RESEARCH, THEORY, TECHNOLOGY AND OTHER DANGEROUS PEDAGOGICAL CONSIDERATIONS

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There are four points we would like to make in this chapter:

1. In mathematics curriculum development at the collegiate level a major role should be played by research in learning. Moreover, this research should be based on theories of learning.
2. Research in learning can be of two kinds. First, it can provide direction for pedagogical strategies and second, it can tell us something about what may have happened when these strategies were implemented. *In fact, we feel that a powerful approach is to combine the two* so that research in learning takes place in the context of a particular and clearly specified set of pedagogical strategies.
3. There are certain characteristics that we feel a theory should possess in order for it to be most helpful for research in learning.
4. It is possible to design and implement an approach to such curriculum development that incorporates and coordinates: a theory of learning, innovative pedagogy including the use of computers and cooperative learning, gathering and analysis of both quantitative and qualitative data. This can be done following a paradigm that relates to the cognitive, social and affective domains while maintaining a strong mathematical content.

A large part of our justification for these assertions lies in the fact that we can point to research studies, based on the implementation referred to in 4., that provide supporting data for our position. In a nutshell, one argument for conducting theoretically based research in learning and applying it to curriculum development is that it can work.

In Section 1 we will describe how we combine the use of research to drive pedagogy and to assess learning that may have taken place so as to form a synthesis of curriculum development and research. Note that, generally speaking, the term "curriculum" refers to both content and pedagogy. Our investigations and reflections have convinced us that the most important reforms that are needed in collegiate mathematics education are in pedagogy and this is where we have focussed our efforts. For this reason, we will use pedagogy interchangeably with curriculum.

In Section 2 we will present our arguments for the use of theory in mathematics education research and also describe what we consider to be the essential aspects of a theory of learning.

In Section 3 we will describe APOS Theory which is is the tool that we use in making our theoretical analysis.

In Section 4 we will outline the pedagogy that is used in our approach. We will show how our choice of innovative strategies was directed by their appropriateness to our theoretical learning paradigm. For example, we use computers not only for their computing powers, nor just for their visualization tools, as is often done. We will describe another use of software that makes it possible to enhance learning processes that are predicted by our theory.

Finally, In Section 5 we will indicate our methods of gathering data and point to the results that have been obtained in using our approach.

Throughout the chapter, we will present several examples from our work in precalculus, calculus, discrete mathematics, and abstract algebra. Current projects concern linear algebra and differential calculus. Although we are far from having studied more than a portion of these areas in any depth, the breadth of our work covers the whole of the first two years of collegiate mathematics.

1 A paradigm for research and curriculum development in collegiate mathematics education

Our paradigm consists of three components and their interactions. The three components are: theoretical analysis, design and implementation of instruction, and collection and analysis of data. The structure is illustrated in Figure 1.

The theory that we use (called APOS Theory and discussed in Section 2) is a constructivist theory and its first role is to propose mental constructions that students might make in order to develop their understanding of a particular mathematical topic. The design of instruction focuses on means of helping students make these constructions. Of course the specifics of the mathematical content plays an essential role, but the instruction is driven by the desired mental constructions. We will say a little in Section 3 about the pedagogical strategies employed.

LOCATE FIGURE 1 HERE.

After the instruction is implemented data is collected (see Section 4 for a brief description of the kinds of instruments used) and analyzed.

The relation between the data analysis and theory is bi-directional. On the one hand, the theoretical analysis tells us what questions to ask of the data. On the other hand, the analysis of the data may call into question some of the aspects of the original theoretical analysis and call for changes in it, or even in the theory itself.

The cyle is then repeated, presumably with a new theoretical analysis leading to revised instructional treatment and, hopefully, better data. This continues until it is considered that the approach leads to satisfactory levels of student learning of the topics being considered.

We refer the reader to Asiala et al (1996) for more details on this framework.

2 Theories of learning and their role in education research

As we mentioned above, the most straightforward reason for using theory in essential ways is that it can lead to relatively high levels of student learning and understanding. For example, in Weller et al (in review), there is a summary of papers reporting on the effectiveness of using theory in the way described in this chapter.

Another reason for using theory is that it can give direction to an otherwise confusing endeavor. For example, in the paradigm discussed in the previous section, theory is used at each stage to direct activities. Theory provides goals for instruction; it tells what questions to ask of the data; and it forms the basis for assessing student performance.

Finally, theory can provide a coherent basis for generalization. Of course, there is no proof, but if one has a coherent theory that predicts performance on the basis of certain conditions and if students who meet those conditions do indeed learn, then there is some justification for generalizing to other students the results obtained in specific studies.

2.1 Characteristics of a theory

In a research field as new as mathematics education, with almost no generally accepted theories but attempts by several researchers (including us) to establish a theory, it is important to pay attention to the characteristics that a theory should possess.

Following is one list with some indication of how the theory considered in this chapter meets those conditions.

Support prediction. It should be possible to make statements of the form: If such and such happens (does not happen), then learning will (will not) take place. In our paradigm, the theory provides specific mental constructions, organized in what we call a *genetic decomposition* for learning a particular concept. We then assert that if these constructions are made by a student, then he or she is likely to develop an understanding of that concept. Our research reports present results indicating the extent to which these predictions hold.

Possess explanatory power. It should be possible to use a theory to explain why a student did or did not learn. In our case, we can make such explanations in terms of whether the proposed mental constructions are or are not met.

Be applicable to a broad range of phenomena. It is not enough to construct a new theory for each situation. It should be possible for a theory to be built around one (or a small number) of phenomena and then be used to study other phenomena, not closely related to the original. APOS Theory was originally developed in the context of mathematical induction and predicate calculus. It has been applied, often with little or no modification to functions, derivatives, integrals, binary operations, groups, cosets, quotient spaces and many other topics.

Help organize thinking about learning phenomena. The reduction made by APOS Theory of a wide class of learning phenomena to specific constructions of actions, processes, objects and schemas, along with various methods of making such constructions, provides a powerful organizing tool for thinking about these phenomena.

Serve as a tool for analyzing data. Our method of analyzing data as described in Asiala et al (1996) consists of comparing, at the finest grain level practicable, students who succeed and students who do not succeed in understanding a mathematical point. The researcher then tries to find explanations of the differences (by looking at interview transcripts, written comments, etc.) in terms of making or not specific constructions of actions, processes, objects and/or schemas.

Provide a language for communication about learning. It appears that in the literature there is widespread use of terms such as action, process, object, schema, interiorization, encapsulation, thematization. In many cases this usage was a direct influence of APOS Theory. It seems that many others, who do not use APOS Theory, find these terms useful.

2.1.1 How a theory should be used.

This is of course a matter of opinion, but it is an essential part of any framework within which research is conducted. In our view, theories of learning mathematics, like theories in the physical sciences, do not tell us what is reality, the truth. In other words, a theory is neither correct nor incorrect. Rather a theory is a tool for doing research and improving teaching. Therefore, rather than its correctness, we are interested in its effectiveness.

One conclusion that we have come to is that one should not change theories in midstream. Some researchers look at their data and for each point, select a theory to use in working with it. We feel that there is value in choosing a theory at the outset of a particular study and then sticking to this choice in a disciplined way until the study is completed. Of course, it is perfectly reasonable to use different theories for different studies.

3 APOS Theory

In this section we will sketch the general theory in its present stage of development and describe the nature of mental constructions that the theory proposes along with some examples.

The original source of APOS theory was Piaget's epistemology of mathematics learned from infancy through adolescence (see, for example, Beth & Piaget, (1966)) and an attempt to apply his mechanism of reflective abstraction to learning post secondary mathematics.

APOS theory begins with a statement of what it means to learn and know something in mathematics.

> An individual's mathematical knowledge is her or his tendency to respond to mathematical problem situations by reflecting on them in a social context and constructing or reconstructing mathematical actions, processes and objects and organizing these in schemas to use in dealing with the situations.

There are, in this statement, references to a number of aspects of learning and knowing. For one thing, the statement acknowledges that what a person knows and is capable of doing is not necessarily available to her or him at a given moment and in a given situation. All of us who have taught (or studied) are familiar with the phenomenon of a student missing a question completely on an exam and then really knowing the answer right after, without looking it up. A related phenomenon is to be unable to deal with a mathematical situation but, after the slightest suggestion from a colleague or teacher, it all comes running back to your consciousness. Thus, in the problem of knowing, there are two issues: learning a concept and having it available for use when needed.

Reflection is an important part of both learning and knowing. Mathematics in particular is full of techniques and algorithms to use in dealing with situations. Many people can learn these quite well and use them to do things in mathematics. But understanding mathematics goes beyond the ability to perform calculations, no matter how sophisticated. It is necessary to be aware of how these procedures go, to get a feel for the result without actually performing all the calculations, to be able to work with variations of a single algorithm and to understand relationships among algorithms.

It is a controversial point, but this theory takes the position that reflection is best performed in a social context. There is evidence in the literature (see Vidakovic (1993), for example) of the value to students of social interaction and there is also the cultural fact that almost all research mathematicians feel very strongly the need for interactions with colleagues before, during, and after creative work in mathematics.

APOS theory asserts that "possessing" knowledge consists in a tendency to make mental constructions that are used in dealing with a problem situation. Often the construction amounts to reconstructing (or remembering) something previously built so as to repeat a previous method. But progress in the development of mathematical knowledge comes from making a reconstruction in a situation similar to, but different in important ways from, a problem previously dealt with. Then the reconstruction is not exactly the same as what existed previously, and may in fact contain one or more advances to a more sophisticated level. This whole notion is related to the well known Piagetian dichotomy of assimilation and accomodation (Piaget, (1972)).

Finally, the question arises of what is constructed, or what is the nature of the constructions and the ways in which they are made? It is when we talk about this that our theoretical perspective, which may appear applicable to any subject whatsoever, becomes specific to mathematics. We will deal with this question in the next paragraph.

3.1 Mental constructions for learning mathematics

As illustrated in Figure 2, understanding a mathematical concept begins with manipulating previously constructed mental or physical objects to form actions; actions are then interiorized to form processes which are then encapsulated to form objects. Objects can be de-encapsulated back to the processes from which they were formed. Finally, processes and objects can be organized in schemas.

Actually, there is a potentially misleading aspect of this description in that there may be too much of a suggestion of a linear progression from action through process to object and, ultimately, schema. In fact, although something like a progression can be discerned, it often appears more like a dialectic in which not only is there a partial development at one level, passage to the next level, returning to the previous and going back forth, but also the development of each level influences both developments at higher and lower levels.

LOCATE FIGURE 2 HERE

In describing briefly the main mental constructions, we will use the example of cosets of a subgroup, Lagrange's Theorem and quotient groups of a group.

Action. A transformation is considered to be an action when it is a reaction to stimuli which the subject perceives as external. This means that the individual requires complete and understandable instructions giving precise details on steps to take in connection with the concept.

One example of an action conception comes from the notion of a (left or right) coset of a group in abstract algebra. Consider, for example, the modular group $[\mathcal{Z}_{20}, +_{20}]$ — that is, the integers $\{0, 1, 2 \ldots, 19\}$ with the operation of addition mod 20 — and the subgroup $H = \{0, 4, 8, 12, 16\}$ of multiples of 4. It is not very difficult for students to work with a coset such as $2 + H = \{2, 6, 10, 14, 18\}$ because it is formed either by a listing of the elements according to some rule ("begin with 2 and add 4") or an explicit condition such as, "the remainder on division by 4 is 2". This is an action conception. Something more, (such as a process conception) is required to work with cosets in a group such as $\mathcal{S}_n$, the group of all permutations on n objects where simple formulas are not available. Even in the more elementary situation of $\mathcal{Z}_n$, students will have difficulty in reasoning about cosets (such as counting them, comparing them, etc.) which requires an object conception.

According to APOS theory, all of these difficulties are related to students' inability to interiorize these actions to processes, or encapsulate the processes to objects.

Although an action conception is very limited, it is an important part of the beginning of understanding a concept. Therefore, instruction should begin with activities designed to help students construct actions.

Process. When an individual reflects on an action scheme and interiorizes it, then the action can become perceived as a part of the individual who can establish control over it.

For cosets, a process understanding consists of thinking about the formation of a set by operating a fixed element with every element in the subgroup. It is not necessary to perform the operations, but only to think about them being performed. Thus, in the case of the coset $2 + H$ above, thinking about its formation as adding (mod 20) 2 to each element of H, whether or not the computations are actually made is moving towards a process conception of coset.

Object. When an individual reflects on operations applied to a particular process, becomes aware of the process as a totality, realizes that transformations (whether they be actions or processes) can act on it, and is able to actually construct such transformations, then he or she is thinking of this process as an object.

In the course of performing an action or process on an object, it is often necessary to de-encapsulate the object back to the process from which it came in order to use its properties in manipulating it.

In the case of cosets, given an element x and a subgroup H of a group G, if an individual thinks generally of the (left) coset of x modulo H as a process of operating with x on each element of H, then this process can be encapsulated to an object xH. Actions on cosets of H, such as equipping the set of cosets with a binary operation (quotient group), or counting their number, comparing their cardinality, and checking their intersections (Lagrange's Theorem) can make sense to the individual. Thinking about the problem of investigating such properties involves the interpretation of cosets as objects whereas the actual finding out requires that these objects be de-encapsulated in the individual's mind so as to make use of the properties of the processes from which these objects came (certain kinds of set formation in this case.)

Schema. Once constructed, objects and processes can be interconnected in various ways: for example, two or more processes may be coordinated by linking them through composition or in other ways; processes and objects are related by virtue of the fact that the former acts on the latter. A collection of actions, processes, and objects can be organized in a structured manner to form a schema which may also include previously constructed schemas. The structure of a schema has coherence in the sense that an individual understands, implicitly or explicitly, which phenomena the schema can be used to deal with.

Schemas themselves can be treated as objects and included in the organization of "higher level" schemas.

For example, sets can be formed as objects and linked with binary operations to form pairs which may or may not satisfy certain properties. All of this can be organized to construct the schema for coset. Groups and rings and other such mathematical objects might be organized in a schema called algebraic structures.

The idea of schema is very important for our story in that it is one of the few cases in which difficulties in analyzing data have led to significant revision of the theory. In research studies connected with the chain rule (Clark et al, 1997), using properties of the derivative to draw the graph of a function (Baker et al, 2000) and understanding sequences of numbers (Mathews et al, to appear), researchers found the mechanisms of actions, objects and processes inadequate. Moreover, the idea of schema had only a superficial description as given in Dubinsky (1991) where the development of a schema is not explained. The notion of the *triad* developed by Garcia & Piaget (1983) could be seen to be very closely related to that of schema and it turned out that it could be used to provide better explanations of what a schema is and how it is constructed.

In the triad notion an individual's understanding of a concept develops through three stages: the *intra* stage in which there is a focus on single objects (which might be encapsulated processes or thematized schemas); the *inter* stage in which there is a construction and understanding of transformations between these objects (such as the relations of Skemp (1976) or interiorized actions); and finally *trans* in which there develops a coherence among the transformations in that the individual constructs an organized system of transformations including both those which have been experienced and those that are only potential, so as to understand and be ready for, explicitly or implicitly, new situations. It is this coherence that gives substance to our notion of schema and connects the various actions, processes, objects and other schemas.

Thus, for example, in the concept of the chain rule, an individual might begin at the intra level by being able to take the derivative of individual epxressions obtained by replacing the variable in a given expression with another expression. Then at the inter level there are certain “rules” such as the “power rule” or the “inside-outside” rule; finally at the trans level, all of these individual rules are encompassed by a single operation of taking the derivative of the composition of two functions. The result is a schema in which the individual understands that if a function can be writtten as a composition of functions whose derivatives are known, then the derivative of the original function can be obtained.

In this way, APOS Theory connects with the Triad Theory through the notions of coherence and schema, to form a more powerful mechanism for the investigation of learning advanced mathematical concepts.

The Triad Theory is not the only case in which APOS Theory relates to other theories of learning found in the research literature. Other examples include the operational/structural characterization of Sfard (Sfard, 1991) which is very similar to our process/object analysis, the concept-image/concept-definition dichotomy of Tall and Vinner (Tall and Vinner, 1981), the procept notion of Gray & Tall (1991, 1994), and the didactical engineering of the French school (see, for example, Farfan (1997) and references therein.) In some cases such as the use of the Triad to develop the schema concept discussed above and the work of Arnon (1998), in combining APOS Theory with Nesher's Theory of Learning Systems, a synthesis of two different but related theories leads to advances in our understanding of how learning happens and how it can be improved.

4 A theory- and research-based curricular approach

The pedagogical strategy we have designed and used is a structure that we call the *ACE Teaching Cycle*. It is a methodology that makes use of cooperative learning and students writing computer programs. In this design, the course is broken up into sections, each of which runs for one week. During the week, the class meets on some days in the computer lab and on other days in a regular classroom in which there are no computers. Homework is completed outside of class. We have the students working in cooperative groups in all of these activities.

Following is a description of the three components of this structure with some indications of the pedagogical goals of each component and examples for the topic of coset.

Activities. The course meets in a computer lab where students work in teams on computer tasks designed to foster the specific mental constructions proposed in the genetic decompositions of the course topics. The lab assignments are generally too long to finish during the scheduled lab and students are expected to come to the lab when it is open or work on their personal computers, or use other labs to complete the assignment.

In the abstract algebra course there were a large number of computer activities through which students could construct, on the computer, many of the basic processes and objects of elementary group theory such as: examples of binary operations, examples of groups; tests for closure, associativity, existence of an identity, inverses, and commutativity.

After some work with such constructs, the students were given a rather difficult task. They were asked to write a computer program that would accept a set and a binary operation that formed a group and return a new binary operation that could accept any of the four combinations of inputs that were either elements of the set, or subsets. This binary operation would then determine what the inputs were and, as they dictated, return the product (in the group) of the two elements, a left coset, a right coset or the set of all products of elements of the two sets.

This is an extremely difficult task for the students, but the program they must write is actually very simple. As a result, all of their struggles to get this program working correctly led them to confront important mathematical issues and construct deep meanings for concepts related to cosets. Moreover, succeeding in the tasks required by this program coresponds to making certain specific mental constructions. We can see this by looking at a program that solves this problem and is about the same as the programs which the students tended to write..

```
PR := func(G,o);
       return func(x,y);
               if x in G and y in G
                  then return x .o y;
               elseif x in G and y subset G
                  then return { x .o b :  b in y};
               elseif x subset G and y in G
                  then return { a .o y :  a in x};
               elseif x subset G and y subset G
                  then return { a .o b :  a in x, b in y};
               end;
              end;
      end;

oo := PR(S5, comp);
GH := {{g .oo h :  h in H } :  g in G };
K .oo L;
```

How does the construction of such a program foster students' construction of knowledge according to the APOS Theory? For example, in accepting a binary operation (the parameter, `o`) as part of the input and returning a `func` that is a binary operation, the students are led to think of binary operations as objects. Working out the steps of the `func` requires that they de-encapsulate this general object conception and return to thinking of a binary operation as a process. Treating the inputs `x`, `y` of the `func` that the program constructs and using them either as elements of the group or as cosets, leads students to encapsulate a process conception of a coset to an object conception.

In the last three lines, the call to `PR` defines the new operation, computes the set of all cosets of the group `G` by the subgroup `H` and computes the coset product of the two cosets `K` and `L`.

One can see that the technology we implement does not make use of innovative or sophisticated technological tools. Although relatively simple from the point of view of technology, it is not at all simple or easy from the point of either teaching or learning strategies. But APOS predicts its support to students' development of mathematical concepts, and research supplies evidence supporting this prediction. The way we use technology in our pedagogy makes the kind of technology and the ways we use it indispensable in our pedagogy, as well as difficult to replace by other computerized tools.

Class. The course meets in a classroom where students again work in teams to perform paper and pencil tasks based on the computer activities in the lab. The instructor leads inter-group discussions designed to give students an opportunity to reflect on the calculations they have been working on and further construct their own meaning for their mathematical experiences. On occasion, the instructor will provide definitions, explanations and overviews to tie together what the students have been thinking about.

In the case of cosets, constructing the lines of the program that form left and right cosets leads, acccording to APOS Theory, to students constructing in their minds a process conception of cosets. Returning the two cosets as the result of the operation fosters development of an object conception. Finally, writing code to form the product of all pairs of elements from two cosets relates to de-encapsulating these objects back to the processes from which they came. There are also many other examples of constructions in this short program that relate to process and object conceptions of other mathematical entities such as group as object and binary operation as both process and object.

After writing and working with the above program PR, students can find it meaningful when they are asked in class to talk in their groups about the possiblity of the intersection of two cosets, the number of elements in a coset, and set of all elements that are in at least one coset. Each of these questions has a complete answer for which the formal proof is rather easy for students who have constructed meanings for all these ideas. It is not difficult for the instructor to lead a discussion that brings out, not only the facts, but reasonable proofs of what amount to the steps in a proof of Lagrange's theorem. Furthermore, the computer experiences lead many students to think about cosets as objects and the result of PR as a binary operation on them. In class, they are asked to consider if the group axioms are satisfied and the notions of normality and quotient groups emerge.

Exercises. Relatively traditional exercises are assigned for students to work on in teams. These are expected to be completed outside of class and lab and they represent homework that is in addition to the lab assignments. The purpose of the exercises is for students to reinforce the ideas they have constructed, to use the mathematics they they have learned and, on occasion, to begin thinking about situations that will be studied later.

5 Gathering Data and Results

Gathering and analyzing data in our approach is a complex matter. This is for several reasons. One reason is that as we indicated in Section 1, our analysis of data is both driven by, and drives, our theoretical analyses and possibly even the development of the theory itself. Another is that we do not feel that a choice between gathering quantitative or qualitative data is appropriate. In our view, the best research will combine the two forms of information. This introduces complexity both in the interaction of two diferent types of results and the special needs of qualitive research which can be less straightforward than quantitative investigations.

We have already discussed the interaction between data and theory in Section 1. Regarding the qualitative/quantitative approaches, we don't just use both, but try to develop a synthesis of the two. For example, one can administer a written instrument to a large number of subjects. We do this at times and analyze the results statistically. But we also use such data in the following way. Often it is possible to partition a large number of subjects into a relatively small number of categories by putting together people in a category whose responses on the written instrument are similar. Then we can apply the more time-consuming and laborious qualitative methods, such as in-depth interviews, to one or two representatives in each category. Our assumption is that the interview transcripts we obtain in this manner both tell us a great deal about the student being interviewed and are also, to a greater or lesser degree, representative of all of the students in the same group.

Another purpose of combining the two kinds of data is triangulation. An in-depth interview can bring out a perhaps unexpected phenomena that needs to be explained. One can use written instruments to prepare for an explanation for example by considering questions such as: With how many subjects does this phenomena occur? What are its variations? What relationships does it have to other features of the subjects? Triangulation is of course, of critical importance in qualitative research. In addition to using combinations of different forms of data, we also take advantage of the fact that research in this approach is generally conducted by a team. This allows us to have different people on the team interpret qualitative data independently. Some of our best insights have been obtained from discussions designed to resolve the inevitable differences in interpretations. Triangulation consist in that when several researchers can agree on an interpretation, its validity and reliability is enhanced.

Again returning to Section 1, we recall that our data, both quantitative and qualitative tries to answer two kinds of questions: Have the students made the mental representations that the instruction was designed to foster? and, To what extent do the students appear to understand the mathematics? Of course, the critical issue for our approach is whether there seems to be a causal relation from the former to the latter.

The answers we can give to such questions are contained in a number of studies of courses conducted according to our approach through which a very large amount of data was collected. Descriptions of methodologies and summaries of results are given in Weller et al (in review). There is not sufficient space here for any details, but perhaps the reader will be interested in the final paragraph of that report.

> Our results seem to point to the success of this theoretically-based approach as a valid tool by which students learn advanced mathematical concepts.

6 Conclusions

In this chapter we tried to lay out our ideas about the relations between theory and research in mathematics education, and the cotribution their appopriate use can make to the development of mathematics education as a scientific discipline. Their implementation in the practice of teaching can contribute to their mutual growth, in the sense that the data collected and analyzed in research based upon a specific theory can be used to reconsider and reshape that theory. New teaching strategies, such as cooperative learning and special uses of technology, are designed and implemented in the light of the growing theory.

Researched based on a specific theory is designed for several purposes:

- To throw light on the way pedagogical strategies work;
- To supply evidence of the effectiveness (or lack of it) of these strategies for

improving learning;

- To improve out understanding of how specific mathematical concepts might develop; and,

- To improve the theory.

We have described APOS (Action, Proces, Object, Schema) Theory as an example of the above ideas and an illustration of their feasibility. The main teaching strategies that are used in conjunction with APOS Theory are: having students write programs in a mathematical programming language — in this case, ISETL; cooperative learning; and the ACE Cycle of programing Activities, Class discussions and reinforcement Exercises.

These strategies are used to enhance the construction of the actions, processes, objects and schemas related to every mathematical knowledge-domain, as proposed by APOS Theory. As both the research and the teaching strategies we described were theory-based, we suggested certaain criteria for judging a theory of learning. These are: support prediction, possess explanatory power, be appliable to a broad range of phenomena, help organize thinking about learning phenomena, serve as a tool for lanalyzing data, and provide a language for communication about learning.

Finally, we pointed to evidene suggesting that APOS tends to meet these criteria. This theory is indeed used by a growing number of researchers who are involved in research of the kind described above, investigating an increasing number of mathematical topics. Innovative teaching strategies such as cooperative learning and certain ways of using technology fit well with APOS theory and the evidence we referred to suggests that the resulting pedagogy can be highly effective.

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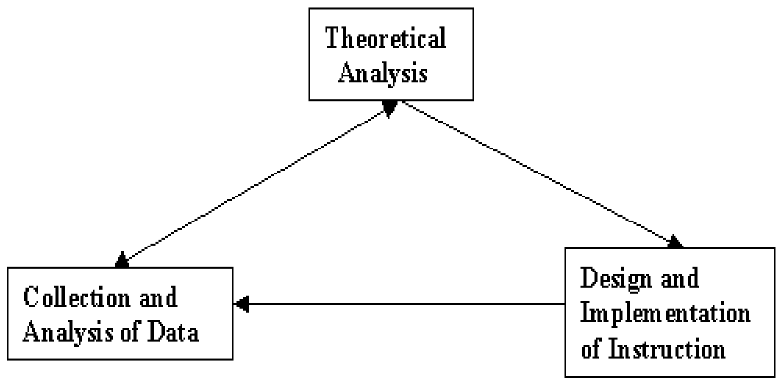
Theoretical Analysis
Collection and Analysis of Data
Design and Implementation of Instruction

Figure 1

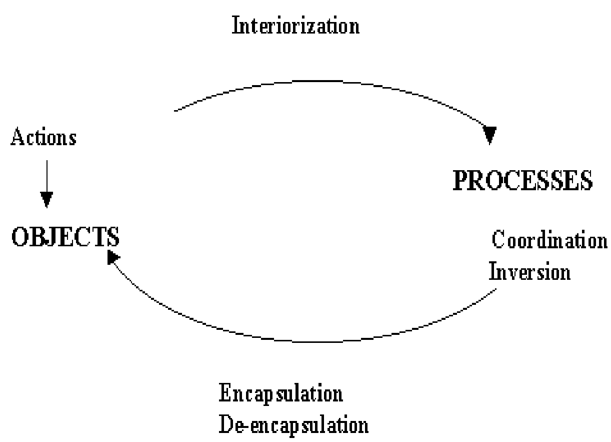
Interiorization
Actions
PROCESSES
OBJECTS
Coordination
Inversion
Encapsulation
De-encapsulation

Figure 2