Is Calculus Obsolete?

Ed Dubinsky
Purdue University, Education Development Center, University of New Hampshire

So why do we need calculus if we have all this technology? I mean, if you want me to find all the relative maximum values of a function, I'll just draw the graph, zoom in on the humps, and get as close an approximation as you want.

Do you ever hear this from your students? What do you answer? You might complain that the graphical approach gives only an approximate answer but, if the student is any good, he or she might ask you why an "exact" answer like

$$\left(\frac{1}{\sqrt{2}+3}\right)^4$$

is any better, more accurate, or more insightful than

$$0.002634$$

or

$$0.002633823309767369$$

Of course, the question has many answers. You might talk about partial derivatives and solving optimization problems for functions of several variables, or the applications of calculus to higher math like real analysis or differential geometry. Most of the students in our classes, however, are, unfortunately, thinking about less, not more math. All of us would like our students to be interested in and appreciate the power, beauty and subtlety of a set of ideas like finding the shape of a curve by studying derivatives. I don't think we can get there by pointing to things they are not likely to care about (and, I am afraid, that might include, for many students, so-called "real-world" problems). We need to look for more effective justification for studying powerful, but not immediately accessible concepts.

One thing that most students care about in our present educational culture is getting the right answer. So what about trying to argue for the inadequacy of studying a graph just with a computer or graphing calculator by suggesting to your students that *you might not get the right answer*.

Consider, given a specific function, the problem of drawing its complete graph, showing, in particular, all turning points, changes of concavity, zeros, and asymptotes. I would like to illustrate what might happen if a student was capable of dealing with such a problem *only* by using a technological tool to draw its graph. I also assume that the tool will draw the graph on some default domian (usually $[-10, 10]$) but that the user may specify any interval for the domain.

I want to consider two examples f, g and just to let you, the reader share the students' inability (or unwillingness) to apply any other concepts, I won't tell you right away what the definitions of the functions are.

So here is what the graphs look like on a default domain $[-10, 10]$[1]

PUT PLOTS 1,2 HERE SIDE BY SIDE

A pretty reasonable first guess is that these two functions are the same, but with different scales — probably the same expression with very different coefficients. It is pretty clear that the x-axis is a horizontal asymptote and that a little more needs to be done to make sure about what happens near the y-axis.

Let's zoom in and look at the two graphs in the domain $[-1, 1]$.

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They still look very much the same and it appears as though the y-axis is a vertical asymptote.

At this point, a student might concentrate on f and keep looking at smaller and smaller intervals around zero. After a while, the student might conclude that he or she has the whole story, The x-axis is a horizontal asymptote, the y axis is a vertical asymptote, there are no turning points, and the concavity is always down. This is actually correct because the function f is indeed given by

$$f(x) = -\frac{1}{x^2}, \quad x \neq 0$$

I think that most of us would be pretty happy in an elementary course if a student made such an investigation and gave this answer.

What about g? To save space, perhaps you will take my word for it that for some time, as the domain interval gets smaller, you get the same graph. If a student made such an analysis and concluded that the graph of g was pretty much the same as that of f, but with different scaling, or perhaps a higher exponent, you would have to be satisfied. Same analysis, same answer.

But a really persistent student might go as far as the interval $[-0.1, 0.1]$ and get this.

PUT PLOT 5 HERE

[1] The reader will notice that this particular tool plots only a discrete set of points and does not try to connect them with a continuous line. The sampling of points is "ineractive" in the sense that in regions where the function changes more rapidly, more points are sampled. I think there are good pedagogical reasons for this kind of graphing protocol, but that is another story and you don't have to agree. The points I wish to make in this article are just as valid (maybe even more so) if your tool connects the points somehow.

Not terribly informative. The problem is that the graph has a point at around $(0, 10^{42})$ (can you see it?) but most of the points are much closer to the x-axis, so you don't see very much. It does say that g is probably not very much like f. It also suggestes that there is something going on around $x = 0$ (look at how the points cluster) so maybe the student might try the domain $[-0.02, 0.02]$.

PUT PLOT 6 HERE

Now we are getting somewhere. After looking around, the student might conclude that the graph looks something like

First hand drawn picture

The description might then be that again the x-axis is a horizontal asymptote. The y-axis is also a vertical aymptote, but with f it was on the negative side and with g it is on the positive side. Moreover, there are two more vertical asymptotes at $x = \pm a$ where a is some fairly small positive number. Again there are no turning points, but the concavity does change from up to down at the $x = -a$ asymptote and from down to up at $x = a$.

This is certainly a much richer result of thoughtful use of the graphics tool. It has two serious drawbacks.

1. As far as I know, there is no graphing tool that will create a graph such as the above which was drawn by hand. The reason is that in order to show the various features, very different scales must be used. A scale that is appropriate to show one feature (such as large positive values close to $x = 0$) will cause a squeezing in other places (such as near $x = 0.0004$) that may conceal important features. Also, one cannot show, on the same graph the behavior near $x = 4 \times 10^{-1}$ and near $x = 4 \times 10^{-4}$.
2. The result is wrong.

Now we can reveal the definition of g. It is given by

$$g(x) = \frac{1}{x^{12}} - 2\left(\frac{10^3}{x}\right)^6$$

A little calculus will tell you very quickly that there is a critical point at $x = 0.001$, but again you have to be careful in making a graph. If you use an interval like $[-0.0015, 0.0015]$ then again the large values near 0 will squeeze out all the action near 0.001. But if you realize (from the expression) that this graph will be symmetric about the y-axis, then you can look at only one side and go a little away from 0.

You might think that it was enough to notice the symmetry and use the graphics tool to look at domains of the form $[b, c]$ where b is a small positive number. You can try it yourself and I think you will find that if you don't hit on an interval in which c is very close to 0.001, then you still don't see much. The contribution from calculus is to suggest such values for c.

Some experimenting leads to a choice of a domain interval like $[0.0009, 0.0011]$ which gives the following graph.

PUT PLOT 7 HERE

Again the vastly different scales prevent graphics software to present a single picture of the curve and it is necessary to analyze pieces and then put them together. It seems that only by using concepts from calculus, possibly with the help of graphing software is a student likely to come up with anything like the following solution to the original problem.

Second hand drawn picture

As a final comment, the reader might think this is a problem made up to illustrate my point. Not so. I made up the coefficients[2], but the basic expression for g describes a chemical reaction between two molecules in a certain substance. I took it from a Chemistry book. I wonder if that matters.

[2] with some help from Andy Gleason