Conceptions of Area: In Students and in History

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Introduction. Calculus as a course poses several difficulties for students, and the aim of this and other related studies is to assess carefully the specific aspects of the topics that pose conceptual problems for students. Computation of the definite integral is a standard topic in calculus courses, and the method of computing its value, used in a majority of calculus books, is the familiar method of Riemann sums. Our topic of investigation is students' understanding of this method.

While we expected that students might have trouble computing definite integrals, it was indeed an unexpected surprise to observe responses among some students that were clearly not based on the Riemann sum approach. In fact, we observed a coherent intuition of computing the area in students' minds that was very different from the intuition the experimental calculus course was trying to develop. Our students were not alone in having this intuition: they share it with one employed by mathematicians like Archimedes, Cavalieri, Wallis, and Roberval.

We will refer to the two methods of computing (partially or completely) the definite integral as the "chopping up method" and the "method of indivisibles", where the chopping up method means the classical Greek method of exhaustion refined later to the Riemann sum approach, and the method of indivisibles means thinking of the area under the curve as the sum of all vertical parallel lines contained in it.

We will describe the research setting of the experiment, sketch the historical development of the indivisibles idea, outline some aspects of the historical relationship between both methods, and present some of our findings on students' ideas about the two. We will conclude by directing attention to the work conducted by some teachers of mathematics to teach mathematics from original sources, and offer some pedagogical suggestions.

Research Setting. Data for this research was collected during the fall semester of 1992. The participants were 32 engineering, science and mathematics students who had, during the previous year, taken two semesters of single variable calculus at a large midwestern university. Twenty-one of them had completed at least one semester of the reform calculus course *Calculus, Concepts, Computers and Cooperative Learning* (C^4L) taught to about 145 students. Ten students had completed a two-semester calculus sequence taught to about 2,000 students as a standard lecture with recitation course. There was one student who took both the C^4L and the standard calculus course. The instructional method in C^4L classes included guided discovery learning activities suited to cooperative learning and a computer environment. For more details about this approach see **[10]**.

The students for the study were chosen to provide a variety of abilities as well as to have a reasonable number of representative students from both groups. Each of the 32 students completed a calculus integral interview conducted by some of the researchers in this study. The interview consisted of ten questions about the concept of the integral, and each interview lasted for an hour on the average.

We will give a analysis of certain aspects of the responses to questions 2, 4 and 9 from the interview. A full analysis of the remaining items will be presented elsewhere. The interview questions on which this study is based are:

Question 2. Compute the following

$$\int_{-2}^{2} \frac{1}{x^2} dx.$$

Question 4. What is the mathematical meaning of

$$\int_{-3}^{-1} \frac{1}{x^2} dx?$$

Question 9. Suppose now that you have a region S in space which is a body of density ρ which has a different value at different points in the region. Write a formula for the mass of this body.

Initially the students were given the opportunity to answer each question without prompting. Based on their responses the interviewer asked more questions or provided hints or clarification. The interviewer encouraged any kind of response from the student (verbal, written, graphical) that might help to explain her or his ideas.

Historical Connection. The ideas of indivisibles and chopping up form one of the fundamental dichotomies in the history of science, namely between the atomic and continuous nature of space and matter. The indivisibles of Archimedes and Cavalieri can be seen as atoms of space while the indivisibles of Roberval, and the infinitesimals of Leibnitz and Cauchy are the arbitrarily small continuous elements of space. Whenever we are observing the indivisibles intuition, we are observing the atomic point of view in action, and the chopping up intuition corresponds to the infinite divisibility of space.

We will sketch the mathematical strengths of the ideas based on the indivisibles intuition among mathematicians of the past, point out their weaknesses as well, and outline certain aspects of their relationship with the chopping up intuition. After establishing the historical framework, we will illustrate the ideas of these mathematicians with the contemporary thoughts of our students as recorded in the interviews.

It is interesting to observe that both intuitions of area, chopping up and indivisibles, were employed as early as the third century BC by Archimedes. In his *Quadrature of the Parabola* **[1]**, Archimedes found the area under a parabola by employing an exhaustion technique that later led to the notion of passing to the limit. The method of exhaustion consists in dividing (chopping up) the area of a parabolic segment into a series of triangles which, as their number increases, exhaust the area in the sense that a remaining area can be made as small as one wishes **(Prop. 24)**[**1**][1]

PLACE FIGURE 1 ABOUT HERE

On the other hand, his route of attack in *The Method* [**2**] is substantially different in that he looks at the segment of a parabola as being made up of all straight lines which originate at the base of the segment and terminate at the arch of the parabola. This view seems to be a manifestation of a general view presented in *The Method* where each figure is seen as made up of all the lines in it and it became known later as the method of indivisibles. He used the principle of the lever to balance every line of a parabola with a corresponding line in a suitably chosen triangle whose area was known. The area under the parabolic segment was then obtained as the sum of all the lines which made up that segment, which in turn was equal to the sum of all lines of the triangle with known area.

PLACE FIGURE 2 ABOUT HERE

It is important to note that Archimedes himself did not consider the technique presented from *The Method* as a sufficient demonstration (or a proof) but rather a heuristic method of investigation that needed a formal proof to be acceptable. According to Knorr [**7**], the formal structure of the Greek proof did not allow for the presence of indivisibles.

As the manuscript of *The Method* was lost until 1906, Archimedes' technique had to be rediscovered. It is interesting to note that both methods, the method of exhaustion and the method of indivisibles arrived again on the European intellectual scene more or less at the same time - in the mid 1620s. Gregory St. Vincent employed infinitely many infinitely thin rectangles to compute areas of some irregular figures [**12**], while Cavalieri [**5**] employed the notion of the sum of lines contained within these figures.

St. Vincent's idea was based on the notion of infinite divisibility of space and was developed later by Fermat and Cauchy. It became the basis of the modern concept of the definite integral. At the basis of Cavalieri's work, on the other hand, was the following assumption: a surface consists of an indefinite number of equidistant parallel lines and a solid is made up of a set of equidistant parallel planes. The plane figure is seen then, in some sense, as the sum of these parallel lines and the solid as the sum of those parallel planes contained therein.

These interpretations enabled Cavalieri to define the concept of equal and similar figures and to compute the areas of figures whose corresponding lines are in a constant ratio. By considering certain special curves of the type $\frac{y}{a} = \left(\frac{x}{b}\right)^n$, he was able to reduce a whole range of quadratures

[1] Although Roberval used Cavalieri's term, "indivisibles" for the concepts he discussed, yet his ideas, as can be seen later in the text (Pg.4), being connected with the infinite divisibility of the space signal the transition to what later was called infinitesimals..

and cubatures to the determination of the sums of the powers of the lengths of lines in a triangle. He was also able to obtain a table of integrals of the form $\int_0^1 x^n dx$ for positive integral n [**3**]**Pg. 129-132**.

Despite its successes, Cavalieri's method was strongly criticized by Tacquet [**13**] among others, who declared that "a geometrical magnitude is made up of homogenea — a solid of solids, a surface of surfaces, a line of lines - not of heterogenea, or parts of lower dimensions as maintained by Cavalieri. Lines cannot be generated by points, solids from surfaces; and indeed no finite quantity can be generated from indivisibles." Thus again, formal and philosophical aspects of mathematical understanding did not allow for the indivisibles intuition to develop further, since the notion of indivisibles began to disappear quickly from mathematics.

In fact, primarily because of this criticism, mathematicians of that time such as Roberval, Wallis, and Pascal, although very sympathetic to Cavalieri's ideas, created an alternative path to deal with indivisibles. For both Roberval and Wallis, indivisibles, the lines of Cavalieri, came to be seen as a sort of limit of infinitely divisible inscribed rectangles. Thus, an interesting bridge had been built between the two opposing views on the structure of mathematical space.

Technically, Cavalieri made this transition easier for them because, he "was careful only to compare figures in which the distribution of indivisibles was uniform . . . the summation of lines and planes in his sense is therefore formally equivalent to the addition of rectangular elements of equal thickness,

$$\frac{\sum l(1) * b}{\sum l(2) * b}\frac{\sum l(1)}{\sum l(2)}$$

where b approaches 0."[**3**]

There was however, a deeper justification for seeing indivisibles as obtained from a great number of equally narrow rectangles. For Roberval, an adherent of the theory of the infinite divisibility of matter and of a mathematical continuum [**14**], a line may be divided into an infinite number of parts but there will be no ultimate part, and after the division the resulting portions will still be lines, showing all the characteristics of a line. Or in his own words: In order to draw conclusions by means of indivisibles, it is necessary to assume that every line straight or curved, can be divided into an infinite number of parts. Now, inasmuch as every line is terminated by points, instead of lines we shall use points, and then instead of saying that all little lines are to a given thing in a certain ratio, we shall say that all these points are to a given thing in the said ratio. [**9**]

It is interesting to note that although Roberval is using the word "indivisibles" he sees them in reality as infinitesimals, indicating a subtle but definite change of his thought towards the infinite divisibility of the mathematical continuum.

Thus, the departure from the ideas of Cavalieri with the subsequent prevalence of the chopping up approach can be seen as the victory of the notion of the infinite divisibility of the mathematical continuum over its atomistic counterpart. John Wallis created a mathematical counterpart to Roberval's ideas and was able to use it to associate numerical values with the indivisibles of Cavalieri or infinitesimals of Roberval. He asserts, in agreement with Cavalieri's ideas, that the "sum of the lines making up the triangle bears to the sum of those making up the rectangle on the same base and of the same altitude a ratio equal to $\frac{1}{2}$, since this ratio can be manifestly written as,

$$\frac{0+1+2+\cdots+n}{n+n+n+\cdots+n}.$$

Moreover, this ratio persists if n be infinite, i.e., if the aggregate of the lines become identical with the area of the triangle." [**11**]. This is illustrated in Figure 3.

PLACE FIGURE 3 ABOUT HERE

Wallis' assertion can be checked easily. If we place the Wallis triangle so that its height of length 1 is positioned on the x-axis and its base of length 1 is perpendicularly bisected by that axis, we can compute the sum of the lengths of n lines in a triangle to be

$$S_n = \sum_{i=0}^{n} \frac{1}{2}\frac{i}{n}2 = \sum_i \frac{i}{n}$$

This method can be extended without much difficulty to a general function f (sufficiently regular, positive and bounded by 1 on the unit interval) if we think of the region between the graph of f and the x-axis as replacing Wallis' triangle. Then the "sum of the lines making up the triangle" is replaced by the sum of the lines making up this region and the ratio of that to the sum of the lines making up the rectangle is

$$\frac{\sum_{i=0}^{n} f(\frac{i}{n})}{\sum_{i=0}^{n} 1} = \sum_{i=0}^{n} f(\frac{i}{n})\frac{1}{n+1}$$

.

The notion of dividing a figure into uniform rectangular pieces serves as the point of departure for computations of an area in the context of both indivisibles and chopping up intuitions. In the first case, sending n (the number of uniform rectangles) to infinity fills up the areas with an infinite number of lines whose sum of lengths can be computed as the limit of Wallis' formula; in the second case, sending n to infinity exhausts the area by increasing the number of rectangles. The area in that case is obtained as the limit of Riemann sums.

It is surprising, therefore, that while both techniques give the same mathematical results, only one of them, chopping up, survived and was developed. The reason for this may be that one of the main supports for the continuity of mathematical space came from Newton's work in physics, where he was a believer in the continuity of time. A quantum theory of space-time, the conceptual equivalent in physics of the indivisibles of Cavalieri, was not in existence.

Reflections on historical intuitions. The reader may be intrigued, as we are, by what these giants of early mathematics and science might have been thinking when they determined the area of a region, or the ratio of its area to a containing rectangle, by considering the ratio of the ordinates of the curve defining the figure to the ordinates of the rectangle. Perhaps they dealt only with the simplest case, a figure bounded by a straight line in which the ratios of the ordinates is constant, and then extrapolated to higher powers. On the other hand, given their intellectual power, we may wonder if they had more general formulations. Of course it is impossible to determine the intuitions of people in the past, even if there were any evidence, which there does not seem to be. Therefore we can only conjecture as to their intuitions.

Consider a continuous function f on the unit interval whose maximum is 1, so that its graph is contained in the unit square. For $x \in [0,1]$ we may then consider $f(x)$ to be the ratio of the ordinate of the curve defined by f to the ordinate of the square at x. How might we use this generic ratio in connection with the area A under the curve, which we also consider to be the ratio of the area under the curve to the area of the square? We can think of two intuitions here.

The first is to compute the "average" ratio of the ordinate. We might do this by fixing a positive integer n and selecting points $x = \frac{i}{n}$, $i = 0, \ldots, n$. The average ratio of these ordinates is,

$$\frac{1}{n+1}\sum_{i=0}^{n}\frac{f(\frac{i}{n})}{1} = \frac{n}{n+1}\sum_{i=0}^{n} f(\frac{i}{n})\frac{1}{n}$$

and as n goes to ∞, this converges to the Riemann integral of f, which is the area under the curve.

PLACE FIGURE 4 ABOUT HERE

A second possibility is to "add proportions". There are many situations in which the sum of proportions is the proportion obtained by adding the numerators and adding the denominators. Thus if a student gets 2 out of 3 problems correct on one exercise set and 4 out of 5 correct on another, then the "sum" of these proportions may be thought of as the total proportion of correct answers,

$$\frac{2}{3} \oplus \frac{4}{5} = \frac{2+4}{3+5} = \frac{6}{8} = \frac{3}{4}.$$

Similarly, if one thinks of the region under a curve as made up of the totality of the ordinates to the curve (and there *is* evidence that this idea was in the minds of the mathematicians we are quoting) **[1]**, then it is reasonable to imagine that the proportion of the area under the curve to the area of the rectangle, is the sum (in the above sense) of the proportions of the ordinates to the curve to the ordinates to the rectangle. Of course, there are "too many" of these ordinates for computation, so one might take a representative sample, or try to "exhaust" the totality of ordinates by taking n of them and passing to the limit. In either case, a reasonable choice is n ordinates spaced evenly at the points $\frac{i}{n}$, $i = 1, \ldots, n$.

One might then intuit that the sum of the proportions

$$\frac{f(\frac{i}{n})}{1},\ i = 1, \ldots, n$$

is obtained by adding numerators and denominators which gives

$$\frac{f(\frac{1}{n}) + f(\frac{2}{n}) + \ldots f(\frac{n}{n})}{1+1+\ldots+1} = \Sigma_{i=1}^{n} f(\frac{i}{n})\frac{1}{n}$$

,

which is exactly the usual Riemann sum (using right endpoints).

Student Responses. To begin with, we present an excerpt which illustrates how students understand the classical Riemann construction. The detailed discussion of this aspect of our investigation will be presented in [**6**]; the excerpt is included here to assure the reader that our teaching of calculus is not completely lost on our students. I indicates the Interviewer and A, B, C, D, and E are students.

I: How else do you find this area?

A: That area? You could sit here and do the, uh, the Riemann sums where you draw little triangles, or draw little rectangles.

I: Mmm-hmmm.

A: ...or trapezoids or whatever you wanted and, uh, once you got those, add them up.

I: Mmm-hmmm.

A: ...add each little box to get a, uh, a sum. And another thing you could do was as those rectangles get smaller and smaller you're getting a better and better approximation.

I: Mmm-hmm.

A: And the limit as you went to infinity of those, all right, actually limit as you went to those boxes getting to zero like, uh, you know, you, you're getting smaller and smaller, eventually you're going to that number which two-thirds.

The excerpt indicates students' ideas and understanding of the Riemann construction (the chopping up method) in estimating the value of the definite integral. The student describes the process of forming a single Riemann sum, and applies the limit to that sum. A common aspect among many responses of this type was the idea of exhausting the area under the curve by inscribing more and more polygons. For example, we encountered a student using arguments that are close to Archimedes' method of exhaustion when he indicated that when drawing "little boxes" he is focusing on "eliminating empty space, or space that you don't want".

For the indivisibles intuition, it is important to remember that not a single piece of mathematics had been taught to students based on it and it is not surprising that students' thinking in that line is very pure, if a bit raw. The excerpt below shows an intuition very similar to that of Archimedes in *The Method*, when the student sees an area as the sum of ordinates of the function, and the volume as the sum of planes:

B: Um, well the integral is, is um basically the sum of $f(x_i)$ for i in this case in $[-3,-1]$, not necessarily integers but every number, every single point between -3 and -1 is going to have an $f(x)$ value.

I: Okay

B: And if you add all those together

I: Um-hum

B: You should get an area under the curve.

....

B: Now, if you took on this one you'd have to take like an integral to find the base of this thing, and that you have that you'd want to integrate like this, like vertical for well, however you have your plane, but let's say if you cut out like a, a plane, like a bunch of horizontal planes..

I: uh-huh

B: that are parallel, then you gonna want to add all those planes together.

The next excerpt shows the indivisibles intuition of a student that is very similar to that of Wallis. In Carl Boyer's words "he [Wallis] assumed at the outset, as had Cavalieri, that a plane figure may be regarded as made up of an infinite number of lines – or rather, as he preferred, of an infinite number of parallelograms, the altitudes of which are equal, that of any of them being $\frac{1}{\infty}\ldots$" [**4**]. The student struggles here with the question of going from an approximate to the exact value of the integral. Note the students' phrase "to be able to work with basically a rectangle with no width, just height".

C: Right. Um. Well, the Riemann Sum breaks this up into n, an infinite number of rectangles. And it's difficult to use the theory behind it. It's difficult for me. Um... basically the Riemann Sum was just summation of the area of a number of rectangles where you would always have a number of some error, because they wouldn't fit directly to each point.

I: How would you get the closest...

C: Um...

I: ...possible area?

C: Closest possible area would be by taking the length of a line segment from the x axis to the function itself. And that would give you an infinitely many..and many areas to add up. And that's what the definite integral gives you. It just allows you, you know, to be able to work with basically a rectangle with no width, just height. So, you calculate, you know, the length of a line segment. So, you add the lengths of all the individual line segments and, um, get an area. (Pause) You can't, I am trying to recall the theory behind how the definite integral gives you summation of rectangles. I don't remember that. Hmm...

The following student demonstrates an intuition very similar to Roberval's, and very distinct from that of Wallis in that he sees the lines as the limit of rectangles.

I: How do you go from a Riemann sum to make it equal to the $\frac{2}{3}$ we got here?

D: Make these rectangles infinitely small, smaller, smaller and smaller, I mean almost until they're a line, they're a unit and then you are just adding up these units and the smaller – this empty area is the more exact estimation until you get to a point where there is no empty space to be accounted for that gives an exact number.

I: So what did we do [to] the Riemann sum?

D: We added them all up we took the values of sum, we made them infinitely small giving us infinitely many number of values and we added them up.

I: Okay, let's look at it, say we drew 10 rectangles on our picture and we had to figure out the area for each of these ten rectangles, and we add that up and we get the number and like you said depending on whether you draw them above the curve or below the curve you get more than $\frac{2}{3}$ or less than $\frac{2}{3}$, then if you drew 100 rectangles, each rectangle would be...

D: smaller and it would have less of an empty space to be accounted for.

Among the students who confronted the issue of adding up infinitely many (indeed, uncountably many) values, such as the lengths of all vertical lines making up the figure, a few made beginning steps towards dealing with the difficulty. One way to make the indivisibles intuition rigorous and come to the standard Riemann sum formulation is to consider only some of the lines, perhaps spaced out evenly, compute their average length, multiply by the length of the base and pass to the limit as the number of sample lines increase. Here is an example of a student's initial steps in this direction:

E: I mean, um, volume. And I don't remember the exact notation, but what I would do it would give you the summation of, or I don't want it to give the summation. I want it to return to the,. .I can't think of the word I am thinking of but the average, I want it to, I want to find the mass at every point, and then take the average of all that, you know what I mean? Like for different spots, for different spots, it's gonna give you a slightly different value probably and the correct answer would be somewhere in the middle of all those values.

Conclusion. The rediscovery of the indivisibles intuition as occurring in students' thinking while they analyze the notion of an area under an irregular figure is a witness to the persistence of the atomic point of view on the structure of space (and possibly time). The physical analogies that seem to influence the understanding of the structure of mathematical space are also important to take note of in one's thoughts. While the Archimedean approach in *The Method* was the full utilization of the lever principle - one of the first non-trivial laws of statics – the approach of Cavalieri germinated in the environment of views of people like physicists and mathematicians, Oresme, and Galileo all of whom were holding, according to Boyer[**4**], the atomic point of view and saw a plane as the sum of the indivisible lines. Boyer actually hypothesizes that the first velocity diagram and the computation based on it of the distance as the area under the graph was formulated by Oresme with the help of indivisibles.

It seems clear that a thorough and systematic investigation into the contemporary indivisibles intuition among learners is called for. As this intuition can occur in students' mathematical thinking, it is natural to design instruction that tries to take it into account. However, intuition similar to that of Roberval is not derived from this source, and must come in other ways, perhaps, students' life experiences. Investigations into the nature of this intuition will help us to design proper instructional strategies. It may be that one of the proper ways to develop the indivisibles intuition and to show its natural character is to start at the common ground of dividing the domain of the function into n equal regions. Focusing on the number and lengths of lines which, when n tends to infinity, cover the area, leads to the Cavalieri-Wallis approach. The focus on the sum of areas of rectangles and the exhaustion principle lead to Riemann sums. Thus both the chopping up and indivisible points of view could be presented and, quite possibly, developed further.

It was fascinating for the researchers to observe an insight demonstrated by the students different from that developed by the classroom instruction. This could be due in part to the nature of the instruction, to the visual nature of developing concepts via the computer. However, it is even more fascinating to see the students mimic in a small measure some of the logical arguments of several great mathematicians. It indicates that a strong intuition cannot be destroyed by the instruction received in the classroom, in this case the Riemann sum approach. It is not often that instructors of mathematics have the opportunity to demonstrate these differing conceptions to students. Instruction might be organized that does take into account students' natural intuition on this topic, which provides a rigorous method of evaluating a definite integral. Such an introduction in a Calculus course may well serve as a strong motivating influence to students that are inclined mathematically. In fact, mathematics courses based on historical works are being taught, not just as History of Mathematics courses [**8**].

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