilibration/reequilibration dichotomy. The student must see the problem in the situation and be disturbed by it if learning is to take place. The social context refers, at least, to the role of cooperative learning. But the most important component of this statement, and the one that makes it specific to mathematics, is the part about specific constructions and I turn to this now.

7 Constructions for mathematical knowledge

I will discuss in a little more detail the construction of mental actions, processes and objects that was mentioned earlier. The overall scheme for these constructions is illustrated in Figure 1.

Action. An action is a transformation of objects which is perceived by the individual as being at least somewhat external. That is, an individual whose understanding of a transformation is limited to an action conception can carry out the transformation only by reacting to external cues that give precise details on what steps to take.

Figure 1: Construction of actions, processes, objects

For example, a student who is unable to interpret a situation as a function unless he or she has a (single) formula for computing values is restricted to an action concept of function. In such a case, the student is unable to do very much with this function except to evaluate it at specific points and to manipulate the formula. Functions with split domains, inverses of functions, composition of functions, sets of functions, the notion that the derivative of a function is a function, and the idea that a solution of a differential equation is a function are all sources of great difficulty for students. According to our theoretical perspective, a major reason for the difficulty is that the learner is not able to go beyond an action conception of function and all of these notions require process and/or object conceptions. (See [3] for an elaboration of these issues.)

Another example of an action conception comes from the notion of a (left or right) coset of a group in abstract algebra. Consider, for example, the modular group $[\mathcal{Z}_{20}, +_{20}]$ — that is, the integers $\{0, 1, 2 \ldots, 19\}$ with the operation of addition mod 20 — and the subgroup $H = \{0, 4, 8, 12, 16\}$ of multiples of 4. As is seen in [13] it is not very difficult for learners to work with a coset such as $2 + H = \{2, 6, 10, 14, 18\}$ because it is formed either by an explicit listing of the elements obtained by adding 2 to each element of H or applying some rule, e.g., "begin with 2 and add 4" or an explicit condition such as, "the remainder on division by 4 is 2". Understanding a coset as a set of calculations that are actually performed to obtain a definite set is an action conception. Something more is required to work with cosets in a group such as $\mathcal{S}_n$, the group of all permutations on n objects where simple formulas are not available. Even in the more elementary situation of $\mathcal{Z}_n$, students who have no more than an action conception will have difficulty in reasoning about cosets (such as counting them, comparing them, etc.) In the context of our theoretical perspective, these difficulties are related to a student's inability to interiorize these actions to processes, or encapsulate the processes to objects.

Although an action conception is very limited, actions form the crucial beginning of understanding a concept. Therefore, our learning-theory-based pedagogical approach begins with activities designed to help students construct actions.

Process. When an action is repeated, and the individual reflects upon it, it may be interiorized into a process. That is, an internal construction is made that performs the same action, but now, not necessarily directed by external stimuli. An individual who has a process conception of a transformation can reflect on, describe, or even reverse the steps of the transformation without actually performing those steps. In contrast to an action, a process is perceived by the individual as being internal, and under one's control, rather than as something one does in response to external cues.

In the case of functions, a process conception allows the subject to think of a function as receiving one or more inputs, or values of independent variables, performing one or more operations on the inputs and returning the results as outputs, or values of dependent variables. For instance, to understand a function such as the one given by $\sin(x)$, one needs a process conception of function since no explicit instructions for obtaining an output from an input are given; in order to implement the function, one must imagine the process of associating a real number with its sine.

With a process conception of function, an individual can link two or more processes to construct a composition, or reverse the process to obtain inverse functions [3].

In abstract algebra, a process understanding of cosets includes thinking about the formation of a set by operating a fixed element with every element in a particular subgroup. Again, it is not necessary to perform the operations, but only to think about them being performed. Thus, with a process conception, cosets can be formed in situations where formulas are not available. (See, for example, [13].)

Object. When an individual reflects on operations applied to a particular process, becomes aware of the process as a totality, realizes that transformations (whether they be actions or processes) can act on it, and is able to actually construct such transformations, then he or she is thinking of this process as an object. In this case, we say that the process has been *encapsulated* to an object.

In the course of performing an action or process on an object, it is often necessary to de-encapsulate the object back to the process from which it came in order to use its properties in manipulating it.

It is easy to see how encapsulation of processes to objects and de-encapsulating the objects back to processes arises when one is thinking about manipulations of functions such as adding, multiplying, or just forming sets of functions. In an abstract algebra context, given an element x and a subgroup H of a group G, if an individual thinks generally of the (left) coset of x modulo H as a process of operating with x on each element of H, then this process can be encapsulated to an object xH. Then, cosets are named, operations can be performed on them ([13]), and various actions on cosets of H, such as counting their number, comparing their cardinality, and checking their intersections can make sense to the individual. Thinking about the problem of investigating such properties involves the interpretation of cosets as objects whereas the actual finding out requires that these objects be de-encapsulated in the individual's mind so as to make use of the properties of the processes from which these objects came (certain kinds of set formation in this case).

In general, encapsulating processes to become objects is considered to be extremely difficult ([24], [25], [26]) and not very many pedagogical strategies have been effective in helping students do this in situations such as functions or cosets. A part of the reason for this ineffectiveness is that there is very little (if anything) in our experience that corresponds to performing actions on what are interpreted as processes.

8 Characteristics of this theory.

We are now in a position to see how this theoretical perspective relates to the characteristics mentioned above.

Coherent and parsimonious. We can see how this theory can be used to explain the data presented in our excerpts from interviews. The question of parsimony is somewhat negative (is there anything that does not need to be there) and subjective so I will leave that for the reader to decided.

For Nathan, the powers of elements and the order of an element are barely understood on the action level. We can see that he is unable to reason about them.

Ted definitely has an action conception of these mathematical concepts and this allows him to arrive at a conclusion which solves half of the problem. Because, however, he has not developed a process conception of these manipulations of group elements, he can not reverse them or think about a potentiality so as to consider the possibility of arriving at the identity earlier. Hence, he cannot solve the other half of the problem which is to show that 6 is the smallest positive power of ab that is equal to the identity.

It seems clear that Mitch is performing the calculation of powers in his mind and can think about both actualities and potentialities. This is how he gets the second half of the problem ("you're going to have to go that far"). But he cannot step back from the process and reason about it see how it works for elements a, b of arbitrary order.

In the case of Jocelyn, we have a very sophisticated description of the process and it is reasonable to suspect that she is able to think of this process as an object which can be "indexed" by the values of the orders of a and b.

Generalizable and predictive. I have tried to illustrate the generalizability of this theory by showing how it can be used in a number of different mathematical contexts related, for example, to functions and groups. In our research studies ([3, 4, 5, 12, 13]) we generally present a description of how a concept may be learned based on this theory — and then revise it after analyzing the reactions of students and their difficulties. Although the data often leads to making substantial revisions of our descriptions, we find that more often than not, the theory predicts the mental process that seem to be in the data.

Another indication of generalizability is that many other researchers have been able to use this language.

Applicable and effective. We will see in the next two sections how this theory is applied to develop pedagogical approaches. For its effectiveness, I refer to specific research studies ([3, 4, 6, 11]).

9 A pedagogical strategy

I have already described the ACE Teaching Cycle which structures the pedagogical approach we have derived from our theory. The most important part of this system is the computer activities which are designed to get students to make the specific constructions of actions, processes and objects contained in the theory as applied to the particular concepts in question. I will give some examples of this in the next section.

Other features of the theory are also implemented in our pedagogy. For example, cooperative learning provides a social milieu that is conducive to conceptual development and working in groups tends to help students consider alternative methods and become aware of the structures they are constructing ([27]). The fact that their computer activities require them to deal with problems for which the required mathematics has *not* yet been introduced in the course gives students an opportunity to discover mathematical concepts and build their own understandings of them. Finally, our system of testing allows for group tests and tests without time limits to try to deal with the fact that students may have knowledge and understanding that is not revealed in traditional examinations.

10 Computer constructions

In this section, I give some examples of constructions on the computer which students are asked to make and describe how they relate to actions, processes and objects. The computer work is done in a mathematical programming language called ISETL. The syntax of this language is very close to standard mathematical notation and the reader should have little difficulty in interpreting the syntax. For further information about ISETL and its use in education, see [9].

We begin our examples with some constructions related to elementary group concepts. Students are given only the vaguest descriptions of certain situations (such as "using a binary operation keeps you in context") and asked to write computer programs that test if a given binary operation satisfies the condition. Most of the students come up with something like the following (although usually not with these names.)

```
is_closed := func(S, op);
             return forall a, b in S| a .op b in S;
      end;

is_assoc := func(S, op);
             return forall a, b, c in S|
                a .op (b .op c) = (a .op b) .op c;
      end;

id := func(S, op);
             return arb({ i:  i in S|
                forall x in S| i .op x = x and x .op i = x});
      end;

inv := func(S, op);
             return {[a,b] :  a,b in S| a .op b = id };
      end;
```

They apply these programs to several examples illustrating various possibilities. Then they are asked to use the properties to distinguish certain of the examples and this leads to the following program and its application to modular groups and groups of symmetries.

```
is_group := func(S, op);
               return is_closed(S, op) and
                      is_assoc(S, op) and
                      is_defined(id(S,op)) and
                      forall x in S | exists x' in S | [x,x'] in inv;
            end;


Z5 := {0..4};
plus5 := func(a,b); return (a+b) mod 5; end;
is_group(Z5,plus5);

S5 := [a,b,c,d,e] :  a,b,c,d,e in 1,2,3,4,5 | #a,b,c,d,e=5;
comp := func(x,y,); return [x(y(i)) :  i in [1,2,3,4,5] ]; end;
is_group(S5, comp);
```

The above examples are designed to help students construct actions corresponding to group properties, interiorize them into processes (by writing computer functions that implement them) and to encapsulate these processes into objects by including them in a list that checks the property of being a group.

In the next example, we show computer activities designed to help students understand cosets first as processes formed by multiplying a fixed element of a group with every element of a subgroup and then as objects in the sense that it is possible to apply an operation (product) to two cosets to obtain another. As a result of such activity students tend to be more successful with Lagrange's theorem and quotient groups [1]. They are given the following task to perform.

> Write an ISETL `func PR` that accepts a set and a binary operation and returns a `func` that accepts two inputs which can be any combination of elements of `G` and/or subsets of `G`. Your `func` is to determine which of the four cases the inputs fall in and then return the "generalized group product" of the two inputs. That is, if the inputs are both elements, then the product is the usual group product. If the first input is an element `g` and the other is a subset `S` then the product is the set of all products `g .o x, x` in `S`. Similarly for the other two cases.

This is a very difficult problem for the students and they struggle with it for a long time. In the end, and in some cases with some help from the instructor, most students come to something like the following. It is interesting to note how simple is the actual program. Although there is no evidence, I am convinced that the difficulties students have with this task are not because of any programming obstacles, but because they are in process of making non-trivial mental constructions.

```
PR := func(G,o);
       return func(x,y);
               if x in G and y in G
                  then return x .o y;
               elseif x in G and y subset G
                  then return { x .o b :  b in y};
               elseif x subset G and y in G
                  then return { a .o y :  a in x};
               elseif x subset G and y subset G
                  then return { a .o b :  a in x, b in y};
               end;
              end;
      end;

oo := PR(S5, comp);
```

Finally, we give an example related to mathematical induction. Here the difficulty is first that propositions must be objects so that they can form the range of a function. Next, a function must be understood as a process which can transform any kind of object, say an integer, into any kind of object, say a proposition. These two difficulties are dealt with in the context of the following problem by asking students to write a computer function that implements the statement in the following problem.

> Show that, in a casino with chips worth \$5 and \$9, any sufficiently large amount of money can be represented.

They do not have too much difficulty in writing the following program and they learn to begin an induction problem by constructing, on the computer or in their minds, a function which transforms positive integers into propositions.

```
P := func(n);
       return (exists x,y in {0..n} | 5*x+9*y=n);
     end;
```

It turns out that not all types of propositions are understood as objects at the same time or with the same degree of difficulty. The hardest, of course, is the implication. Students are asked to write a computer function that will convert a proposition valued function of the positive integers into another such function in which the proposition has been changed to the implication from n to $n+1$.

```
impl_fn := func(P);
              return func(n);
                        return P(n) impl P(n+1);
                      end;
           end;
```

After writing this code and applying in induction situations, students seem to stop having trouble realizing that in induction, one is not trying to prove the original statement, but only that its truth for one integer implies its truth for the next integer. Once this is achieved, the domino or ladder ideas become meaningful for the students.

11 Conclusions

I have tried in this article to survey the ideas of Piaget on mathematics education and explain how they can be applied to collegiate level mathematics. I have indicated how this work is based on a theoretical perspective largely influenced by Piaget's theory and driven by the reactions of students to mathematical problem situations. Finally I have shown how this theory can be applied in developing pedagogical strategies, the results of which are reported in the literature.

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