UNDERSTANDING THE LIMIT CONCEPT: BEGINNING WITH A COORDINATED PROCESS SCHEMA[1]

Jim Cottrill, Ed Dubinsky, Devilyna Nichols, Keith Schwingendorf, Karen Thomas, Draga Vidakovic

Many authors have provided evidence for what appears to be common knowledge among mathematics teachers: the limit concept presents major difficulties for most students and they have very little success in understanding this important mathematical idea. We believe that a program of research into how people learn such a topic can point to pedagogical strategies that can help improve this situation. This paper is an attempt to contribute to such a program.

Specifically, our goal in this report is to apply our theoretical perspective, our own mathematical knowledge, and our analyses of observations of students studying limits to do two things. First, we will reinterpret some points in the literature and second, we will move forward on developing a description, or genetic decomposition, of how the limit concept can be learned.

In discussing the literature, we will suggest a new variation of a dichotomy, considered by various authors, between dynamic or process conceptions of limits and static or formal conceptions. We will also propose some explanations of why these conceptions are so difficult for students to construct.

In describing the evolution of a genetic decomposition for the limit concept, we will give examples of how we used our analysis of interviews of 25 students from a calculus course to make appropriate modifications.

Outline of paper.

This paper is organized in five sections. We begin with very brief descriptions of our research paradigm and our theoretical perspective which we refer to as the APOS theory. The next two sections are the heart of the paper. They consist of a consideration of some points from the literature and a description of the evolution of a genetic decomposition of the limit concept. Finally we offer some pedagogical suggestions for instruction that could relate to how the limit concept can be learned.

A research paradigm

Our overall paradigm for doing this kind of research has been described in detail in Dubinsky, 1992a, so we need only sketch it here. Essentially, we work from a theoretical perspective to form a first version of a genetic decomposition of the topic(s) we wish to study. Based on this description of how the material in question might be learned, we design instruction which is then implemented. Extensive observations are made of the students in this teaching experiment and the results are analyzed in terms of the theory. The analysis usually leads to a revision of the genetic decomposition and the cycle is then repeated. The

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repetitions are continued for as long as appears to be necessary and thus there is an evolution of one or more genetic decompositions. If the results cannot be reasonably explained in terms of the theory, then revisions of the theory may be necessary.

Insert Figure 1 about here

We will discuss our theoretical perspective in the next section. The main contribution it makes to the design of instruction is to suggest specific mental constructions that can be made in learning the material. The instruction focuses on getting students to make these constructions. One important way in which we try to do this is to have students make constructions on computers, constructions that we feel can lead to their making corresponding constructions in their minds. Our pedagogical strategies also involve cooperative learning and alternatives to lecturing.

In making observations to gather our data, we use a number of methods ranging from standard examinations to in-depth interviews. The data is analyzed to see if the students appeared to make the constructions proposed by the theory. More specifically, we look at a particular mathematical point which some students appear to understand, but others do not. We try to explain the difference in terms of making or not making specific constructions. That is, we ask if (1) there are indications in, say, transcripts of interviews, that students who succeeded with this point seemed to be interiorizing a particular action to a process or encapsulating a certain process to an object while (2) these indications are not present in the data from students who did not succeed with this point. In this case, we add the observed construction to the genetic decomposition.

When a step in the genetic decomposition does not appear to arise in the data, and if we feel that the questions we asked did relate to the particular issue in the step, then we may drop it from the genetic decomposition. This is not a problem if the step arose mainly from theoretical considerations. If, however, it arose from data in previous experiments, then it becomes necessary to revise the genetic decomposition so as to be consistent with data from all experiments, past and present.

In this way, our theoretical perspective guides the analysis of the data at the same time that the data influences our genetic decompositions. Later in the paper, when discussing excerpts from interviews, we will give specific examples of how this works.

As we have indicated, it occasionally becomes necessary to revise the general theory in order to make it more effective in the working of our paradigm. Usually, this is done gradually over the course of several studies. For example, at an early stage of the development of this theory, we considered only two kinds of constructions — processes and objects. We found, however, in several experiments that there appeared to be important differences between students who, for example, insisted that a formula be present before they were willing to say that a function existed in a certain situation and those who could imagine a transformation without specific calculations (see Breidenbach et al, 1992, Dubinsky & Harel, 1992 and Schwingendorf et al, 1992). These seemed to correspond to differences between students' abilities to think about inverting or composing functions. We also found differences between students who appeared to be in control of a set of manipulations and those who seemed to be controlled by them. It seemed to us that the notion of *action* and the idea of interiorizing an action to a process (see below for explanations of these ideas) effectively captured these differences and so we added them to our theory.

More relevant to the present paper is the idea of a schema which is just in development. In the past we have described the construction of an object as an encapsulation of a process and asserted that, in this theory, this is the only way that objects are formed. But consider the concept of limit. Although the present paper is somewhat preliminary, we believe that we will eventually be able to interpret this concept as a coherent collection of actions, processes and objects and we will call that collection a *schema*. It seems to be possible to apply actions to a schema. For example one can think of the limit schema in relation to two functions and then in relation to a combination of functions and compare the results. As one progresses through calculus and into more advanced mathematics, one will encounter other versions of this notion: limit of a sequence, limit of a directed set, Banach limits, etc. Comparing these different notions amounts to performing actions on schemas.

All of this means that these schemas will have to be reinterpreted as objects. This will be, for our theory, a new way of constructing objects and it will have to be included in our analyses.

Given the fact, then, that both our genetic decompositions and the theory on which they are based are continually evolving, one might ask what the value is, at a given point in time, of a particular genetic decomposition. We certainly do not suggest that it is a "true" description of what is going on in the minds of the students we are observing, nor that it is in any sense "proved" or even established. We only claim that it is a tentative description that has the following kind of value:

- It provides a method for making sense out of a large amount of qualitative data.
- It provides a language for talking about the nature of learning particular topics in mathematics.
- It has the potential to suggest pedagogical strategies that could improve the extent to which this learning takes place.

The first two points are matters of judgement, but the third can be evaluated and it is part of our paradigm that the instruction based on our genetic decompositions eventually be evaluated to see what effect it has on student learning. We are not yet ready for this assessment in our investigations of the limit concept and so we focus in the present study on the first two points.

We are, in fact, in the midst of the first iteration through the paradigm's circle. Using our knowledge of the mathematics in question and the general theory (see the next section) a preliminary genetic decomposition was constructed, and instruction based on it was designed and implemented. Interviews of 25 students were conducted after they had gone through this instruction and the transcripts of the interviews were analyzed. Based on this analysis, a revised genetic decomposition was developed. In the section, "Evolution of a Genetic Decomposition," we will present both versions and illustrate how our analyses led us to the revision.

The APOS Theory

Our theoretical perspective arises out of an interpretation of Piaget's constructivism. This is described in detail in Dubinsky, 1992b so we will concentrate here on the modifications that have occurred since that was written. The main modification has to do with an increased role for schemas. Schemas have been discussed before in our theory (e.g., Dubinsky and Lewin, 1986), but it has not been until doing this research that we have begun to seriously consider them in the construction of specific mathematical concepts.

The following statement forms the basis for our perspective:

> Mathematical knowledge is an individual's tendency to respond, in a social context, to a perceived problem situation by constructing, re-constructing, and organizing, in her or his mind, mathematical processes and objects with which to deal with the situation.

We observe three general types of mathematical knowledge. These are actions, processes, and objects, and they are organized into structures, which we refer to as schemas. (Thus, we have come to refer, informally, to our theory as APOS theory.)

An *action* is any physical or mental transformation of objects to obtain other objects. It occurs as a reaction to stimuli which the individual perceives as external. It may be a single step response, such as a physical reflex, or the act of recalling some fact from memory. It may also be a multi-step response, but then it has the characteristic that at each step, the next step is triggered by what has come before, rather than by the individual's conscious control of the transformation. In this case, the results of the various steps in the response are at least partially controlling the individual (by triggering memory, for example), who is, in this sense, a part of the transformation. When the individual reflects upon an action, he or she may begin to establish conscious control over it. We would then say that the action is interiorized, and it becomes a process.

A *process* is a transformation of an object (or objects) which has the important characteristic that the individual is in control of the transformation, in the sense that he or she is able to describe, or reflect upon, all of the steps in the transformation without necessarily performing them. Once an individual has constructed a process, it can be transformed in several ways. A process may be reversed, or it may be coordinated with other processes. In some cases, this coordination leads to a new process (as in composition of functions). In others, the processes are linked (perhaps with other constructs) to form a schema. As the individual reflects upon the act of transforming processes, they begin to become objects.

An *object* is constructed through the encapsulation of a process. This encapsulation is achieved when the individual becomes aware of the totality of the process, realizes that transformations can act on it, and is able to construct such transformations. Objects can be de-encapsulated to obtain the processes from which they came, and it is often important in mathematics for an individual to be able to move back and forth between an object and a process conception of a mathematical idea.

A *schema* is a coherent collection of actions, processes, objects and other schemas, which are linked in some way and brought to bear upon a problem situation. As with processes, an individual can reflect on a schema and transform it. This can result in the schema becoming a new object. Thus, we now see that there are at least two ways of constructing objects — from processes and from schemas.

Objects can be transformed by higher-level actions, leading to new processes, objects, and schemas. Hence we have a mechanism which may be envisioned as a spiralling of action, process, and object within expanding schemas.

Some Comments on the Literature

There is general agreement in the literature that students have trouble with the concept of limit, whether it be in the context of functions and continuity or of series and sequences. See, for example, Artigue, 1992; Cornu, 1981, 1992; Davis & Vinner, 1986; Ervynck, 1988; Le & Tall, 1993; Monaghan, Sun & Tall, 1994; Robert, 1982; Sierpińska, 1987; Tall, 1981, 1982; Tall & Vinner, 1981; and Williams, 1991. Moreover, many of the difficulties encountered by students in dealing with other concepts (continuity, differentiability, integration) can be related to their difficulties with limits (Orton, 1983; Tall, 1992).

We have not, however, found any reports of success in helping students overcome these difficulties. Some authors (Tall, 1992; Le & Tall, 1993; Monaghan, Sun & Tall, 1994) report that even using technology has not been successful in doing so. Unfortunately, we are not able, at this point in our project, to report any greater degree of success in helping students learn this concept.

What we can try to do is to cast some aspects of these reports in the language of the APOS theory and thereby, it is hoped, provide a framework in which to look for pedagogical solutions.

Cornu (1992) gives a good summary of various theoretical viewpoints in which limits have been considered. He also forms a synthesis of these ideas with some of his own. He comes close to casting the overall situation in the context of the APOS theory when he writes:

> That the limit concept is essentially difficult may be seen in the way that it is defined in terms of an unencapsulated *process*: "give me an $\epsilon > 0$, and I will find an N such that ..." rather than as a *concept*, in the form "there exists a *function* $N(\epsilon)$ such that ..." [p. 163, emphasis in original]

We would like to pursue this line of argument.

Some authors such as Cornu (1980, 1981) and Sierpińska (1990, 1987) report that a high percentage of students have what they call a static view of mathematics and can only deal with a very specific calculation that is placed before them. One may interpret such a static view of limit as indicating a pre-conception, or perhaps action conception. We would include in this the idea of some of our students that the limit of a function at a is the same as the value of the function at a, or the value of the function at a point very close to a — always provided there is an explicit formula for computing these values. One distinction between pre-conception and action would be in whether the student tends to evaluate several points or only one before coming to a conclusion about the limit.

Cornu (1981) points out that the limit concept is, for most students, the first topic in which mathematics is not restricted to a finite computation that gives a definite answer. In our view, this is precisely a distinction between an action and a process. Indeed, we would argue that once a calculation involves an infinite number of steps, it can only be understood through a process conception. Since an action is something which can be (and usually is) performed externally, when a student states or thinks "and so on ...", he or she has moved to process by definition — since the calculation is not actually performed, but only contemplated. Thus we would say that a student who cannot go beyond calculating a finite number of values of the function at points near a has an action conception of a variable approaching a fixed quantity, which has not been interiorized to a process.

We find in the literature that there is general agreement that process or operational conceptions must precede the development of structural or object notions (Sfard, 1987). There are, however, two different views of the relationship students have to a process conception of limit, which is sometimes referred to as a dynamic conception. Some authors seem to indicate that a dynamic conception is easy and natural for students to develop (Tall, 1992, 1981). According to this view, the main difficulty is for students to pass from a dynamic conception to a formal understanding of limits (Ervynck, 1988; Williams, 1991). There is even a suggestion (Tall & Vinner, 1981; Williams, 1991) that students' dynamic conception hinders their movement towards developing a formal conception. A group of authors (Le & Tall, 1993; Monaghan, Sun & Tall, 1994) proposed a 'functional/numeric computer paradigm' and a 'key stroke computer algebra paradigm' as a transitional environment which could help students to move from a dynamic to a formal understanding of limits. Preliminary attempts using these paradigms have not proved successful.

Other authors seem to feel that developing a strong dynamic conception is necessary for a formal understanding, and that the latter must build on the student's dynamic conception. In this view, the difficulty comes in constructing the dynamic conception and this difficulty is the obstacle to understanding (Orton, cited by Artigue, 1992; Davis & Vinner, 1986). The most commonly cited difficulty that students have in constructing a dynamic notion of limit is their perception of a limit as something that is never actually attained. Mamona–Downs (1990) suggests that students' understanding of the concept of continuum could offer some explanation for this misconception. Other typical student errors are that the limit must be approached monotonically and that the limit value is bounded. Williams (1990) suggests that, while it is possible to avoid the formation of these misconceptions, it is less easy to help the students move from a dynamic to a static notion of limit.

We will argue in the next section that the standard informal, dynamic notion of *the values of a function approaching a limiting value as the values in the domain approach some quantity* is more complicated than may have been thought. It is not just a single process, but a coordinated pair of processes which is, in fact, a schema. Our observations lead us to agree with the position that constructing this schema should be the first order of business for most students, that most students have not succeeded at making such a construction, and that this helps explain their difficulties with the limit concept.

On the other hand, there is in the works we have cited, a tendency to set up a dichotomy between dynamic and static conceptions of limit with the latter being identified with the formal ϵ-δ definition. The APOS theory and our own understanding of the limit concept allows us to speculate on a different interpretation. According to this view, the schema of two coordinated processes (as $x \longrightarrow a, f(x) \longrightarrow L$) is reconstructed to obtain a process described as $0 < |x-a| < \delta$ *implies* $|f(x) - L| < \epsilon$. This is a (mental) process in the sense of going from the hypothesis to the conclusion. The next step is to encapsulate this process into an object and then apply a two level quantification schema (forall ϵ there exists δ such that ...). We see this as still very much dynamic, involving processes, but in a different way from a dynamic schema of two coordinated processes described above.

According to this formulation there are (at least) two reasons for student difficulties with the formal concept. First, the coordinated process schema is difficult in itself and not every student can construct it immediately. Second, it is necessary to have an understanding of quantification (Tall & Vinner, 1981; Sierpińska, 1990, 1987). Unfortunately, there is evidence suggesting that students do not have a conception of quantification which is sufficiently powerful to deal with the formal limit concept (Dubinsky, Elterman, & Gong, 1988).

Evolution of a Genetic Decomposition

Our description of this evolution will consist of four parts: a preliminary genetic decomposition; an indication of the instruction that was implemented; a revised genetic decomposition; and a discussion of how our observations influenced the changes that were (and were not) made.

Preliminary genetic decomposition

Our description of what might occur is organized in six steps that occur only very roughly in the given order and with a great deal of "backing and filling" as the student constructs the concept of limit.

1. The action of evaluating the function f at a few points, each successive point closer to a than was the previous point.
2. Interiorization of the action of Step 1 to a single process in which $f(x)$ approaches L as x approaches a.
3. Encapsulate the process of 2 so that, for example, in talking about combination properties of limits, the limit process becomes an object to which actions (e.g., determine if a certain property holds) can be applied.
4. Reconstruct the process of 2 in terms of intervals and inequalities. This is done by introducing numerical estimates of the closeness of approach, in symbols, $0 < |x-a| < \delta$ and $|f(x) - L| < \epsilon$.
5. Apply a quantification schema to connect the reconstructed process of the previous step to obtain the formal definition of limit. As we indicated in our comments on the literature, applying this definition is a process in which one imagines iterating through all positive numbers and, for each one called ϵ, visiting every positive number, calling each δ this time, considering each value, called x in the appropriate interval and checking the inequalities. The implication and the quantification lead to a decision as to whether the definition is satisfied.
6. A completed ϵ-δ conception applied to specific situations.

Instructional treatment

The instructional treatment of limits took place during the first six to seven weeks of a first semester experimental calculus course which is part of a general curriculum development program in which the authors have been involved. The design of the course takes into account preliminary genetic decompositions of the topics covered. In this case, students performed several computer activities during the first weeks of the course prior to a two week unit focusing on the concept of limit. The pedagogical strategy used in this experimental course consists of a combination of computer activities (designed to help students make the mental constructions in the genetic decomposition), classroom tasks without computers followed by discourse (aimed at getting students to reflect on what they had done on the computer and could do with pencil and paper), and exercises (to help students reinforce the knowledge they had constructed). All of the student work was done in teams of three, four or five students. The limit concept was reconsidered throughout the students' subsequent study of calculus, in particular with derivatives, integrals, and sequences.

There were five kinds of computer activities embedded in the usual topics of approximation, one- and two-sided limits, rules for computing limits, properties of limits, the formal definition of limit and applications such as limits of difference quotients related to slopes of tangent and instantaneous rates of change.

1. Computer investigations of approximation.

Students wrote computer code to compute, for example, the average rate of change of a falling body over a small interval of time. They were able to use their code to investigate what happens as the length of the time interval takes on increasingly small values.

2. Graphical investigations of the limit concept.

Values of the difference quotient for small differences were an important class of examples in all of the activities leading up to the concept of derivative in this course. For example, the students were asked to consider the following freezing point function F of a mixture of two chemicals A and B as a function of the mole fraction m of chemical A which is given by the expression,

$$F(m) = \begin{cases} 5m^4 - 23.6m^3 - 146m^2 + 129m - 17 & \text{if } 0 \leq m \leq 0.76 \\ 45.056(m - 0.76)^{\frac{1}{3}} - 11.98132480 & \text{if } 0.76 < m \leq 1 \end{cases}$$

The students were asked to estimate the slope of the tangent to the curve of this function at $m = 0.76$ in two ways: first by using the computer to produce a graph and then measuring the slope of the secant; and second by using a computer algebra system to obtain a formula for the difference quotient using small differences.

3. Computer constructions of a value approaching a limit.

It was hoped that students would tend to interiorize the action of trying values increasingly close to the limit point by constructing computer code that would evalute a given function at each of a finite sequence of points, appropriately chosen for this purpose. To this end, it was suggested that they use code like,

```
a := 2;
s := [a + ((-1)**n)/(1000*n) :  n in [1..20]];
for i in [1..20] do
print f(s(i));
end;
```

for various choices of f and a. The choices required them to look, not only at functions given by rational, trigonometric and transcendental expressons, but by less traditional functions, in some cases, having split domains.

In this kind of activity, students were asked to predict the value of the limit and to write out descriptions of various activities in connection with this situation such as the values of $s(i)$, the values of $f(s(i))$ and the role of $(-1)^n$. They were also asked to think about various "ways of approaching a".

Although this code was given to students, they were asked to construct their own modifications to implement the idea of approaching from one side.

4. Computer constructions of the limit concept.

Students studied and modified a program, provided by the instructor, which approximated values of limits. This program used a sequence of domain values (on both sides of the limit point) and then used values of the function at some terms of the sequence of domain values to estimate the limit of a given function at a point. The program was applied to the same examples that had been studied previously.

Again, students were asked to make modifications corresponding to limits from the left and limits from the right.

5. Investigations of ϵ-δ windows.

Finally, in aiming at an understanding of the formal definition of limit, we asked students to perform tasks in which they were given the vertical dimensions of a window on the computer screen and were asked to adjust the domain scale of the graph of a function so as to prevent the graph from leaving the screen through its upper or lower edges. The students could choose the dimensions and the computer would draw the "box" and the graph using the horizontal dimension of the box as domain. The student could see whether or not the vertical range of the graph forced it outside the top or bottom of the box.

This was done over a period of several weeks in which students were gradually taken from studying cases in which the window had a fixed dimension through varying the width of the window so as to see different results, and finally to varying the height of the window to a point where it was possible to find a width so that the graph of the function stayed inside the window.

Again, this approach was applied to examples which the students were using in other problem situations and it included both cases in which the limit exists and cases in which it does not.

Revised genetic decomposition

In analyzing the transcripts of our interviews, we decided to make two major revisions in the genetic decomposition. The other points seemed to relate reasonably to the data, or did not appear for one reason or another.

The first revision was to add a step that seems to come even before the action of evaluating the function at several points approaching the point at which the limit is to determined. This is a very static activity and it is only an action in the sense of the manipulations required to evaluate a function at a single point.

The second revision has to do with the nature of the process conception of limit. Although this is not discussed explicitly in the literature, one gets the impression that most people are thinking about a single process. Our observations suggested that there may be more than one process involved and Step 3 in the revised genetic decomposition reflects this finding.

We will discuss below the relationship between the data and the following genetic decomposition.

1. The action of evaluating f at a single point x that is considered to be close to, or even equal to, a.

2. The action of evaluating the function f at a few points, each successive point closer to a than was the previous point.

3. Construction of a coordinated schema as follows.

   (a) Interiorization of the action of Step 2 to construct a domain process in which x approaches a.

   (b) Construction of a range process in which y approaches L.

   (c) Coordination of (a), (b) via f. That is, the function f is applied to the process of x approaching a to obtain the process of $f(x)$ approaching L.

4. Perform actions on the limit concept by talking about, for example, limits of combinations of functions. In this way, the schema of 3 is encapsulated to become an object.

5. Reconstruct the processes of 3(c) in terms of intervals and inequalities. This is done by introducing numerical estimates of the closeness of approach, in symbols, $0 < |x-a| < \delta$ and $|f(x) - L| < \epsilon$.

6. Apply a quantification schema to connect the reconstructed process of the previous step to obtain the formal definition of a limit.

7. A completed ϵ-δ conception applied to specific situations.

The revised genetic decomposition and the observations.

We will organize this part of the discussion by using the same numbers as in the revised genetic decomposition. Where a step is the same or similar to what appeared in the preliminary version, we will attempt to explain why we did not change it. The notation Step n will refer to the revised genetic decomposition.

1. We introduced this Step 1 because we found students who did not seem to be thinking beyond a single action. For them, the limit of a function f at a point a was an extremely static notion that was not really very different from the the value of f at a. In the following are some responses from Charles which are typical of what was heard from these students. At first we see Charles working with the greatest integer function which, in the course, was given the name *floor*.

   I:[2] *OK, what I'd like you to do is determine L. At $a = 0.5$, what is the limit of $f(x)$?*
   Charles: *Zero.*
   I: *And why do you think it is zero?*
   Charles: *Because the floor of 0.5 is zero.*

   This excerpt is just one example of several situations in which Charles reiterated his understanding that the limit of a function at a point means the value of the function at that point. In talking about the limit in general, he was quite explicit about the equality of $f(a)$ and the limit at a, the only exception being when the limit does not exist.

[2]We use the notation I: for the interviewer in all excerpts even though the actual interviewing was done by several people, all members of our research team.

Charles: *Yeah, after you plug in a in the function, you come out with what L equals.*
I: *Ok, if its ... will that always happen, or are there cases where...*
Charles: *Uh, well, you can plug in and get no limit for infinity.*
...
I: *Well, which ok, um, if $f(a)$, does $f(a)$ have to equal the limit?*
Charles: *No. Well, f ... Yeah, $f(3)$, yeah it does.*
I: *It does?*
Charles: *Yeah, it would have to equal the limit.*
I: *Now, are there cases where it wouldn't?*
Charles: *Like if it had an undefined limit, or no limit at all?*

He does this even at jump points. Indeed, in considering the function *floor* at the point $a = 1$ he refers to the jump but still considers the limit to be the value.

Charles: *Is that where the jump is, between zero and one, because there is nothing between zero and one?*
I: *Yeah. So what I'm going to ask you the question again. This floor function at one, what do you think the limit will be?*
Charles: *At one, the limit will be one.*
I: *Why will it be one?*
Charles: *Because the floor on one is one.*

In the following excerpt, Jean is quite explicit about this. When asked about limits in general she explains her idea for the limit at -2 of the function f given by $f(x) = x+2$. At all times she focuses her attention on the static situation of evaluation of the function at a single point, first -2 and then "a point very close to negative 2". Notice that although she refers to points on both sides of -2, she always speaks of a single value, or two values, never indicating a process of looking at several values approaching 2.

I: *So when you're actually looking at a limit situation and trying to determine if a limit does or does not exist, what are you doing?*
Jean: *First I plug the negative 2 into my function $f(x)$ to see if it is defined.*
I: *Mhm.*
Jean: *If it is, then that is the limit. If negative 2 does not exist, or if it's not defined at negative 2,...*
I: *Mhm.*
Jean: *...then I would take a point very close to negative 2, maybe on each side of negative 2*
I: *Mhm.*
Jean: *to see if those two values are very close to the same number.*

Thus, for Jean as well, the limit at a is the same as the value at a, if the latter exists. If it does not exist, then Jean's response at the end of this excerpt suggests that she determines the limit of f at a by taking two numbers on either side of a and seeing if the values are close to the same number. Using these two values may be the beginning of a notion of comparing the limit with the limit from the left and the limit from the right. What is important here is that, again, there is no suggestion that she is thinking about anything other than a single pair of values.

It is possible that a student thinking like Jean might fall into a difficulty coming from an emphasis on the mathematical fact that if the limits from the left and from the right both exist and are the same then the limit exists and is equal to this common value. The difficulty could arise in that a student might determine the two one-sided limits without any process conception and, if the same answer is obtained, make an unwarranted conclusion that the limit exists.

2. Some students made statements similar to those of Charles and Jean at one point in the interview, but later gave responses that indicated they may have been in transition to Step 2. Following is an excerpt from an interview in which the student seems to be struggling between a static evaluation and the beginning of a more dynamic action of several evaluations. The discussion is about the general situation of the limit of a function at a point a.

   Notice that at first, Chester appears to be uncertain about whether to "...plug in the value of a .....", or replace his x's with a. He does, however, give indications that he is thinking about several values in the domain.

   > **Chester:** *I guess, um,... the limit of the function would be... if you plug in ... the ... as x approaches the a value, if you plug in the value of a into... into... if you plug it into your x's in your function, then that's what your limit is.*

   And a few minutes later, talking about an alternative explanation, Chester gives a clear description of a domain process, but then at the end, reverts to using the single value a to obtain the limit.

   > **Chester:** *Well, that meant to me is... is I plugged in numbers... well, I guess it's kind of the same... as I plugged in numbers getting closer and closer to a, it approached my limit L.*
   > **I:** *Mhm. OK, that's what I thought you meant here.*
   > **Chester:** *Yeah, that's what I, OK, so I guess I didn't really change anything. I just went directly and said... plugged in a for x to equal my L.*

   Here is an example of a student who appears to have taken Step 2 and may even be at Step 3(a). Donald is talking about the limit of the function *floor* at 3. He refers to approaching and gives several explicit values for x that suggest he has constructed a domain process in his mind. He says nothing, however, that tells us about any range process.

   > **Donald:** *Well, I mean like if you say a is... or a is 3 and x is approaching it, I mean, you know, you made 2.91 and 2.92 and 2.93, that's x approaching it.*

3. In constructing the preliminary genetic decomposition of the limit of a function f at a point a, we considered that the action of successive application of f to a few points, each one closer to a than the previous one would be interiorized to the process of *as x approaches a, f(x) approaches L*. This was not easy for students and the excerpts we will give suggest that the construction may proceed in three steps.

   (a) In the following excerpt, Norton is talking about the limit in general and he considers explicit successive values of x, starting quite far from 3 and then taking two values closer to three, finally referring to "smaller intervals" which suggests he may have interiorized the action connected with explicit values. He then seems to jump to the statement that "the function will approach a limit" and it is not clear that he is talking about values of the function. In his next statement he talks of "...it approaching a limit L" and again, it is not clear what "it" refers to. It is possible that he is thinking about the values of the function, but there is a difference in how he speaks about the range compared to how he referred to what is going on in the domain. That is, Norton seems much more explicit about the process of x approaching a than he is about the values of $f(x)$ approaching L.

Norton: *Um, if you start, if, if a was, say, 3 and you started with x as 10, you go 10, 9, 7, 6, whatever, then you get 3.2, 3.1, and get smaller intervals between x and a, the function will approach a limit and the limit will be L.*
I: *... How close do things have to get before you're willing to call L a limit?*
Norton: *Oh, tricky. Um ... If, well, it should – you want get, start getting smaller intervals between x and a and if you do a sequence of points, you will see it approaching a limit L. If it doesn't approach a certain number, then the limit doesn't exist. But in this function, it says that it has the limit L, so as your x approaches a, you should, the function should assume the limit L.*

A few minutes later, he repeats this difference in considering limits of the function *floor*. Here he again omits giving any specific values of the function although he does mention several numerical values of x along with the general process in which "x is approaching one-half". The only thing he does in the range is to evaluate the greatest integer function at one-half. Here, he is quite explicit that "it" refers to one-half and not the various values of x he has considered. Norton's reluctance to deal with values of the function is not due to any difficulty with this particular function as he has indicated earlier in the interview that he knows quite well how to evaluate the greatest integer function for specific values of x.

Norton: *Alright, um, for the first one, as x is approaching one-half, the limit of that should be 0. Because you plug, you start with ... x is at any value, like 4 or whatever, and then you go all the way down to one-half, by plugging it into the function, then you get the limit of the greatest integer of one-half. And the greatest integer of one-half is 0, so the limit is 0.*

Thus it could be that in the domain, Norton is interiorizing the action of successively replacing x by various values to a process of x approaching a, but in the range, he is still thinking about a single value, $f(a)$.

(b) Once a student has constructed a domain process, there are several possibilities for what can happen next. It can happen that a student has difficulty progressing beyond this point to a range process. For example, in the following excerpt, Kerrie is asked to explain what she understands by "the limit of f at a". She appears to be thinking about x approaching a dynamically, but in the range, she speaks only of applying f to a single value of x close to a.

Kerrie: *OK, it means that when the f(x) function gets close to a, the value will be L.*
I: *Ohh, can you say that again?*
Kerrie: *When you put the, as x goes to a, you put a value very close to a into your function f(x) and you'll get a value that's about L.*

Another possibility is that, in addition to constructing a domain process, a student might construct a range process somehow (e.g., by pushing buttons on a calculator) but does not appear to use the function to coordinate it with the domain process. The two processes seem to exist independently and the function appears to be forgotten.

In the following excerpt, the interviewer has just asked Rose about $\lim_{x\to 0}\cos(\frac{1}{x})$. First we see that she appears to construct a domain process. When she says that "...it comes closer and closer to one", she seems to be constructing a range process, but becomes confused when the interviewer tries to probe for the connection between these two processes.

Rose: *What am I going to do here? I am going to look at the graph. Ok it's undefined here at zero.*

I: *Why?*
Rose: *Because it says x can't equal zero right here.*
I: *Good point.*
Rose: *Ok, so...hum...Ok as it approaches here,* (calculates) *as it approaches zero from the right, it comes closer and closer to one. So...,*
I: *Why do you say that?*
Rose: *Because, I plugged into the calculator, and plugged in a bunch of stuff and it gets closer and closer to one.*
I: *Tryin' something different, you tried two points.*
Rose: *Right.*
I: *And what happened...*
Rose: *Um, well, I know it's becoming...ok...*(pause)*...ok, let me try something...I don't know. I'm not getting anywhere.*
I: *Why not?*
Rose: *Because.*

Then the interviewer gives Rose some positive feedback and she jumps to discussing what happens with the values of the function, but does not indicate the relation between the two processes.

I: *I'll let you know, these numbers you are getting are correct. You're doing the right sort of button pushing and all that.*
Rose: *Oh,that's good...You see. Well from what...Is this one of those oscillating function things?*
I: *Well it's cosine and cosine of something, it's gonna oscillate, yes. But what do you mean by "one of those oscillating things?"*
Rose: *Um, it means that it'll just go back and forth between at this point, I would assume one and negative one, as it gets closer and closer and closer to...the limit. Right? Yeah.*

The interviewer attempts to prompt Rose to see that the behavior in the range comes from applying the function to what is happening in the domain. It appears, however, that for her these are different processes and although she may begin to see that there is a connection between them, she does not appear to understand that it is the function and its process which makes this connection. Instead, she focuses on the physical curve in the graph of the function.

I: *Why would a function do that?*
Rose: *Why would it? I don't know.*
I: *You asked me if it's one of those, and I'm trying to bounce the question back to you.*
Rose: *Why would it?*
I: *Or can you figure out...think through maybe, if this one is or not...what makes you think it is?*
Rose: *Well because I keep pushing in different numbers in my calculator and at some points it'll go closer to one. And at other points it'll get closer to negative one and then sometimes I'll get somewhere in between the two. And it keeps, it doesn't go just near one number, it keeps going back and fourth. And so I know that's what an oscillating function does. Or a discontinuous oscillating thing. And since I know that, I just know that when something oscillates it just gets closer and it the...the curves get closer and closer together as it approaches the limit.*
I: *What is the limit?*
Rose: *Zero.*

(c) Here is an example of an excerpt which suggests that the student is thinking about constructing the range process by applying the function to the domain process. The role of the function in forming the range process is expressed in his various references to taking the reciprocal of a number and to applying the cosine function.

Robert: *...If you have a number very close to zero, but not equal to zero...,*
I: *yeah,*

Robert: *one over that number would be a very large number of some, of some value that we don't know. Which would give you a cosine of some value which we don't know. Which would mean it would be between one and negative one, because those are the, the range of the cosine.*
I: *Yeah.*
Robert: *So the limit is obvious between one, obviously between one and negative one. But where it is between those you can't really, it's very, very hard to tell. And from what I know it's impossible to tell, simply because you can't, we, I have no way of determining what a, one over a very, very small number is, except that it's a very large number. I have no way of determining what exactly that number is. Unless I use real examples, real number examples which...you know, are kind of, not good here.*
I: *So what are you saying about the limit?*
Robert: *I'm saying that the limit doesn't exist.*

Robert's next comments suggest that he is thinking of a domain process ("...as you get very close to 0...") and is trying to apply the function to obtain a range process.

I: *Ok, and just to clarify the basic, most... I mean you just told me why, but now I'm, I want to get a sense of it again.*
Robert: *Yeah.*
I: *Why it doesn't, why you're saying it doesn't exist.*
Robert: *It doesn't exist because as you get very close to zero, one over zero, one over very close to zero, is, it's, I want to say infinity. Which it is sometimes but in this case you can't really say, "infinity." Because you need the cosine of a number.*

The interviewer and Robert then negotiate what happens when x is positive or negative and Robert returns to what we take to be a description of a coordinated process, especially in view of his very last sentence.

I: *As x is getting smaller, one over x is getting bigger though?*
Robert: *Yeah,*
I: *Assuming x is positive.*
Robert: *Right, well even if x were negative, the cosine of a negative value equals the cosine of that same value. So it doesn't matter whether x is positive or negative.*
I: *Ok.*
Robert: *For this one, I think.*
I: *Ok, you're right it's...*
Robert: *Yeah, yeah. That's right, yeah.*
I: *symmetric, yeah.*
Robert: *So, um, even, it doesn't matter which direction you're coming from, you, you're going to get a very large number.*
I: *All right.*
Robert: *So it's just, you don't know the cosine of a very large number. It changes as that number changes by just the slightest bit.*

4. Turning now to object conceptions of the limit concept, we consider the possibility of a student thinking, for example, about the limit at a of the sum of two functions f and g. We assume that the student has a conception of forming this sum by coordinating the processes of the two functions and encapsulating the resulting process to obtain the function $f + g$.

   First, the student should indicate an awareness that there are three objects, $\lim_{x\to a} f(x)$, $\lim_{x\to a} g(x)$, and $\lim_{x\to a}(f + g)(x)$ which are to be computed by applying the limit schema. This can be done in two ways. In one way, the limit schema is de-encapsulated and applied to f, g and the two resulting limits are added. Alternatively, the function $f + g$ is computed and the limit schema is de-encapsulated and applied to it. The mathematical fact is that these two methods give the same result.

There were some students who treated the limits as objects, but instead of de-encapsulating to get at a process conception, they merely applied a generalized distributive law.

> **Calvin:** *You can almost say that's like, a, uh, you could say those ... you could almost say right here that if you ... if a person was looking at this ... say, uh, almost like substitution here, say U equals the limit as x approaches a of that. And then carry that U like this ... of x plus g(x) ... and think of it as a variable. Then multiply across; Uf(x) plus Ug(x) ... then when you substitute what U is equal back in there ... you get all that into the equation right there.*

Other students did not give any indication that they were thinking about limits as objects in any way, but simply went directly to the processes and coordinated them.

> **Paul:** *Um, the limit, if you want to find the limit, of that one big graph, you would take points a little to the left and a little to the right, together, added together, which is what's going on here. You're taking a limit of the big graph equals the limit of the points a little to the left and a little to the right of the little graph, or the two little graphs.*

Finally, some students did appear to come close to expressing what we described above. In the following excerpt, Barbara does not indicate the de-encapsulated process very clearly, but she may have skipped this in attempting to go on to a formal proof, which she at least begins.

> **Barbara:** *Well, I said that, um, the limit of this had to be the whole thing, the whole, the whole function as like f of, the function f plus the function g ...*
> **I:** *Mhm*
> **Barbara:** *We had to find the limit of that whole thing.*
> **I:** *Mhm*
> **Barbara:** *And for this one you have two little limits ...*
> **I:** *OK*
> **Barbara:** *... that we had to work individually and then add them.*
> **I:** *OK, so what is this notation down here?* (The interviewer points to her expression $f(x) + g(x) - \epsilon = f(x) - \epsilon + g(x) - \epsilon$ together with $a - \delta$)
> **Barbara:** *The limit of this, the f function plus the g function...*
> **I:** *Mhm*
> **Barbara:** *...minus ϵ, and then your point a, $a - \delta$.*
> **I:** *OK, what about the other side? I'm sorry.*
> **Barbara:** *Oh.*
> **I:** *I was just waiting for the right hand.*
> **Barbara:** *And then this is the function $f - \epsilon$.*
> **I:** *Uh-huh.*
> **Barbara:** *Um, and $a - \delta$ that ... function plus the g function minus ϵ would be ... and $a - \delta$ and you add those two together...*
> **I:** *Mhmm-hmmm.*
> **Barbara:***...and that should equal this one.*

In the following excerpt, Jolene may be thinking about limits as objects when she refers to "the first one" and "our second one". She does give some indication of what might be a de-encapsulated process.

> **Jolene:** *Ok, there's the first one.*
> **I:** *Mhm-hmmm.*
> **Jolene:** *And here's our second one so, ...* (brief pause) *... and so, looking at my two examples, the graphs cross at a certain point. So, you're coming in on both sides on one function it's gonna go to a certain point, and you're coming in both sides on the other function it's gonna go to a certain point.* (brief pause) *All right, when you add them it's not*

gonna change the graphs. So, they're still gonna come into the same limits. So, if you add them first, then take the limit, it's not gonna change the limit. Or if you took the limit and then added them, it wouldn't change it because ... just add the two functions doesn't change the values of ... of your graph ... of your function.

5. Only a few of the students that we observed gave any indication of passing very far beyond the first four steps of this genetic decomposition. In general, they had only the vaguest notion of the standard inequalities involved in the ϵ-δ description of limit. A typical response regarding a formal definition of limit was the following.

   I: *What about this ... this formal definition of limit?*
   Donald: *I never understood it in the first place, but I ... I know what it is, but ...*
   I: *Just tell me what you remember.*
   Donald: *Um, what was it? I know it has to deal with, uh, ... epsilon and delta ... this much I know.*
   I: *What about epsilon and delta?*
   Donald: *Um, that ... it ... it ... um, epsilon is what you add onto L ...*
   I: *Mhm*
   Donald: *So, basically, the difference between $f(x)$ and L would be ... epsilon. The difference here would be delta.*

   Donald illustrated his statements with a picture showing a generic graph and some parts of the standard ϵ-δ picture.

   Some students tried to explain the situation more in terms of intervals than inequalities, but they did not get much further.

   Joseph: *... so in order for this to be true, like I said over there, if you move an epsilon away from this point either way, then you should be able to move a delta from this point either way, it should be the same. So, if you move the epsilon to the left you should be able to move the delta to the left, and the epsilon, you know, the same epsilon, if you move from the left to the right, you should be able to move the same delta, you know, right or...*

   Following is a student who does seem to have captured the limit process in static intervals, but neither he nor any of the other students we observed went much beyond this to apply a quantification schema.

   Bob: *Uh, you wanted to figure out which. uh,... something $a-\delta$, $a+\delta$, so that the whole uh, function* (draws a graph and a box at the usual place and points to relevant portions of it), *so that $a-\delta$ and $a+\delta$ were within the limits of $L+\epsilon$ and $L-\epsilon$.*
   I: *OK, so in other words you want this...*
   Bob: *Yeah, the graph would be in between the limits of $L+\epsilon$ and $L+\epsilon$.*

6. There were no students who progressed to the point where we could ask questions that indicated their thinking relevant to the last two steps of the preliminary genetic decomposition. We repeat them in the revised version, although they might be dropped for the present since there is no evidence for them.

Pedagogical Suggestions

The main pedagogical suggestion that we would make based on the considerations in this paper has to do with constructing a domain process and range process and using the function to coordinate them. We propose certain computer activities that could help students construct a domain process separately and then, in their minds, use the function to construct the range process.

To help achieve this, we have constructed a computer tool called `LimitProcess`. This tool requires students only to continually punch a key and interpret what they see on the screen. What appears each time they press this key is a number. This number in fact converges to a certain number which the students do not know. There are variations in which the convergence is from one side, two sides, or monotonic; and variations in which the limit is fairly obvious such as 0 or less obvious, such as $\sqrt{3}$ or π or e.

The students are asked to write an explanation of what they think is going on and to answer certain specific questions. These questions ask them to predict an ultimate value, compare the different modes of convergence, and, given a small number such as 0.001 to decide how many times they must press the key before the number they see differs from their predicted value by less than 0.001 and remains so. (The system is set up so that as they press the key, the number will get close to the limit fairly quickly, but then jump away for some time before settling down to stay within an interval. This is then repeated for smaller intervals.) Here are some examples of how this would be done.

The student enters the code

```
LimitProcess();
```

and then follows instructions to repeatedly type `1` and note the output each time. The process can be halted at any time by typing `0`. What the student sees on the screen each time is a number. The numbers that appear are approximations to a single number. The approximations improve, but not in a single direction. They might oscillate and there will be an occasional number that is farther away from the limit than numbers which have appeared recently, although these aberrations also eventually approach the limit. The student works with a few variations (referred to as `P1`, `P2`, and `P3`) related to speed and direction of approach and then the following tasks are set.

1. Write an explanation of what you think is going on.
2. Explain what would happen ultimately if you continued to type `1`.
3. Compare and contrast what happens with processes `P1`, `P2`, and `P3`.

After this initial period of exploration, the following activity is given to the student.

> Denote by x the current value which appears on the screen when you enter `1`, and denote by a the ultimate value that you would get if you entered `1` indefinitely.
>
> Answer the following questions as you repeatedly enter `1`.
>
> 1. Write down your best estimate of the value of a.
> 2. How many times must you enter `1` before it happens that $|x - a| < 0.01$?
> 3. Once this inequality is satisfied, does it continue to hold?
> 4. How many times must you enter `1` before it appears as if the inequality will always hold?
> 5. Repeat the previous three questions with 0.01 replaced by 0.0001.

The next step in using `LimitProcess` is to introduce a function. Students are given examples of functions, either expressed with formulas or embedded in situations (from which the student must extract the formula). The task is to write a computer program that implements the function. This program is given to `LimitProcess` (as an optional parameter) and then, when the key is punched successively, the screen displays both a value in the domain and the corresponding value of the function. Activities similar to the ones without the function are performed. Their purpose is to get the students to construct the idea of a process coming from successive applications of a function to the numbers generated in the domain process.

Finally, the student is instructed to denote the two outputs which appear each time the key is pressed by x and $f(x)$ respectively. The following tasks are then set.

> Implement the process for some time and then estimate the ultimate values of the outputs x and $f(x)$ which appear on the screen. Call these ultimate values a and L respectively. Now, answer the followng questions.
>
> 1. Set $n = 2$. How many times must you enter `1` before you have the inequality $|x-a| < 10^{-n}$ and this inequality appears as if it will continue to hold. Repeat for values of $n = 3, 4, \ldots$.
> 2. Set $k = 2$. How many times must you enter `1` before you have the inequality $|f(x) - L| < 10^{-k}$ and this inequality appears as if it will continue to hold. Repeat for values of $k = 3, 4, \ldots$.
> 3. Try to fill in the dots so as to make a statement (as general as possible) of the form,
>
> $$\text{If } |x - a| < \ldots \text{ then } |f(x) - L| < \ldots$$

The students are asked to perform this task with examples of functions which have a removable singularity at the point in question, functions which arise from calculating the difference quotient of a function at a certain point, and piecewise continuous function in which the limit does or does not exist at the break.

As this paper is written, there are some experiments in progress in which the use of `LimitProcess` as described here is added to the other instructional treatments described earlier in this paper. We have not yet investigated the effect of using this additional pedagogical tool.

Limitations

A genetic decomposition is a tool which we use to make sense of data relating to a student's understanding of a concept. The two versions of our genetic decomposition presented here are grounded in the APOS learning theory, our own understanding of limits, and an analysis of the interviews. In no sense are we claiming that our data "proves" anything about our genetic decompositions. We only claim that the former illustrates and does not contradict the latter. Our work serves as a preliminary to the design of what we believe will be a more effective pedagogy to help students learn and understand the concept of limit. Some indications of our intentions for the design of more effective instruction were discussed. We plan to continue this research with a study of students who use our re-designed instruction in an attempt to further refine and improve our genetic decomposition of the limit concept and hence our future pedagogical strategy. We will also investigate the upper levels of our genetic decompositions.

Conclusions

We have tried to contribute to knowledge of how the concept of limit can be learned by analyzing some points in the literature and by providing a genetic decomposition of the limit concept as called for by Cornu, 1992. In trying to fit our observations to the APOS theory we have felt the need to pay more attention to the idea of schema than in our previous work with this theory. In this sense, our data has led us to revise the theory, or at least to make a modest change in the relative emphasis on various components of it. This change is in keeping with the very latest ideas of Piaget as expressed in the last book which he wrote (Garcia & Piaget, 1983).

Our analysis suggests that what has been called a dynamic conception of limit is much more complicated than a process that is captured by the interiorization of an action. As opposed to some researchers who believe that a dynamic conception may hinder progress towards the development of a formal understanding of the limit concept, we believe that the difficulty in moving to a more formal conception of limit is at least partially a result of insufficient development of a strong dynamic conception.

Our considerations lead us to the position that the formal concept of limit is not a static one as is commonly believed, but instead is a very complex schema with important dynamic aspects and requires students to have constructed strong conceptions of quantification. We conjecture that it is the requirement of constructing a schema involving the coordination of two processes together with the need for a sophisticated use of existential and universal quantification, rather than the "formal" nature of the standard ϵ-δ definition of limit that makes the limit concept so inaccessible to most students. Research on how students learn concepts of quantification, together with refinements in our genetic decomposition of limit could contribute to the design of effective instruction which may help students learn what is universally agreed to be the very difficult concept of limit.

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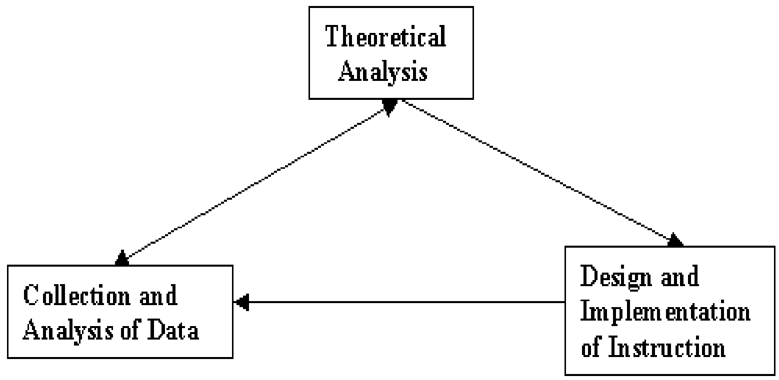
Theoretical Analysis
Collection and Analysis of Data
Design and Implementation of Instruction

Figure 1