Running Title: An APOS analysis of infinity issues

Some historical issues and paradoxes regarding the concept of infinity: An APOS analysis, Part 2

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SOME HISTORICAL ISSUES AND PARADOXES REGARDING THE CONCEPT OF INFINITY: PART 2

Abstract

This is Part 2 of a two-part study of how APOS Theory may be used to provide cognitive explanations of how students and mathematicians might think about the concept of infinity. We discuss infinite processes, describe how the mental mechanisms of interiorization and encapsulation can be used to conceive of an infinite process as a completed totality, explain the relationship between infinite processes and the objects that may result from them, and apply our analyses to certain mathematical issues related to infinity.

KEYWORDS: APOS Theory, limit, Encapsulation, History of mathematics, Human conceptions of the infinite, Infinite processes, Infinitesimals, Natural numbers

1 Introduction

The main purpose of this study is to attempt to go beyond the standard mathematical resolutions of various dichotomies and paradoxes of the infinite to obtain cognitive explanations, i.e., descriptions of how one might think about these issues and their resolutions. These explanations are given in terms of APOS Theory. The basic ideas of this theory, its uses in mathematical education research and curriculum development, its role in this investigation, and the potential pedagogical advantages of the use of a theory of learning have been discussed in Part 1 (Dubinsky et al., 2005).

In Part 1, we considered the distinction between potential and actual infinity, the difference between actual infinity and the notion of attainability, paradoxes related to infinity, and the relationship between finite and infinite phenomena. Here, in Part 2, we use APOS Theory, together with the analyses made in Part 1 and preliminary findings of several research studies, to discuss the following issues related to infinity: the nature of infinite processes (Section 2.1); an infinite process as a completed totality (Section 2.2); mental objects that may result from infinite processes (Section 2.3);

conceptions of the set $\mathbf{N}$ of natural numbers (Section 3.1); the relation $.999\ldots = 1$ (Section 3.2); and infinitesimals (Section 3.3).

These problems present potential stumbling blocks in an individual's development of an understanding of infinity. Using APOS Theory we propose cognitive explanations and resolutions that are expected to lead, eventually, to the design of pedagogy that may help students develop mathematically useful conceptions of various aspects of infinity.

As in Part 1, our sources include historical texts (not always in the original, but as reported by various scholars), the experiences of mathematicians (including ourselves) in thinking about and teaching the concept of infinity, and preliminary information from ongoing empirical studies as sources for how mathematicians, philosophers, and students may be thinking about the infinite.

It is important to note that the mental mechanisms of interiorization and encapsulation and the mental structures of action, process, object, and schema, which are the components of the cognitive analyses we make in Sections 2 and 3, can be used to describe the "everyday" activities of people doing mathematics. These include constructing the concept of number, establishing the commutativity of addition, understanding multiplication, conceptualizing mathematical induction, and formulating the concept of function (see Dubinsky, 1991 for a discussion of how these mechanisms may apply in understanding mathematical concepts). In that sense, it appears as though such structures are available to many people involved in mathematical work not explicitly related to the concept of infinity. Thus, our use of these structures to explain how people may think about infinity does not represent the introduction of totally new tools, but the extension to (not very) different ways of using these cognitive tools in the understanding of infinity concepts. For more information about APOS Theory and a summary of how it has been used in mathematics education research, see Dubinsky and McDonald (2001).

After completing our analyses, we conclude this report in Section 4 by summarizing our results and pointing to future efforts in our research and curriculum development project on the concept of infinity.

2 Infinite processes and objects

There seems to be general agreement, both from a cognitive and a mathematical point of view, that various aspects of infinity involve some sort of process, that is, some form of ongoing mental activity. We examine some underlying questions regarding an individual's conception of infinity as a process. Then we explore the sense in which it is possible for an individual to conceive of an infinite process as a completed totality and how an infinite process, which does not involve a final object per se, can nevertheless result in an object. In our APOS analyses of these issues, we make use of the mental mechanisms of interiorization and encapsulation, as well as the notion of transcendent object—a term introduced in a nearly completed study by A. Brown, M. McDonald and K. Weller—to explain the relationship between infinite processes and the objects that may ultimately result from them.

2.1 Infinite processes in mathematics

In practice, one can speak of an infinite collection without explicitly referring to a specific process. Yet, it is reasonable to assume the existence of an underlying mental process of placing elements into a set, or gathering elements together, as did Cantor (from the 1955 translation). We may consider whether infinite mental processes underlie our conceptions that a line consists of an infinity of points, that time is endless, or that space extends infinitely far in all directions. Similarly, we can ask whether mental processes support our conceptions of functions and limits of functions defined over a real interval, as well as the notion of infinite cardinality itself.

In many of these instances, we see evidence of the infinite as a process. The data in Cottrill et al. (1996) suggest that the limit of a function is conceived as the coordination of two processes, a domain process and a range process. Since the domain and range of a function are often infinite sets, we have an example of an important mathematical concept that incorporates aspects of mathematical infinity that are first conceived as processes.

Definitions of the infinite throughout history were generally framed in terms of an ongoing process that cannot be completed. For example, Aristotle considered the infinite to be an activity that was ongoing but untraversable (Moore, 1999).

Recent research by Tirosh (1999) suggests that school-aged children think of the infinite in terms of processes and do so regardless of whether the context is numerical, geometrical, or material. For instance, when asked whether the set of all melodies that can be composed is infinite, the children in her study typically responded by saying something like: "It is always possible to add more melodies, even by adding one note at the end of a known melody" (p. 343).

The observations of Tirosh are reiterated in Monaghan (2001), where he writes: "When children talk about infinity, their language repeatedly reflects infinity as a process: 'This goes on and on. It's infinite.', seeing infinity not as a thing but as the act of going on and on" (p. 245). Monaghan goes on to note that "infinity as a process" is not only used to define infinity, but is "also used as an evaluatory scheme to determine whether a question has an infinite answer" (p. 245).

More recently, in an ongoing study (Arnon, Brown, Dubinsky, McDonald, Stenger, Vidakovic & Weller), mathematicians were asked to describe their conceptions of the power set of the natural numbers, $P(\mathbf{N})$. Several of the respondents imagined listing the one-element sets, two-element sets, three-element sets, and so on. Although there was an almost immediate realization that this would not yield all of the subsets of $\mathbf{N}$, there appears to be a natural tendency for some people to construct a process. Something similar has been observed by Brown, McDonald, and Weller in a nearly completed study of infinite iterative processes. In this study, students were asked to determine whether an infinite union of power sets, $\bigcup_{k=1}^{\infty} P(\{1, 2, \ldots, k\})$, is equal to the power set of the natural numbers, $P(\mathbf{N})$. All of the students interviewed tried to make sense of the infinite union notation by constructing an infinite process.

Our conceptions of the infinite appear to be rooted in underlying processes, even though we may not always speak of the infinite in terms of these processes. It is interesting to note that most discussions of infinite processes, such as those found in Aristotle's writings and Tirosh's study, all involve iteration. Might this imply that infinite processes are always iterative? This remains an open question with various views discussed in the literature. (For example, Poincaré, as noted in the 1963 translation; Moore, 1999; Lakoff & Núñez, 2000, p. 157.) All we will say about this topic at present is that it appears to be a different question for countable processes than for uncountable processes.

Another open question we can only raise here has to do with the meaning of the notion of "all" as in "for all $\epsilon > 0 \ldots$". Some teachers of calculus feel that of the two formulations of the limit concept:

$$\text{For all } \epsilon > 0, \text{ there exists } \delta > 0 \text{ such that if } 0 < |x - a| < \delta, \text{ then } |f(x) - L| < \epsilon,$$

and

$$\text{For all } n \in \mathbf{N}, \text{ there exists } \delta > 0 \text{ such that if } 0 < |x - a| < \delta, \text{ then } |f(x) - L| < \frac{1}{n},$$

students tend to find the second easier to understand than the first. Could this be because one can understand the meaning of "For all n" as an iterative process, whereas no such process is apparent regarding "For all $\epsilon > 0$"?

In proving that a proposition holds for *all* elements of an infinite set, one typically selects an arbitrary element of the set, supplies a proof for that element, and then asserts that the proposition holds for every element. In the case of a countably infinite set, the ability to transfer the meaning of "all" from an arbitrary element to every element, or the ability to construct the meaning of "all" in one's mind, may be possible because there is an underlying constructive process, namely the act of iterating. However, if there is no apparent mental process underlying one's conception of a set, as may be the case with uncountable sets, the meaning of "all" is not entirely clear. For instance, Quine (1970) discusses the meaning of existential quantification at length, but for universal quantification, he mentions only that it is obtained by negating existential quantification. In this general case, it is not clear to us how to construct mentally the meaning of "all." These questions all call for further empirical research.

2.2 Can the infinite be conceived as a completed totality?

Whether infinite processes are iterative or not, can an individual have a sense that all the steps of an infinite process are present all at once, even if he or she cannot actually take all the steps? That is, can one conceive of an infinite process as a totality?

In many instances, the solution of a mathematical problem calls for something to be transformed. In order to transform something, the process underlying its mental construction must be conceived statically, that is, the underlying process must be thought of as a cognitive object. Before this can occur, the process must be seen by the individual as a totality, a whole capable of being acted upon. In situations involving the transformation of infinite processes, the question that arises is how an individual can think statically about something that is, at least in temporal terms, forever dynamic. In this section, we consider this question, acknowledge that there was no historical agreement on its answer, and then present some examples where human beings seem to, or at least need to, view an infinite process as a totality.

2.2.1 Background

Again, the question of conceiving an infinite process as a completed totality has been discussed by many authors. These include Aristotle, Archimedes, Cantor, Brouwer, Hobbes, Taylor (see Moore, 1995, 1999, 2002), and Tall (2001). There appears to be strong disagreement on whether such conceptions are possible for human beings.

On the other hand, there is reason to believe that perceiving an infinity of objects and actions on them as a totality may be quite common in mathematical thinking. Consider the sum of two functions defined on a real interval. To conceive of such a sum beyond simply computing the sum of two algebraic expressions, it seems that one must think about all of the pairs of values of the two functions for given domain points and then imagine summing the two range values in each pair. In this view, the individual would think about two infinite sets of numbers and an infinite set of operations all at once, as a totality. In the next section, we present some examples from recent research that tend to support such an argument.

2.2.2 Examples from recent research

McDonald et al. (2000) studied undergraduate students' conceptions of sequences and found that students *could* think of infinite lists as completed totalities and single entities on which "they could perform actions ... such as comparing two lists" (p. 84).

Brown et al. (1998) examined how undergraduate students might come to understand binary operations, groups, and subgroups. They found evidence of a mental construction in which the underlying set to which the group axioms are applied is infinite and is conceived as a totality.

Stenger, Vidakovic, and Weller are currently analyzing undergraduate students' thinking on the tennis ball problem described in Part 1[1]. One of the students who was successful in solving the problem showed evidence of seeing the process as a totality when he remarked that the nth ball describes the movement of all of the balls.

Brown, McDonald, and Weller asked 13 undergraduates to determine whether $\bigcup_{k=1}^{\infty} P(\{1, 2, \ldots, k\})$ is equal to $P(\mathbf{N})$. To solve the problem, the students considered set inclusion relationships. The key point in seeing that the equality does not hold is observing that an infinite element of $P(\mathbf{N})$ is not an element of the infinite union. This turned out to be difficult: only one student in the study accomplished this. She explained that the set of natural numbers could not be an element of the infinite union because the infinite union is "the union of an infinite number of finite sets," and each individual power set contains only finite sets. In making these observations, the student gave evidence of seeing the infinite collection of power sets as a totality.

These examples show instances of the ability to conceive of infinite processes as completed totalities. According to APOS Theory, seeing a process as a completed totality is a prerequisite for encapsulating the process into a cognitive object. Now we turn our attention to the issue of how mental objects can result from infinite processes.

2.3 Infinite processes and objects

How might one obtain cognitive objects from infinite processes? Here we contrast the notion of "final objects" that has appeared in the literature with an alternative explanation based on APOS Theory.

2.3.1 The notion of "final objects"

In the case of a finite process, an object may be obtained at each step of the process. We finish the process with one last step, and thus obtain a final object. Such is not

[1]We recall that in this version, we suppose as given an infinite set of numbered tennis balls and two bins of unlimited capacity. We imagine that we place balls 1 and 2 in the first bin and then immediately move 1 to the second bin. Then we place balls 3 and 4 in the first bin and move 2 to the second. And so on, ad infinitum. What are the contents of the two bins when this is finished?

the case with an infinite process, as there is no final step and, consequently, no "final object."

Lakoff and Núñez (2000) maintain that our understanding of mathematical notions of infinity is based on the establishment of a conceptual metaphor, the *Basic Metaphor of Infinity* (p. 159), that links the target domain of processes that go on and on with the source domain of completed finite iterative processes. They argue that the mechanism of conceptual metaphor enables an individual to conceptualize the "result" of an infinite process, the state at infinity, by thinking in terms of a process that does have a final state. However, the support for this claim depends on the methodology of mathematical idea analysis, a technique whose adequacy for explaining mathematical thought has come into question (Schiralli and Sinclair, 2003).

Thinking about infinite processes in terms of processes that do have a final state, even metaphorically, can lead to problems. For example an individual might think a final object is actually produced by the process, similar to the types of situations reported by Cornu (1991), Monaghan (2001), and Mamona-Downs (2001). Another possibility is that an individual might try to construct the state at infinity by mimicking what happens for related finite processes. For instance, in the study of infinite iterative processes being conducted by Brown, McDonald, and Weller, one of the students simplified the union of the first k power sets to obtain $P(X_k)$, where $X_k = \{1, 2, \ldots, k\}$. She then reasoned incorrectly that it must follow that "$P(X_\infty)$" is the final result of the infinite union. A similar phenomenon occurred in the tennis ball study being conducted by Stenger, Vidakovic, and Weller. Several of the students reasoned that if, after n steps, the second bin contains balls numbered 1 through n and the first bin contains balls numbered $n + 1$ through $2n$, then at 12:00 noon the second bin will contain balls numbered 1 through ∞ and the first bin will contain balls numbered $\infty + 1$ through 2∞.

2.3.2 The object resulting from an infinite process

The issue of how one conceptualizes the state at infinity and its relation to the process from which it originates, or at least precursors of it, is found in history. For example, Nicholas of Cusa, in considering an infinite sequence of equilateral polygons inscribed in a circle, noted that increasing the number of sides would reduce the error, but would never yield the circle. (Nicholas of Cusa, as quoted on p. 11 of the 1954 translation.)

Galileo made similar comments, but went further regarding a process of subdivision and the method for determining the state at infinity, which he viewed as transcending the process of subdivision. After acknowledging that there is no final subdivision, he proposed a method for "separating and resolving the whole of infinity [*tutta la infinitaà*], at a single stroke" (Galilei, as quoted on p. 48 of the 1968 translation).

In our view, this "resolving the whole of infinity at a single stroke" corresponds to the APOS notion of encapsulation. As Galileo suggests, the object that results from the "single stroke" is not produced by any individual step. Instead, it transcends and stands outside the process.

The issue of determining the state at infinity is motivated by an individual's perceived need or desire to perform an action, which may amount to determining "What's next?" or "What is the ultimate result?" Prior to performing this action, we suggest that an individual needs to see the infinite process as a completed totality, after which he or she performs an action on the process by encapsulating it to obtain the state at infinity. In making the encapsulation, the individual needs to realize that the state at infinity is not directly produced by any step of the process. Instead, as the individual reflects on the process, he or she realizes that the object associated with the state at infinity reflects the totality of the process rather than any of its individual aspects. This object is obtained through encapsulation, which is Galileo's "single stroke". It is in this sense that the state at infinity stands apart from or transcends the process.

There are many instances in which this seems to occur in mathematical thinking. For instance, in the addition of two numbers, say 2 and 6, the number 8 is obtained from the process of adding 2 and 6, but is not the object that results from the encapsulation of this process. Instead, the encapsulation allows the addition to be considered as an object that can be acted on; for example, it could be compared to other processes like $6 + 2$ and $8 - 6$. The number 8, like other natural numbers, is an object that is constructed by encapsulating processes *other* than the basic arithmetic operations (Piaget, 1952).

The construction of the state at infinity of an infinite iterative process is similar cognitively to the encapsulation of the addition process. For example, in Stenger, Vidakovic, and Weller's study of the tennis ball problem, the empty set (which is the correct answer to what remains in the first bin) never occurs at any stage at which the balls are moved. As in the case of the addition of two counting numbers, the process is first encapsulated, and then the resulting object, which stands outside the process, can be compared with the empty set, an object resulting from a previous mental construction.

Finally, one can see evidence of transcendence in ordinal arithmetic. The first infinite ordinal ω does not occur as the "largest" natural number, as many students think. Rather, it is the limit of the process of enumerating the set of natural numbers (Kamke, 1950). Before one can conceive of such a limit, one must first think of the totality of all of the natural numbers and then ask the question, "What comes after all of the natural numbers have been enumerated?" It is in this sense that ω transcends the process of enumeration.

3 Mathematical problems related to infinity

In this section we use the ideas developed to this point to consider some specific mathematical problems related to infinity.

3.1 Construction of the set $\mathbf{N}$

The simplest infinite process consists of counting, which begins with 1 and in each step adds 1 to obtain $1, 2, 3, \ldots$. We consider that this process leads to the construction of a sequence of sets: $\{1\}, \{1, 2\}, \{1, 2, 3\}, \ldots$. The encapsulation of this process yields the object $\mathbf{N} = \{1, 2, 3, \ldots\}$, the set of natural numbers.

The question of what mental mechanisms and structures can allow one to determine that the cardinality of the set $S = \{-3, -2, -1, 0, \{1, 2, 3 \ldots\}\}$ is 5 was raised in the introduction to Part 1 of this study. According to APOS Theory, after an encapsulation, $\mathbf{N}$ can be seen as a single object and so S can be understood to be equal to the set $\{-3, -2, -1, 0, \mathbf{N}\}$, which has 5 elements.

3.2 The equation $.999\ldots = 1$

Several researchers such as Cornu (1991), Tall (1976), and Tall & Schwarzenberger (1978) have shown that many students think this equality is false. They think that $.999\ldots$ is just less than 1, the nearest you can get without actually reaching it; that the difference between the two is "infinitesimally small"; or that $.999\ldots$ is the last number before 1.

In terms of an APOS analysis, we can propose at least two explanations for the confusion. The first is that the confusion with this infinite decimal may occur for students who are limited to a process conception of $.999\ldots$ (as an infinite sequence of 9's) but have an object conception of the number 1. The difference between the two conceptions is that a process is conceived by the individual as something one does, while an object is conceived as something that is and on which one acts. An individual who is limited to a process conception of $.999\ldots$ may see correctly that 1 is not directly produced by the process, but without having encapsulated the process, a conception of the "value" of the infinite decimal is meaningless. However, if an individual can see the process as a totality, and then perform an action of evaluation on the sequence $.9, .99, .999, \ldots$, then it is possible to grasp the fact that the encapsulation of the process is the transcendent object. It is equal to 1 because, once $.999\ldots$ is considered as an object, it is a matter of comparing two static objects, 1 and the object that comes from the encapsulation. It is then reasonable to think of the latter as a number so one can note that the two fixed numbers differ in absolute value by an amount less than any positive number, so this difference can only be zero.

A second explanation is required for students who have not yet constructed a complete process conception of the infinite decimal. In this case, there would not be an understanding of all of the steps of the process that produce the infinite decimal. For example, the student may actually conceive of $.999\ldots$ as consisting of a string of 9s that is finite but of indeterminate length. Conceptions such as infinitely small differences with 1 could exist without conflict in this situation.

The difficulty with $.999\ldots = 1$ stands in contrast to at least one student's understanding of the equation $.333\ldots = \frac{1}{3}$. In a case study conducted with a real analysis student by Edwards (1997), the student stated that $.333\ldots$ is equal to $\frac{1}{3}$ because one could divide 3 into 1 to get $.333\ldots$. However, the student was adamant that the equation $.999\ldots = 1$ is false, because "If you divide 1 into 1, you don't get $.999\ldots$!" (p. 20). In the case of $\frac{1}{3} = .333\ldots$, the student might have been limited to seeing both $\frac{1}{3}$ and $.333\ldots$ as processes. In the case of $.999\ldots = 1$, the student may see $.999\ldots$ exclusively as a process and $\dfrac{1}{1}$ as a process that does not result in $.999\ldots$. Although a deep understanding of either equation would require encapsulation of the two processes so that a comparison could be made, the student may have been able to accept the equality $\frac{1}{3} = .333\ldots$, because she thought of both as the same process, but could not draw the same conclusion about $.999\ldots = 1$, because she saw $.999\ldots$ as a process that was not the same as the process of dividing 1 by 1.

3.3 Infinitesimals

The well-known controversy regarding infinitesimals centered around the difference quotient which can be expressed in the form $f(x + o) - f(x) : o$. The issue in the writings of Newton, Berkeley and many other disputants focused primarily on the nature of the "small" increment o. How could o be both regarded as nonzero, so that it was permissible to divide by it, and disregarded because it has no contributing value later in the calculation?

In our proposed cognitive resolution of this controversy, we will use some language from the study of the cognition of the limit concept (Cottrill et al., 1996) mentioned in Section 2.1. In that paper, an analysis of student data suggested that the limit L of a function f at a domain point a is understood as the coordination of two processes, a domain process and a range process. These processes are coordinated by the function in that the domain process x is transformed by f to the range process $f(x)$. In the language of Brown, McDonald, and Weller, the domain point a is the transcendent object of the domain process, while the limit L is the transcendent object of the range process.

Consider then, the following comment by Newton.

> ...those ultimate ratios with which quantities vanish, are not truly the ratios of ultimate quantities, but limits towards which the ratios of quantities, decreasing without limit, do always converge; and to which they

> approach nearer than by any given difference, but never go beyond, nor in effect attain to, till the quantities are diminished *in infinitum* (as quoted on p. 39 of the 1934 translation).

In the language of Cottrill et al. (1996), we suggest the interpretation that the quantities which form the denominator represents the domain process and the ratios themselves represent the corresponding range process (with the processes coordinated by the function and the difference quotient). Thus, in terms of an APOS analysis, we suggest that, in his most formal work, Newton intended o and $f(x+o)-f(x)$ to represent domain processes of approaching 0 and, in turn, the difference quotient as representing a range process of approaching an "ultimate ratio." But his critics, such as Berkeley with his "ghosts of departed quantitites", insisted that o and the difference quotient itself must always be viewed as static objects. Our view is that the confusion was caused by these critics not being able to distinguish between an object directly produced by the process and an object that is brought into being by encapsulating the process as a result of applying the action "What is the ultimate value of the range process?"

We then propose, from APOS Theory and using more modern terminology, that in the expression $\dfrac{f(x+dx)-f(x)}{dx}$, the symbol dx represents a process. The action of finding the ultimate value of this process leads to its encapsulation and the selection of 0 as the transcendent object. The process of dx approaching zero is coordinated by f and the difference quotient to obtain a new process. As one imagines the completion of this latter process, it can be thought of in its totality, and the action of determining the ultimate value of the quotient leads to an encapsulation. The ultimate ratio (Newton's phrase), which results from this encapsulation, is the derivative. The limit is a mathematical expression of the cognitive encapsulation. Because the derivative stands outside of, and reflects the totality of, the difference quotient process, the issue of whether Berkeley's ultimate "evanescent increment" is finite, or infinitely small, or "yet nothing" is moot. Specifically, the derivative is not determined by computing the quotient. Rather, it represents a value toward which all of the intermediate states of the process point. This is consistent with Newton's view of the derivative as "the ratio of the quantities, not before they vanish, nor afterwards, but with which they vanish" (as quoted on pp. 38-39 of the 1934 translation).

4 Conclusion

In this paper we have considered the nature of infinite processes: we have used historical texts and research results, both from completed and ongoing studies, to show how infinite processes can be conceived as completed totalities, and to describe the cognitive relation between infinite processes and the objects that may result from them. We used APOS Theory to explain how the mental mechanisms of interiorization and encapsulation might make this possible. Then, we used the ideas developed in our analyses of infinite processes to consider how an individual may construct cognitively the "smallest" infinite set, the natural numbers, and to explain issues that arise in dealing with very small quantities, such as the difference between $.999\ldots$ and 1, and the infinitesimals of differential calculus.

The main contribution that we obtain from an APOS analysis, both in this paper and in Part 1, is an increased understanding of an important aspect of human thought. We feel that the value of this understanding is enhanced by the fact that it is based on a small number of mental mechanisms, mainly interiorization and encapsulation. These two mechanisms have been crucial in the study of many other mathematical concepts and have pointed to effective pedagogy. Now we see that they can also provide plausible explanations of a wide variey of infinity-related issues that are of concern to mathematicians and philosophers and that cause difficulties for students of mathematics. Surely a first step in helping students overcome these difficulties is to understand their nature.

As we expand our analyses to other mathematical topics related to infinity, the next major step in this research program is to develop pedagogical strategies based on these analyses. With regard to infinite processes, such strategies should focus on helping students to interiorize actions repeated without end, to reflect on seeing an infinite process as a completed totality, and to encapsulate the process to construct the state at infinity, with an understanding that the resulting object transcends the process.

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