Learning Binary Operations, Groups, and Subgroups

Anne Brown
Indiana University South Bend
South Bend, Indiana

David J. DeVries
Georgia College & State University
Milledgeville, Georgia

Ed Dubinsky
Georgia State University
Atlanta, Georgia

Karen Thomas
University of Wisconsin-Platteville
Platteville, Wisconsin

Abstract

This paper is one in a series of studies by members of the "Research in Undergraduate Mathematics Education Community', or *RUMEC*, concerning the nature and development of college students' mathematical knowledge. The present paper examines how abstract algebra students might come to understand binary operations, groups, and subgroups. We give preliminary theoretical analyses of what it could mean to understand these topics, expressed in terms of the Action-Process-Object-Schema epistemological framework. We describe an instructional treatment designed to help foster the formation of mental constructions postulated by the theoretical analysis, and discuss the results of interviews and performance on examinations. These results suggest that our pedagogical approach was reasonably effective in helping students to develop strong conceptions of binary operations, groups, and subgroups. Based on the data collected as part of this study, we propose revised epistemological analyses of these topics, and give some pedagogical suggestions related to these topics.

Contents

1 Introduction

This paper reports on a study of the nature of abstract algebra students' understanding of binary operations, groups, and subgroups. The study was carried out according to a very specific research methodology that is being developed by the members of the Research in Undergraduate Mathematics Education Community, or *RUMEC*, for the purpose of studying the learning of collegiate mathematics. Our framework for conducting research has three components: an initial theoretical analysis, an instructional treatment, and empirical data. We will begin by describing our approach briefly; the reader is referred to Asiala et al. (1996) for a complete discussion of each of the three components.

The first step is to make an initial theoretical analysis of the epistemology of the concept of interest. The purpose of the theoretical analysis of the concept is to propose a *genetic decomposition* which is a model of cognition: that is, a description of specific mental constructions that a learner might make in order to develop her or his understanding of the concept. These constructions are called actions, processes, objects, and schemas, so that the theory is sometimes called the *APOS* Theory.

We begin with an explication of the general APOS theoretical perspective, and describe how it was applied in Breidenbach, Dubinsky, Hawks, and Nichols (1992) to characterize student understandings of the concept of function. An *action* is any transformation of (mental or physical) objects to obtain other objects. It is perceived by the individual as being at least somewhat externally directed, as it has the characteristic that at each step, the next step is triggered by what has come before. For example, an individual is limited to an action conception of function if he or she requires that the function be given by an expression containing complete and explicit instructions on what steps to take in order to evaluate the function at a point. Someone with a deeper understanding of a concept may well perform actions when appropriate; if the individual is not limited to performing actions, he or she is said to have moved beyond an action conception of the concept.

When an action is repeated, and the individual reflects upon it, it may be *interiorized* into a *process*. In contrast to actions, processes are perceived as being internal to, and under the control of, the individual. We say that an individual has a *process conception* of a given concept if the individual can think of the concept as a process. Thus, for example, if one is capable of thinking of a function as receiving one or more inputs, performing one or more operations on the inputs, and returning the results as outputs without needing to actually calculate them, then he or she is considered to have a process conception of function. That is, the individual has interiorized the actions which define a function by constructing a mental process that is under her or his own control, rather than simply responding to external cues. Further indications of a process conception include the ability to transform processes through reversal and the ability to coordinate processes.

If an individual reflects on operations applied to a particular process, becomes aware of the process as a totality, realizes that transformations can act on it, and is able to actually construct such transformations, then we say the individual has *encapsulated* the process as a cognitive object, and the individual has an *object conception* of the concept. One who has an object conception is also able to de-encapsulate the object back to the process from which it came in order to work with it. In the case of functions, de-encapsulation is often required when one wishes to perform actions or processes on functions, such as adding or multiplying functions, or forming sets of functions.

In attempting to understand a given mathematical concept, an individual may find it necessary to coordinate several previously constructed concepts. Such a coordination is referred to in our theoretical analysis as an individual's *schema* for the concept. A schema may also be thematized to become an object.

A discussion of the stages through which a schema might develop can be found in Clark et al. (1997). Until now, very little has been done with regard to developing criteria for deciding whether or not thematization of a schema has occurred. Certain criteria that appear to be useful have emerged in our analyses; we report on these in Section 8.1.

Returning to our overall framework, we note that the initial analysis is based primarily on the researchers' understanding of the concept, on their experiences as learners and teachers of the concept, and on other published research. The resulting *genetic decomposition* for the concept forms the basis for the design of instruction that is intended to help students make the proposed constructions. The pedagogical method that drives this instructional treatment is referred to as the *ACE* teaching cycle (Activities, Class discussion, and Exercises); the main strategies of this method include having students construct mathematical ideas on the computer using a mathematical programming language, and having them work in cooperative learning groups for problem solving and discussion of the results of the computer activities.

Implementing the instruction provides an opportunity for gathering data. There are two ways in which the data are related to the theoretical analysis. First, the theoretical analysis directs the analysis of data by asking the question: did the proposed mental constructions appear to be made by the students? Second, the analysis of the data is used to reconsider the genetic decomposition. According to our method of analysis, student responses are compared to find very fine mathematical points which some students seem to understand (or operations that some can perform) but others cannot. Then we try to find some explanation for the difference in terms of some construction of actions, processes, objects and/or schemas. If we can find an explanation that seems to work, then it is used to revise the genetic decomposition. The entire cycle of theoretical analysis, instruction, data collection, and analysis may be and often is repeated in subsequent studies as new insights are gained which lead to a revised theoretical analysis or to changes in the design of instruction. At the same time, data is also gathered to report on the performance of students on mathematical tasks related to the concept in question. Our analysis of this type of data is expressed in mathematical terms, rather than in terms of what mental constructions might, or might not have been made.

Thus the outcome of this approach is, by nature, two-fold. One result of the research is the deepening of the researcher's understanding of the epistemology of the concept. The second result is the creation of pedagogical strategies which are better aligned with the way we believe that students come to understand the concept; these improved strategies should thus lead to increased learning by the students. As a consequence, many of the studies conducted by *RUMEC* contain findings both about epistemological issues and about pedagogical issues, as well as the relations between them.

The reader should note that our framework does not involve trying to establish the truth of the theory, or even to compare the APOS theory with other theories of mathematical knowledge. Our aim is only to develop a theory that provides one possible explanation of the observations that have been made and can thus be used to design instruction that results in improved learning.

The goals of the current study are: to determine to what extent the *APOS* epistemological framework is useful for understanding the mental constructions made by students learning about binary operations, groups, and subgroups; to increase our understanding of how learning about these topics might take place; to evaluate the extent to which our instructional treatment leads to students successfully performing mathematical tasks that require an understanding of these concepts; and to develop a base of information which sheds light on the epistemology and pedagogy associated with these topics.

The structure of this paper mirrors the framework of the research methodology. That is, we start by describing our initial genetic decompositions for binary operations, groups, and subgroups. This is followed by a description of the instructional treatment that was designed to help students make the mental constructions proposed in the genetic decompositions. Next, we discuss the instruments of assessment, and the results of those assessments. Finally, we provide revised theoretical analyses for binary operations, groups, and subgroups, and conclude by making some suggestions concerning pedagogy for these topics and stating some questions for further study.

In addition to the contents described above, we also include a strand throughout the paper concerning one of the problems that students were asked to solve during the interviews—proving that the center of a group is a subgroup. Initially, this problem was viewed simply as an application of the students' knowledge about subgroups but, during the data analysis, it became clear that the students' construction of the center as a subset is also an issue. The students' work on the construction of the subset and showing it is a subgroup is particularly interesting because it involves enhancing and coordinating virtually all of the other schemas discussed in this paper. Hence, our study of the center tends to unify the various epistemological and mathematical components of the paper. The strand on the center follows the structure described above throughout the paper, with the exception that there was no instructional treatment that was designed to help students make the constructions needed, beyond that which focused on subgroups in general.

2 Literature

In this section, we give a brief overview of the literature regarding the teaching and learning of binary operations, groups and subgroups. We have not attempted a comprehensive review; rather, our goal is to give the reader a sense of what is available.

The literature contains many examples of papers which offer suggestions on the teaching of Abstract Algebra. A sampling of this type of paper is given below, but we do not review them in depth because they do not, in general, report on research into how students learn the topics in question. Some give an innovative approach to a particular topic or theorem (see, for instance, Johnson (1983)), while others present alternative ways to structure the class as a whole (Barbut (1987), Freedman (1983), Hirsch (1981), and Leron and Dubinsky (1995)). A few, such as Maruszewski (1991), Simmonds (1982), and Kraines, Kraines and Smith (1990), outline approaches to integrating computer activities into the class.

There are a number of studies of children's learning of these and related topics which appeared during the time of the "New Math" movement. (See, for instance, Branca (1974), Lant (1980), Dienes and Jeeves (1965), and Dienes (1959)). In general, these studies dealt with younger students (in elementary or middle grades) and involved experiments in which students were presented with concrete instantiations of groups and subgroups. The results of these experiments give information about the students' abilities to deal with the specific examples, but do not address the learning of the underlying conceptual mathematics.

Still other studies exist (for instance, Halford (1982), Leskow and Smock (1970), and Burn (1996)) in which the mathematical understandings of very young children are identified and described as being instances of advanced concepts such as binary operation, group, or subgroup. Such studies frequently attribute to the children a deep understanding of the underlying mathematical concept on the basis of the observed use of a concrete example (see, for example, the comment by Burn (1996) who asserts that quotient groups are not difficult because the additive properties of even and odd "provides a description of a quotient group of the integers which is familiar to many school children.")

Recently, a few studies have been done which investigate the ways in which students learn these concepts in various settings. Hart (1994) studied the proof-writing behavior of mathematics majors in two levels of undergraduate abstract algebra courses and in a beginning graduate course. While the goal of the study was to investigate proof-writing rather than the learning of topics in elementary group theory, the proofs were chosen with the intent of reflecting the level of understanding of elementary group theory possessed by the student. Students were provided with a page of relevant facts from group theory in order to reduce the amount of variation in proof-writing performance due to ability to recall the necessary facts. On the basis of performance on certain proofs, the students were classified according to levels of conceptual understanding of the concepts of elementary group theory. It was found that "the amount of academic experience with abstract algebra does not necessarily reflect the level of understanding" (p. 56).

Dubinsky, Dautermann, Leron, and Zazkis (1994) studied the learning of topics in group theory in the context of a six-week summer workshop for high school teachers. They found that understanding of groups and subgroups may progress somewhat simultaneously. It was observed that "understanding may move from seeing groups and subgroups as primarily sets of discrete elements, to a stage where the operations as well as the group elements are incorporated into the necessary definition" (p. 273).

3 Preliminary Theoretical Analyses

This paper reports on the second iteration of the research and curriculum development cycle described in the introduction, following up on Dubinsky, Dautermann et al. (1994) and, as such, the preliminary genetic decompositions used here are based on the results of that study. In the following sections, we describe the preliminary genetic decompositions for group and subgroup that directed the instructional design and data collection of this study, and then compare it with the results of the analysis in the previous study.

In keeping with its status as a separate, unifying strand, we will consider the center separately from the other schemas.

3.1 Binary operation

The essence of our initial theoretical analysis is that a binary operation is a function (of two variables), and hence our genetic decomposition will be very close to a genetic decomposition for the function concept. (See Breidenbach et al. (1992)). Following is a description of various conceptions of binary operation in terms of the *APOS* framework.

Action. The student can perform a binary operation only when given an explicit formula as in modular arithmetic.

Process. The student can think in terms of a process for a generic binary operation in which two objects come in, something is done to them, and a new object comes out.

Object. The student is able to distinguish between different binary operations, and/or consider more than one binary operation defined on the same set. The student is able to de-encapsulate a binary operation in order to work with it as a process.

Schema. The student has a schema for binary operation that can be invoked and used in dealing with mathematical problem situations.

3.2 Group

The concept of group can be understood as a schema that consists of three schemas: set, binary operation, and axiom. The schemas of set and binary operation have been thematized to form objects and they are coordinated through the axiom schema.

The axiom schema includes the general notion that a binary operation on a set may or may not satisfy a property, which is essentially the process of checking the property. It also includes four specific objects obtained by encapsulating the four processes corresponding to the four group axioms. Checking an axiom consists of coordinating the general notion of satisfying a property with the specific process for the axiom (de-encapsulated from the object) and applying it to a particular binary operation and set. In doing this, the binary operation and the set are de-encapsulated to their processes and the three processes (axiom, binary operation, set) are coordinated to establish that the axiom is satisfied. The four instances of this operation are coordinated into the total process of satisfying the axioms.

The group schema is thematized to form an object to which actions can be applied. Examples of such actions include determining that a particular set and binary operation form a group, checking various properties a group might have, and considering whether two given groups are isomorphic.

An important component of one's group schema is the ability to consider a generic group as well as particular examples of groups.

3.3 Subgroup

The concept of subgroup can be understood as a coordination of three schemas: group, subset, and function. The function and subset schemas are coordinated to obtain the process of restriction of a function to a subset of its domain. This process is then coordinated with the binary operation in the group schema to obtain the restriction of the binary operation to a subset. Finally, the axiom schema in the group schema is applied to the pair consisting of the subset and the restriction of the binary operation to that subset.

3.4 Center of a group

In constructing the center of a group, the group schema is maintained as a generic group, producing a set and a function. The defining condition of the center of a group (as the set of all elements of the group which commute with every element of the group) is applied to construct the required subset of the generic group. The subgroup schema, as described above, is applied to the specified subset to establish the center as a subgroup of the group.

3.5 Comparison with epistemological analyses in previous study.

As we have indicated, this study follows up a part of the study reported in Dubinsky, Dautermann et al. (1994), in the sense that we repeated the research cycle for binary operations, groups, and subgroups. Thus, the preliminary analyses given here are very close to the analyses arrived at in that study. We take as our starting point (which may or may not be the case for the students) the existence of schemas for set, subset and function and focus on the coordinations of these schemas and other schemas that result from those linkings. The analysis in Dubinsky, Dautermann et al. focuses on how these coordinations arise, beginning, for example, with the concept of group and subgroup initially understood almost entirely in terms of the underlying set with the binary operation brought in later. This was seen particularly in students thinking that a group whose underlying set was a subset of another group would also be a subgroup. For example, many students in that study thought that $\mathcal{Z}_n$ is a subgroup of $\mathcal{Z}$.

One difference between the two analyses is that in Dubinsky, Dautermann et al. there is suggested the possibility that the group and subgroup concept may develop simultaneously whereas here we begin with the idea that the group concept is more or less present when the development of the subgroup concept begins. It is possible that this difference may be related to the way in which the subject is made available to the students.

One result of Dubinsky, Dautermann et al. that is not considered here is the idea of a generic group as an equivalence class (under isomorphism) of pairs, each consisting of a set and a binary operation.

4 Subjects

The subjects in this study were undergraduate students at a large midwestern university who had taken or were taking a first course in abstract algebra which was designed for mathematics majors but was not the honors course. The main group of subjects consisted of 31 students who were taking an experimental version of the course during the fall of 1991; a full description of the instructional treatment used in this course is given in the next section. The students were mostly pre-service secondary mathematics teachers, and the instructor and a graduate assistant are two of the authors of this study.

In addition, there were 20 students who had taken an abstract algebra course taught according to standard methods at various times ranging from the same time as the students in the experimental course, back to two semesters earlier. More specifically, 5 of these students took the course in the fall of 1991, 8 in the spring of 1991, and 3 in the fall of 1990. One student had taken it in the spring of 1990 but was taking a second undergraduate course in abstract algebra at the time the data for this study was collected. There were 3 students who were not asked when they took the course.

It is possible that some of the other students in either group were taking courses that touched on concepts in abstract algebra before or at the time the data was collected. Finally, two of the 31 students in the experimental course had previously taken the standard course.

5 Instructional treatments

As mentioned earlier, the overall structure of the experimental course was based on the *ACE* teaching cycle; our implementation utilized both computers and cooperative groups. Students were grouped into permanent teams consisting of three or four members each, and the majority of course work was completed in groups. Material was broken into topical sections, each of which ran for approximately one week. One two-hour class session per week was spent in a computer laboratory, and two one-hour class sessions per week were held in a classroom with no computers. In the computer laboratory, students completed computer activities using the mathematical programming language **ISETL**. In order to stimulate reflection, the computer activities usually dealt with concepts that had not been formally studied in class. These concepts were then discussed in the successive class meetings.

To encourage further exploration of the concepts, students were assigned homework to be completed outside of class; both computer exercises and traditional exercises were included in the assignments. The course uses a textbook that was written explicitly to support this pedagogical approach, *Learning Abstract Algebra with ISETL*, by Dubinsky and Leron (1994).

There are several references to **ISETL** in our description of computer activities. In many cases, the meaning should be clear from the context. Some of the syntax is explained in Dubinsky and Leron (1994) and for others, the reader may wish to consult Dautermann (1992).

We now describe the specific instructional strategies used in the experimental course to help students develop their understandings of each of the topics considered in this study.

5.1 Binary operations

Of course, the concept of binary operation is implicit in all aspects of any abstract algebra course, including the one on which this study is based, because of its key role in the content. All of the work with groups would tend to solidify a student's understanding of the binary operation concept. The binary operation concept is revived when the subject turns from groups to other algebraic structures such as rings and fields. At this point, the student must think about *two* binary operations defined on the same set but having different properties, as well as consider conditions that relate the two operations.

In the experimental course, all of the explicit references to binary operations are in Chapter 1, which is covered in the first two weeks of the course. All of the situations occurred in the context of learning **ISETL** on the computer, and were designed with the initial theoretical analysis in mind. Examples of situations in which students encountered binary operations during this period follow.

- The expression `(x+y) mod(6)` appears (without explanation) in the following selection of code used as an illustration in the textbook:

```
x:= 4; y := 2;
if (x + y) mod 6 = 0 then
      ans := "Additive Inverses!"; end;
ans;
```

  Working with such code is expected to help students construct actions for binary operations other than ordinary arithmetic.

- Several kinds of binary operations appear as **ISETL** predefined operations. These include: `mod, div, min, max, and, or`. The students work with these without explicit mention of the idea of binary operations. This work is intended to lead the student to augment her or his experiential base of binary operations and, ultimately, could strengthen an individual's object conception of binary operation.

- The students construct **ISETL** `funcs` (i.e., functions) to implement various binary operations such as modular arithmetic and composition of permutations. For example:

```
Z20 := {0..19};
op := func(x,y);
      if (x in Z20 and y in Z20) then
            return (x+y) mod 20;
      end;
end;
op(3,5); op(9,16); op(4,20);
```

  The goal of such activities is for the student to interiorize binary operation actions to processes.

- The students use **ISETL**'s infix notation (`.op`) for binary operations. That is, if `op` is any function of two variables in **ISETL**, the expression `a .op b` may be used as an alternative to `op(a,b)`. For example, the student could enter

  ```
  3 .op 5; 9 .op 16; 4 .op 20;
  ```

  to obtain the same results as the last line of the previous activity. Our expectation is that this will help the student make the connection between **ISETL** work and manipulations in ordinary mathematical notation.

- The students form ordered pairs of two elements in which the first is a set and the second is a binary operation on that set. This is expected to help the student interpret the set and the function as objects and also to begin to construct a binary operation schema.

- The students write `funcs` which accept a set and a binary operation on it and return `true` or `false` depending on whether a certain property is satisfied. They are asked to use certain names for these funcs and the names are suggestive, such as: `is_closed`, `is_associative`, etc. Our expectation is that such activities will help the students thematize their binary operation schema.

Formal treatment of binary operations in the text is very brief, consisting of just a few lines. Aside from these lines being contained in a reading assignment, the course did not emphasize this formal treatment.

5.2 Groups

The instructional strategy for the group concept is for students to construct all of the ingredients of the group schema, including the coordination through the axioms (see Section 3.2). In the implementation being discussed here, this work also took place during their study of Chapter 1, concurrent with learning about binary operations and **ISETL**.

In addition, students are expected to construct a mental concept of set as process by constructing set formers and iterating through `forall`, `exists`, and `choose` statements in **ISETL**. For example, the set S_3 of all permutations of $\{1, 2, 3\}$ is constructed as an example of a set former:

```
S3 := {[a,b,c] :  a,b,c in {1,2,3} | #{a,b,c} = 3 };
```

Having constructed the set `Z20` and addition mod 20 in a previous computer activity, students construct the following process that iterates through the set `Z20` to check whether an identity exists:

```
exists e in Z20 | (forall g in Z20 | (e+g) mod 20 = g);
```

They construct sets as objects by working with properties of sets, operations on sets, sets as inputs to a `func` and sets as components of `tuple`s (i.e., finite sequences.)

Chapter 2 of the text begins with some computer activities aimed at making explicit what the group axioms are and how they apply in specific examples. Emphasis in these activities is on the idea that a particular set and binary operation might satisfy some of the axioms, but not the others. Other properties that a group might have, such as commutativity, are also explored.

The discussion in Chapter 2 begins with a formal definition of group as a set together with a binary operation satisfying the four axioms. Thus, the wording of the definition corresponds to the genetic decomposition. Over a three week period, the students apply the definition to two classes of examples, modular groups $\mathcal{Z}_n$ and groups of symmetries S_n, and begin to prove properties of groups.

Of course, the group concept in general and understanding a group as an object in particular pervade all of the remaining work in the course.

One very specific strategy was developed with the intention of helping students construct the notion of a group as a generic object. As shown below, code is written that allows students to use generic group notation (such as denoting the group by G, and the identity by e) during their computer work. The idea being explored here is whether using this generic notation helps students move beyond thinking just in terms of a single concrete example, and begin to see particular examples as simply specializations of the generic group concept.

This strategy comes into play after they have studied groups for a week or two. At that point, they are given the following computer program which incorporates several computer functions they have written themselves.

```
name_group := proc(set,operation);
                   G := set; o := operation;
                   e := identity(G,o);
                   i := |g -> inverse(G,o,g)|;
                   writeln "Group objects defined:  G, o, e, i .";
              end;
```

To see how this program is used, suppose the student has constructed, on the computer, the set `Z12` and a `func a12` which implements the binary operation of addition mod 12. Then they run `name_group` on the pair (`Z12`,`a12`), by writing and executing the following line:

```
name_group(Z12,a12);
```

During the remainder of the session (or until the command is run again with another group and operation), the computer will recognize the symbol `G` as standing for `Z12`, `o` for addition mod 12. It will have found the identity (`0`, of course) and assigned it to the variable `e`. Finally, it will have constructed a `func i` which selects the inverse of an element of the group.

While the students are asked to apply `name_group` to other examples and are advised to use it throughout the course in working with specific groups, it is not clear how many of the students in this implementation of the experimental course actually took this advice.

5.3 Subgroups

Towards the end of the study of the group concept, students are asked to think about subsets of a group, restrictions of the binary operation and the group concept. Then, at the end of the chapter on groups there is a very brief discussion of the subgroup concept including the formal definition of subgroup and examples of subsets of the group D_4 of symmetries of the square.

After approximately 5 weeks of the course, the students begin a chapter in which they spend three weeks studying subgroups. The first week is about the concept of subgroup, the second is about cyclic groups and their subgroups, and the third is about cosets and Lagrange's theorem.

The main approach to the notion of subgroup is to replace the set with one of its subsets, but to keep the "same" binary operation. The notion of restricting a function (of two variables) to a subset is barely mentioned and the formal definition of subgroup is repeated after the students have worked with the concept for some time.

5.4 Center of a group

There was no formal instructional treatment of this topic in the experimental course. However, as part of their homework, students were assigned to show that the center of a group is a subgroup. They were also asked to find the center of each of four groups: $\mathcal{Z}_n$, S_n, and two different subgroups of the group of 2 x 2 invertible matrices with real entries.

6 Instruments

This paper is reporting on a portion of a large-scale study of students learning abstract algebra. In the full study, there were a total of five instruments used to gather data: three written examinations in the course and two sets of interviews. All but the second interview were administered only to students in the experimental course. The interviews were conducted by a team consisting of two of the authors of this paper and four research assistants. In the appendix, we list those examination and interview questions that are related to the topics of binary operations, groups, and subgroups. We also give, for each question, an indication of what we expected to learn from the responses.

The first two instruments were two examinations given during the semester as part of the course. The students took the first exam in their permanently assigned cooperative groups. They were given unlimited time for this closed book examination and each group turned in one exam for which each member of the group received the same grade. The second exam was also closed book and unlimited time, but the students took it individually and did not communicate with each other during the exam. Each student received two grades for this exam: one was the score on her or his own paper and the other was the average of the scores received by all the members of the student's group.

The final examination for the experimental course was given in a traditional way in a two hour period with each student taking the exam individually and receiving only one grade. The exam had three parts: definitions, true/false questions, and a set of 11 propositions from which the student selected any two to prove.

Our data also includes two sets of interviews covering topics from abstract algebra. Audio-taped interviews with 24 of the 31 students in the experimental course were conducted during the last week of the Fall 1991 semester. The second set of audio-taped interviews were conducted during the following semester with 17 of the 31 students from that course, together with 20 students who had taken a standard course in abstract algebra. Transcripts of all sessions were produced to complement the record of written work which the student completed during the interview. The transcripts were carefully read and analyzed in order to produce a list of mathematical issues that arose during the interviews. Focusing on these issues, we obtained results about the mental constructions that students appear to have made, as well as a general statement on performance, all of which is reported, along with the results of the examinations, in the next section.

7 Results

We present two kinds of results regarding student work on the instruments described in the previous section and in the appendix. First, we consider the nature of their responses in terms of the mental constructions proposed in the preliminary genetic decompositions given in Section 3. That is, we are using our data to see whether the students appeared to be making these mental constructions and to see what other constructions they might appear to be making.

The second kind of result is simply a summary of the performance of the students in the mathematical sense. That is, did they answer the questions and solve the problems reasonably well, what typical errors did they make, and of what mathematical concepts did they demonstrate their knowledge?

In each case, we select the relevant interview questions and/or test items listed in the appendix. In discussing each type of result, we consider separately the topics of binary operation, group, subgroup, and the center of a group.

When we report the results of work submitted by teams, we recognize that the work may reflect the knowledge of one member, or it may reflect the knowledge of more than one member. Some of the knowledge may be shared, and some may not be. Therefore, when we refer to a schema constructed by a team, we recognize that it is possible that not all individuals in the team have the same components and linkages in their schemas. In particular, the work submitted by a team may or may not be an application of any single individual's schema for the concept.

On the other hand, for the topic of inverse functions, Vidakovic (unpublished) compared how the concept developed in calculus students who studied the concept individually with how it developed in students who were assigned to work in cooperative learning groups. She concluded that there were no differences in the constructions of this concept for the two types of students. Thus, we might expect to learn something about the individual constructions for the topics under study in this paper by considering the work done by teams.

7.1 Mental Constructions

The main contribution to what we can say about the students' mental constructions comes from the interviews. We also include in this section results from any test item that informed us about the student's possible mental constructions. We omit the true/false questions on the final which are, at most, performance indicators.

For each of the three main topics, we will start with what we learned from the interviews, and support this, when possible, with further results from the written examinations, including the team exam.

7.1.1 Binary operations

First interview, Question 1. The responses of the students to the questions about the binary operation in D_4 are related to both their understanding of binary operation in general and their understanding of symmetries of the square. There is a focus on the latter in another report (see Asiala, Brown, Kleiman and Mathews (in press)), but the two cannot be completely separated since a student's difficulty with understanding a symmetry as an object or process will affect her or his response to these questions about the binary operation in D_4.

In the excerpts of student reponses to the interview questions, we have changed the names of the interviewees to protect their identities. Also, while the interviews were conducted by several different people, we refer to each with the same designation, **I**.

We distinguished five different kinds of responses, three of which contained variations having to do with how the student viewed the product of two symmetries.

1. The lowest level of response, in which the student shows no general understanding of the notion of binary operation, and even has difficulty performing an actual computation, is exemplified by the response of Eli. He started by explaining how to manipulate an actual square in order to perform a single rotation of 90 degrees and then, the following exchange takes place.

   I: Ok. All right. Can you think of another way to do it?
   Eli: Um,...(brief pause)... yes, actually, um, as far as what? As far as product combining? Or...?
   I: No, I mean another way to take these elements and multiply two of the elements to get the outcome? Besides taking the piece of paper?
   Eli: Well, actually, um, well, the product of rotations is um, just, um, the addition of, you know, R_{90} plus, um, combined with R_{270} would be R_0. Um, the ... you can just use the degrees of rotation mod 360.
   I: M-hmmm.
   Eli: Um, you can add them that way...

   We consider this to indicate an action conception of binary operation in general and composition of symmetries in particular, because Eli appears able to refer to only very specific examples, and he describes composition of symmetries only in terms of the explicit procedures of manipulating an actual square and adding angles mod 360.

2. The first progress that was made was to think in terms of two symmetries to which something had to be done, but without reference to the result either in general, or in the context of a particular example. Some students at this level appeared to be using a process conception of symmetries as functions and attempted to compose them, as in the following excerpt.

   Arlene: We'd make a little square, R_0 is, we'd number all of our corners and we'd say look we're going to do R_{90} and then H. We'll make a 90 degree turn and then a horizontal flip and see where we end up, that would be, that was the easiest way to see exactly what it was going to be...

Other students tried to combine two symmetries by thinking of one as acting on the other. This is similar to a situation involving composition of functions reported by Vidakovic (1993) in which, when asked to define the composition of functions, $f \circ g$, one student considered g to be an object to which the process f could be applied.

Lorrie: We used tables where we wrote out each element and we did what like R_{90} does to like R_{180}, you know?

In these responses, we do not see a completed process of taking two symmetries and doing something to produce another symmetry. It is possible that these students lack strong object conceptions of symmetries of a square, which would make it difficult for them to perform actions on symmetries.

3. Some students did indicate an interpretation of two symmetries going in and a symmetry coming out, but only in the context of a specific example, so their conception might be no stronger than action. It is interesting that all of these students seemed more comfortable working with the permutation representation of the symmetry. Although it is hard to follow her calculations, in the following example Kathy does correctly compute R_{90} followed by a horizontal flip to be a flip along the diagonal from the right.

Kathy: Ok, let's try R_{90} and...if I do R_{90} which is 4,1,2,3 and I do a horizontal flip, which is equal to 3,2,1,4 you get 4,1,2,3 times 3,2,1,4....

...

Kathy: This equals 3,2,2,3,3,1,3,1,1,4, which, do we have 2,1,3,4? No. How would we get 2134? Let's see. 2,1 – it'd be 270. Hmmm. If we do 2,1 whoops! 2,1,4,4,2 is that what I had? Oh, yes, I did! Which would have been, how did I do that? Would have been R_{270} with an, um D_r. So, it would have been diagonal from the right? Would that have been it? Yeah.

4. At the next level, we see responses in which the student appears to think of a binary operation as a process that takes two inputs, and produces an output, that is, as a function of two variables. In the following excerpt, Darrel does this when he reports the results of the operation without having to refer explicitly to the details of the calculation.

   Darrel: My H and V, okay, so when you multiply those by themselves you get the identity element. Uh, when you multiply them together, you get the R_{180}, which is the cycle 1,4, cycle 2,3.

   I: Can you show commutativity?

   Darrel: Yeah, and then yeah, if you take HV you get R_{180} and in this case it works out that you get the same thing when you take VH.

   In the following excerpt, Ted refers to the binary operation as a process of interaction between two symmetries, a description that appeared only once in our data.

   Ted: Just use these two, um, I just sort of look at it and picture, you know, what happens to each of them as they interact with each other, so this is where you started with and it ends up, let's see (pause) um, this here is just a, ok, this here has just been turned 180 degrees so then what I do if I was taking this one with this one is just turn this one 180 degrees.

5. Finally, there were some students who just gave correct answers and indicated a strong knowledge without going into details about the nature of that knowledge during the interview.

Next, we turn to the results from the tests.

Test 1, Question 1 Each team was successful in verifying the axiom of closure, and all but one was successful in verifying associativity, with respect to the new operation. At a minimum, this indicates that each team was able to distinguish between two operations defined on the same set, and that they had constructed a thematized schema for each binary operation.

Moreover, in order to verify these group axioms, the teams had to use a link between their schemas for each of the two binary operations. This indicates that each team had constructed a general binary operation schema that contained, in particular, a process for switching back and forth between two operations defined on the same set. Finally, when a team is able to think about different binary operations on a set and decide to work with one instead of the other, we have an indication that the team is reflecting on this schema and seeing it as a total entity, that is, they have thematized their general binary operation schema.

We saw indications of differences in the adequacy of the teams' general binary operation schemas in their work on verifying the identity and inverse axioms for the new operation. Some teams' responses indicated confusion regarding which of the two operations should be used in the calculation of formulas for the identity and inverses for the new operation.

Test 2, Question 3(a). This item is considered both here for binary operations and in the next section for groups. We will focus here on the operation (which is inherited from the original ring), and on the subset specified by the unit condition, which serves as the domain of the restricted binary operation. In the section on groups we will consider the results in relation to the group axioms.

In this context, all responses showed the existence of a binary operation schema including the property of an element being a unit, but there was considerable variation in what an individual student was able to do with this construction. The following categories of responses do not necessarily lie in order of a developmental sequence.

- At the lowest level, a student was able to focus on the operation of "multiplication" in a ring and the idea of a unit and was aware of the properties which this operation could have (in terms of the group axioms), but was unable to check any properties.
- In some responses, students could check associativity, but not closure. They were aware that the identity and inverses existed, but they did not show that that they were contained in R^*.
- Some students dealt correctly with all axioms except for closure.
- At the highest level, students solved the problem correctly, in some cases making minor computational errors.

The responses on this question suggest that after constructing a set and a binary operation, the student still must work to coordinate the various ways in which the binary operation can relate to the set or sets on which it is defined. There appears to be a tendency to construct an environment consisting of a set and an operation and to work as if this is the entire, and only, universe in which things are taking place. In a sense, the student "lives" entirely in this one environment while working on a problem. This can be problematic when a shift in environment is required in order to consider the algebraic structure of a subset, as it is in this problem.

Progress is made when the student recognizes that there is more than one environment, game, or set of rules and, under her or his control, the rules to be used can vary from time to time within the solution of a given problem. We can interpret these conscious choices of environment as actions on binary operation schemas.

Test 2, Question 6. Our interest in this question relates to student understanding of binary operations on quotient groups, in particular, whether the student understands that both the set of cosets and the binary operation are important. Ignoring the binary operation, as some students did, could signal a difficulty in understanding binary operations, or groups, or quotient groups.

The responses to this question fell into three mutually exclusive sets:

a) students who made little progress on the problem,

b) students who found a normal subgroup, but did not mention the binary operation, and

c) students who found a normal subgroup, described the operation on the quotient group, and identified the quotient.

Because none of those who described the binary operation had difficulty calculating and identifying the quotient, it appears that the crux of the problem may consist in constructing the binary operation.

Test 2, Question 7(a). Again the responses to this question showed some students paying attention to the binary operation and working with it correctly, and some never mentioning it. There were also students who considered the operation, but either used an inappropriate one or used the right one incorrectly.

Test 2, Question 7(b). In checking the isomorphism, the question of the binary operation arises in a manner similar to the previous questions. Here the point is that some students compared both the set and the binary operation in thinking about isomorphism, while others consider only the set. There were also some responses that had little in them that indicated progress on the problem.

7.1.2 Groups

First interview, Question 7. In the first interview, the lowest level of response to the question of finding an element of order 6 in a commutative group that contains elements of order 2 and order 3, was for the student to be totally confused and not say anything that can be recognized as progress. One reaction that occurred was that, in struggling to make some sense out of the situation, a student would try to use Lagrange's theorem inappropriately (for example, suggesting that since 6 divided the order of the group, the group must have an element of order 6, a type of response that has been discussed by Hazzan and Leron (1994). For the most part, however, students at this level just fumbled around with incoherent phrases.

Some students appeared able to think about this problem only in the context of a specific example, such as $\mathcal{Z}_{12}$ or S_4, indicating that they may have had difficulty with the concept of a generic group. In the latter case it is, of course, necessary to ignore the fact that the group is supposed to be commutative. In some cases, these students were led to think of a counterexample and they could find one in S_4. A reminder from the interviewer about commutativity sometimes led to the student reconsidering and then solving the problem with a good explanation of why ab has order 6.

An important first step in solving the problem is to think of a candidate for an element of order 6. The most natural choice was ab, but some students discovered that ab^2 must have order 6. In some cases, prompting was necessary before a candidate could be found.

The task of showing that ab is an element of order 6 can be analyzed in terms of actions, processes, and objects.

Action. The first action is to form various products of powers of the given elements a, b without a clear strategy or organization of the data. The written work of the following student was a collection of symbols such as k, lll, k^2l without any apparent coherence. This, together with the following comments, suggests to us that for this student, performing the group operation successively on one or more elements is an action which he is not really able to reflect on at this point.

> **Nathan:** Um. I wasn't really sure. Well, you know, if you have an element 2 times 3 is 6, right, okay, so you're gonna have... You see how I can make that connection, right? How can I make this connection 2 times 3 is 6, right. So you got k times k, you got lll. And if you multiply you get an element that's in there, but not necessarily in order. Well, no, okay. Okay, e times e is e. So k times k times l times l times l, that's gotta be e. That's an element of order 5 isn't it? And this isn't the same element though. So...
>
> **I:** You've got k squared times l cubed.
>
> **Nathan:** Yeah. k squared times l cubed equals e. So, um, if , okay if this were to equal some element, say f to the 6th, if that equals could that... Does that necessarily have to be the case? Well, we know it's commutative, so it's the same thing as l cubed times k squared equals e. I don't know if that makes any difference. Let's see. Um, does that equal f to the 6th, another element? Well, when you multiply numbers.... I don't know. I'm forgetting using properties of exponents here probably. This basic crap. For the same number

you add the exponents. So it's staring me in the face and I'm not seeing it.

A more effective approach to the problem might include organizing the actions into sequences of computations such as $a, a^2 = e$, $b, b^2, b^3 = e$ and $ab, a^2b^2 = b^2, ab^3 = a, a^2b = b, ab^2, a^2b^3 = e$. The following student indicated such an organization in what he wrote down as well as what he says here as he struggles to obtain the desired conclusion. Initially, his calculations are undirected, but the interviewer brings him back to task by repeating the problem and then he proceeds to the desired conclusion.

Ted: I'm trying to think, there has to be more elements . . . *ab* times *ab* squared, because it's commutative, *a* squared *b* cubed which is *e* . . .

I: Okay.

Ted: Because *a* squared is *e* and *b* cubed is *e*.

I: Okay, so those two are inverses, I'm with you so far.

Ted: Um, these two are inverses multiplied by each other.

I: Uh-huh, sure.

Ted: *a* is its own inverse.

I: Okay.

Ted: Um, you multiply, *a*, *a*, anything you multiply together . . .

I: Uh-huh.

Ted: You're not going to get anything new.

I: Okay, so you are saying that, your thought right now is what?

Ted: Is that it does not have to have any more numbers.

I: Okay, so those are the only ones you can definitely say it has to have.

Ted: Yeah.

I: Okay, I'm with you so far. So, the question is, must it have an element of order six?

Ted: Well, we already know the orders of these.

I: Do we know the orders of . . .

Ted: *b* squared, *b* squared times *b* squared is *b* to the fourth times, which is *b* cubed times, 2 *b*'s cubed times *b*, which is *e* times *b*, and times *b* squared, so you multiply that by another *b* squared . . .

I: Uh-huh.

Ted: And that's going to be b cubed, which is e. So then that order is three.

I: Okay.

Ted: Um, ab times ab is equal to a squared b squared and we know that a squared is e so it would be b squared. Times ab . . . is . . .times ab is ab cubed, which is the same as a, multiplied by ab is a squared b which is b.

I: Okay.

Ted: There's probably a faster way of doing this but . .. . um, multiplied by ab is ab squared, multiplied by ab . . . (??) . . a squared b cubed which is e so that's an order six element.

I: So your conclusion is what?

Ted: It does have to have an element of order six.

Another feature of Ted's thinking that was shared by some students is that, although he is clearly forming powers until the *first* occurrence of e, this is not explicit in anything he says or writes.

Process. The student thinks about the computation and expresses it, for example, as the powers cycling back, or as computing powers until there is nothing left over. The following student took the latter view in explaining "why it works". Note that Mitch does seem to be paying attention to the importance of not reaching the identity earlier.

Mitch: Obviously it going to be, the reason why it works is because, um, the least common multiple, is that right, yeah, um, is going to have to, they're going to have to be, you're going to have to go that far, I guess, like, so that they're both equal to e and you don't have either g or h left over. And that's why it has to exist, or that, yeah, that is has to exist because otherwise you won't, g times h, um, and that won't be equal to the identity until, until they're, until g, um, is equal to the identity and h, or not g itself, but, um h times itself is equal to the identity.

Object. The processes become objects of thought. The student thinks about a cycling back in two steps and b cycling back in three steps and realizes that these two processes must be coordinated. The question then is how many steps must be taken before the two processes both cycle back to the identity.

> **Joelyn:** OK so we've got a having a certain order meaning a certain number of times you have to multiply it by itself to get back to the identity. And b having a certain order, the number of times you have to multiply it by itself to get back to the other, So if you have ab together and you take powers of ab ... It's like has a certain no, a has a certain order and b has a certain order, so you are taking ab times itself. So you are going to get to a point where a is going to cycle back to the identity and, you know, b might have a bigger order so it is going to keep getting a bigger order until it cycles back to the identity, while it's doing that a is like going through its cycle and...

For the general case (orders 2,3 replaced by n, k), some students said that the group has to have an element whose order was the least common multiple of n, k and some of these were able to use thinking as just described to explain why.

> **Joelyn:** so there is like going to be a point when they are going to match up. And that will be the least common multiple.

Test 1, Questions 1,2. The responses to Test 1, Questions 1,2 uniformly indicated that the teams had constructed a schema for group consisting of a set and a binary operation, that they could use in working with problems such as these.

Regarding the thematization of this schema there were the following three kinds of responses.

1. Although each team had apparently constructed a group schema, it was possible for a team to not progress very far towards thematizing it. In the response to Question 1, there was confusion between the identity under the original operation and a possible identity in the new operation and, when it was necessary to distinguish between two groups, there were some cases in which there seemed to

be an inability to argue coherently.

2. In some cases, the team appeared to be making progress in thematizing the schema. They were able to compare two situations when the settings were clearly distinct as in Question 2, but when the two binary operations were defined on the same set, as they were in Question 1, some confusion appeared.

3. The work of some teams indicated that they had thematized this schema in that they gave correct answers to Question 1 and all parts of Question 2. They were able to coordinate the use of the schema for two different situations in comparing groups to determine whether they were the same, and also could move easily from one situation to the other.

Test 2, Question 3(a). This item was considered in the section on binary operations. Here, we will focus on the group axioms.

Several responses indicated confusion on the connection between the set and the binary operation as it related to group properties. In various situations students took the position that once a property was present in a situation, then it was present in all aspects of the situation. Thus some students wrote that a subset of a ring is closed because the ring is closed. Other students wrote that the subset has an identity because the given ring is assumed to have an identity.

Interestingly, although this type of thinking appeared to occur with most students and with all three group axioms, very few students displayed it on all three. In fact, most of the students gave an essentially correct argument on at least one of the three. This inconsistency suggests that we should be very careful about drawing conclusions from student responses on this question.

A second difficulty with interpreting student responses occurred in our reading of student work on verifying the inverse and identity axioms. At times, we simply cannot tell, when a point is omitted from a response, whether the student does not understand the point, or just did not think it necessary to mention.

Specifically, several students did not check that the inverse of an element in R^* not only existed in R, but was in R^*. But this point may have been obvious to an individual, since the inverse of the inverse of an element is the element, so that some may just have neglected to mention it explicitly. A similar situation occurred with the identity for R^* in that many students explained why the multiplicative identity of R is in R^*, but neglected to mention explicitly the fact that it is an identity in R^*.

There is an additional point to mention here, regarding student understanding of a subset S determined by a defining property P. Two related issues arise in student work. First, a student needs to be able to conclude that an element is in S whenever it is known to have property P and, second, the student needs to be able to conclude that the element has property P whenever it is known to be in S. In the responses to this question, and in the interviews about the center of a group (reported in a later section), we found evidence that indicated students had more difficulty with the second situation than the first.

7.1.3 Subgroups

We focus in this section on what we learned from student responses to one question in the second interview, since none of the responses to the test items provided useful information about students' possible mental constructions regarding the concept of subgroup.

Second interview, Question 5. Students were first asked to give examples of subgroups of $\mathcal{Z}$ and to make a general statement about their form, which gave them an opportunity to apply their subgroup schema in a particular situation. For those whose schema had been constructed and thematized, we can describe their progress in using it. Some students may not have constructed a very useful subgroup schema and others appeared to have such a schema but may not have thematized it.

Their responses fell into four broad categories which we discuss in some detail. All but the first category indicate a thematized subgroup schema, and analyses of the responses have to do with the examples they gave and the general statement they made.

1. **No useful thematized subgroup schema.** The first excerpt is from the interview of a student whose responses indicate that he apparently does not have a useful subgroup schema.

   I: Um, let's let capital Z denote the group of integers with addition. Okay, so all positive and negative integers with addition. Can you give an example of a subgroup of that group?

   Peter: (Pause.) I know there's a little chart that helps me on this one, in the book.

   I: What does the chart look like?

   Peter: Oh there's a, in the current book, that just has all the groups listed . . . all the way up to like R. Isn't R the one with 360. . . or is that D? It's D, isn't it. It just tells you what each of them are.

   I: Okay.

   Peter: And then . . . (mumbling) . . . and the last one just tells you whether or not it's abelian, it tells you whether or not, yes or no if it's commutative or . . . you know it just gives a bunch of different things.

   After some discussion of the group axioms and much prompting, Peter suggests that the set $\{2, 3\}$ is a subgroup. When the interviewer points out that closure fails, Peter suggest the even integers, but then also suggests the odd integers. No further progress is made.

In the next excerpt, we see a student who has a subgroup schema, but it appears not to be thematized. First, Hillary focuses on the fact that a subgroup is a subset, and considers that $\mathcal{Z}_n$ might be a subgroup of $\mathcal{Z}$ (an error also noted by others who have studied the learning of abstract algebra, such as Dubinsky, Dautermann et al. (1994) and Hazzan and Leron (1994)). Next, she begins to talk about the group properties for the subset, but does not connect the operation on the original group $\mathcal{Z}$ with the operation on the subset. When the interviewer reminds her about this, she acts as if it is knowledge that she has but did not use.

I: Ok, let's move on to the next question about subgroups and ideals in $\mathcal{Z}$.

Hillary: [silently reading the question, which states that $\mathcal{Z}$ is the group of integers under addition and asks the student to give an example of a subgroup of $\mathcal{Z}$.] What is a group of $\mathcal{Z}$? I don't know. I'm not sure. $\mathcal{Z}_4$ maybe, all these elements are in $\mathcal{Z}$. [She is referring to what she has written: $\mathcal{Z}_4 = \{0, 1, 2, 3\}$]

I: How would you check? What do you need to check?

Hillary: Ammm, those, an inverse, identity, and then closed under the operation. If you add all these elements they are also there.

I: Is it closed then?

Hillary: It is closed under the addition. Inverses? Inverse of zero is zero, inverse of one would be three, the inverse... amm, when you take an number and its inverse is equals the identity. So, zero's inverse would be zero, one's inverse would be three, two's inverse is itself two, three inverse is one. So each element has the inverse; then the identity would be zero.

I: What is your operation in $\mathcal{Z}_4$?

Hillary: It's addition mod four...

I: What is the operation in the given group?

Hillary: It's addition.

I: Are those operations same?

Hillary: No, so $\mathcal{Z}_4$ is not a subgroup.

To summarize, the fact that she is able to recognize and correct her errors in response to questioning by the interviewer suggests that she does have a subgroup schema, but initially she did not have ready access to it. We interpret this as an indication that she has not thematized her subgroup schema.

The remaining categories of responses are those of students who show clear evidence of having a thematized subgroup schema. They are organized in terms of an *APOS* analysis of the task of finding the subgroups of $\mathcal{Z}$. At the action level, we see students forming the set of all multiples of a single integer n to produce a subgroup of $\mathcal{Z}$, and possibly repeating this action for certain other values of n. Then, we see progressively more sophisticated examples of responses of students having constructed this action for certain classes of values of n. This culminates in the situation in which a student constructs a process that produces a subgroup of $\mathcal{Z}$ for any n, the encapsulation of which is the set of all subgroups of $\mathcal{Z}$.

2. $n\mathcal{Z}$ for a single value of n and then repeat for other values of n. In this excerpt we see an example of a student who can repeat the action for specific examples, but is unable to make any general statement. After some errors, he responds as follows.

Jeff: Integers. So, that would be the other way around. No. OK. So, it would be... OH! I got it, I got it, I got it.

I: What have you got?

Jeff: $2\mathcal{Z}$.

I: What does $2\mathcal{Z}$ represent?

Jeff: The set of all even integers.

I: Could you write that out for me just so we can discuss this...?

Jeff: Like this? Is that what you mean?

I: Yeah. Whatever $2\mathcal{Z}$ represents to you. I just want to see that actual set. OK. So, it's got zero and positive even numbers in it?...

Jeff: Oh. Positive and negative.

I: OK. You're going to throw the negatives in there. I appreciate that. OK. And you're confident that's a subgroup? Can you name another subgroup?

Jeff: Uh... 3, uh, $4\mathcal{Z}$.

I: OK. Which again...

Jeff: Is the same idea, multiples of 4.

I: So, you say it looks like what?

Jeff: Plus or minus 4, Plus or minus 8, ...

I: OK. Do you remember any general statement from class about the subgroups of $\mathcal{Z}$?

Jeff: Uh... ... I remember... Hm... Oh, I remember, no, yeah. It has something to do with ideals...

The interviewer does considerable prompting, but Jeff remains unable to make a general statement. He does not seem to have interiorized his action of forming a subgroup with a single integer to a process. We did see some students progress towards constructing a process in that they moved from seeing that $n\mathcal{Z}$ is a subgroup for individual values of n to considering $n\mathcal{Z}$ for values of n in various classes such as all the even numbers, or all the primes.

3. **$n\mathcal{Z}$ for a single value of n, possibly repeat for other values of n, and then extend to all values of n.** In this case, we saw evidence of students having constructed a satisfactory process by being able to state that $n\mathcal{Z}$ is a subgroup of $\mathcal{Z}$ for all values of n. For example, here we see Joelyn moving from the action of a single value of n to the general case with ease.

Joelyn: OK, ammm, the subgroup of $\mathcal{Z}$ could have ammm the set of all even integers, that would work, ammm anything in the form $n\mathcal{Z}$ where n in $\mathcal{Z}$ would work.

We also saw cases in which the student went from a single value to considering classes of values of n, other cases in which the student went from considering several individual values to considering all values, and still other cases in which students started with classes of values and then moved to all values of n.

4. **$n\mathcal{Z}$ for all values of n.** Finally, some students just displayed their process by going directly to all values.

The interview question did not ask for more than a general statement so not all of the interviewers pushed to see if the student realized that $n\mathcal{Z}$, n an integer (or positive integer) gave all of the subgroups of $\mathcal{Z}$. However, some did and here we see Diane thinking about her process and, after some difficulty, coming to a reasonable position. We can see in Diane's responses, an interesting visualization of her process for specifying the subgroups of $\mathcal{Z}$.

I: How about back to the subgroups. Can you make a statement about all the possible subgroups of $\mathcal{Z}$?

Diane: OK. All possible subgroups of G–of $\mathcal{Z}$.

I: Of $\mathcal{Z}$. Uh-huh.

Diane: They're all the groups where elements are equal distance from the number 1. We see a number line now. Like 1, $1\mathcal{Z}$ obviously. The $\mathcal{Z}$ itself. $2\mathcal{Z}$, $3\mathcal{Z}$, $4\mathcal{Z}$. Another subgroup of addition is 0 itself.

I: O.K. . . So. . .

Diane: All. . .

I: All the possible subgroups I could come up with could be written in what, what way?

Diane: $x\mathcal{Z}$, where x is any positive, or actually any integer.

I: OK.

Diane: But there's probably more.

I: But there's probably more subgroups?

Diane: No, there couldn't be because then you'd start getting into repeating. If you had a subgroup with, let's say that it wasn't evenly–or even.

There was a misunderstanding on the part of the some of the interviewers about what to ask the students to prove. In some cases, the student was asked to prove that every $n\mathcal{Z}$, is a subgroup of $\mathcal{Z}$ and in other cases, the student was asked to prove that every subgroup of $\mathcal{Z}$ is of the form $n\mathcal{Z}$ for some positive integer n. A few students were not asked for any proof at all at this point.

All of the students who were asked to prove that $n\mathcal{Z}$ is a subgroup of $\mathcal{Z}$ proceeded to check the group axioms. Thus they indicated that they had a subgroup schema which they called upon and used in this situation. Some students omitted associativity, but since this was never pursued by the interviewer it may be that the student did not think to state explicitly that it was inherited. More serious was the omission of checking the existence of inverses. The students' calculations were generally correct although some stumbled with some details.

Among students who were asked to prove that every subgroup of $\mathcal{Z}$ is of the form $n\mathcal{Z}$, most had little or no success, but a few made reasonable progress. However, none was able to give a completely acceptable proof, even with prompting from the interviewer.

While we were not able to learn very much about the mental constructions students might make in proving this result, we can provide two excerpts that give some indication of how students might think about two main issues involved in ar